Suppression of Coherent Radiation by Electrons in a Synchrotron*

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Approximate expressions are obtained for the coherent radiation loss by electrons in a synchrotron in the presence of finite parallel plate metallic shields, such as the pole faces of the magnet. The results would seem to provide a useful interpolation between the two simple limiting cases of shielding by infinite plates and no shielding at all.

INTRODUCTION

As is well known, the coherent radiation loss by electrons in a synchrotron, although independent of energy, increases with decreasing bunch size as the $(-4/3)$ power.[1] In some of the extremely high-energy synchrotrons reportedly under consideration, the bunching is likely to be sufficiently marked so that the coherent radiation loss could become serious. As is also well known, the coherent radiation has a spectrum mainly in the short-wave radio and microwave regions and hence can be suppressed in part by the use of metallic shields.[1] It is our purpose to extend some unpublished results of Schwinger,[2] in which this radiation was calculated assuming the orbit to lie midway between two plane parallel sheets of metal of infinite extent, to the case in which these metallic sheets are finite, as would be the case, for example, if the shielding were produced by the pole faces of the race track magnet itself. Our results, although necessarily rough, would seem to provide a useful interpolation between the two simple limits of shielding by infinite plates and no shielding at all.

POWER RADIATED BY ONE ELECTRON

We begin by writing an expression for the power P_n radiated in the nth harmonic by an electron moving in the $z=0$ plane in a circular orbit of radius R with angular velocity ω; namely,

$$P_n=\mathrm{Re}\left\{4in\omega e^2\int_{-\pi}^{\pi} d(\varphi-\varphi')G_n(R,\varphi,0;R,\varphi',0)\times[1-\beta^2\cos(\varphi-\varphi')]e^{-in(\varphi-\varphi')}\right\},\quad (1)$$

where $\beta=\omega R/c$. In the above, the Green's function $G_n(r,\varphi,z;r',\varphi',z')$, which is to be evaluated on the orbit as indicated, is the outgoing wave solution of

$$(\nabla^2+k_n^2)G_n=-\delta(z-z')\delta(\varphi-\varphi')\delta(r-r')/r,\quad (2)$$

with

$$k_n=n\omega/c=n\beta/R.$$

In addition, G_n must satisfy appropriate boundary conditions if a metallic shield is present. We consider three cases as follows:

I. No Shielding

In this case G_n is just the free-space Green's function $G_n^{(0)}$ which, when evaluated on the orbit, is given by

$$G_n^{(0)}(R,\varphi,0;R,\varphi',0)=\frac{1}{4\pi}\frac{\exp[2in\beta|\sin\frac{1}{2}(\varphi-\varphi')|]}{2R|\sin\frac{1}{2}(\varphi-\varphi')|}.\quad (3)$$

Substitution into Eq. (1) yields, after the angular integration is performed, the well known result[3,4]

$$P_n^{(0)}=(n\omega e^2/R)\left[2\beta^2J_{2n}'(2n\beta)-(1-\beta^2)\int_0^{2n\beta}J_{2n}(x)dx\right].\quad (4)$$

II. Infinite Parallel Plate Shields

In this case G_n must satisfy the boundary condition that it vanish on the metal plates. Taking these plates to be separated by a distance a, with the electron orbit midway between the plates, we thus require a solution of Eq. (2) subject to

$$G_n=0;\quad z=\pm a/2.\quad (5)$$

This function, which we denote by $G_n^{(\infty)}$, is easily derived[5] and can be expressed as

$$G_n^{(\infty)}(r,\varphi,z;r',\varphi',z')=(i/2a)\sum_{j=1}^{\infty}\sum_{m=-\infty}^{\infty}\sin j\pi(\tfrac{1}{2}+z/a)\times\sin j\pi(\tfrac{1}{2}+z'/a)\times e^{im(\varphi-\varphi')}J_m(\gamma_{nj}r_<)H_m^{(1)}(\gamma_{nj}r_>),\quad (6)$$

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[1] L. I. Schiff, Rev. Sci. Instr. **17**, 6 (1946). See also Eq. (23) below.
[2] J. Schwinger, "On radiation by electrons in a betatron," 1945 (unpublished). We wish to thank L. Jackson Laslett who called our attention to this material.
[3] G. A. Schott, *Electromagnetic Radiation* (Cambridge University Press, Cambridge, 1912).
[4] J. Schwinger, Phys. Rev. **75**, 1912 (1949). Our starting point, Eq. (1), with G_n given by Eq. (3), is essentially Eq. (III.7) of this reference.
[5] See, for example, P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), Chap. 7, particularly p. 892. A detailed derivation is given in reference 2.

where

$$\gamma_{nj}=[k_n^2-(j\pi/a)^2]^{\frac{1}{2}}=[(n\beta/R)^2-(j\pi/a)^2]^{\frac{1}{2}},$$

and where $r_<$ is the lesser of the two radii r and r', $r_>$ the greater of the two. Substitution into Eq. (1), then yields after performance of the angular integration,

$$P_n^{(\infty)}=(n\omega e^2/R)(4\pi R/a)\mathrm{Re}\{\sum_{j=1,3,\cdots}^{\infty}[-H_n^{(1)}J_n+\tfrac{1}{2}\beta^2(H_{n-1}^{(1)}J_{n-1}+H_{n+1}^{(1)}J_{n+1})]\},\quad (7)$$

where the argument of all the cylinder functions is

$$\gamma_{nj}R=[(n\beta)^2-(j\pi R/a)^2]^{\frac{1}{2}}.$$

The power radiated into the attenuated modes is of course zero since for these modes the arguments of the cylinder functions, and also therefore the products $H_n^{(1)}J_n$, become purely imaginary. Only those terms for which $j\leq n\beta a/\pi R$ consequently contribute to Eq. (7).

III. Finite Parallel Plate Shields

Imagine now that the shielding plates of Case II, instead of being infinite, extend from an inner radius R_1 to an outer radius R_2 (with $R_1<R<R_2$ of course) as would be the case if the pole pieces of the ring magnet itself were the shielding plates. In this case, the Green's function in the region between the plates must satisfy appropriate (and very complicated) boundary conditions at the surfaces $r=R_1$ and $r=R_2$ in addition to the boundary conditions of Eq. (5). These extra conditions can be satisfied only if a general solution of the homogeneous equations is added to the Green's function of Eq. (6). Thus for this case we must have

$$G_n=G_n^{(\infty)}+F_n,\quad (8)$$

where we write F_n in the form

$$F_n=(i/2a)\sum_{j=1}^{\infty}\sum_{m=-\infty}^{\infty}\sin j\pi(\tfrac{1}{2}+z/a)\times\sin j\pi(\tfrac{1}{2}+z'/a)e^{im(\varphi-\varphi')}\times[A_{mj}H_m^{(1)}(\gamma_{nj}r)+B_{mj}H_m^{(2)}(\gamma_{nj}r)],\quad (9)$$

where the factor $(i/2a)e^{-im\varphi'}\sin j\pi(\frac{1}{2}+z'/a)$ is included for convenience. The important fact is that F_n is a general solution of

$$(\nabla^2+k_n^2)F_n=0,$$

and satisfies Eq. (5). The coefficients A_{mj} and B_{mj} are exactly determinable only upon consideration of an extremely difficult, if not insoluble, boundary value problem. However, F_n represents essentially reflected waves at the boundaries R_1 and R_2, and hence these coefficients can be roughly estimated from physical arguments. In particular, we shall seek to stay on the safe side by looking for something like an upper limit to the power radiated. As a first step, we assume that as far as the propagating modes are concerned, the power radiated is not less than it would be for an infinite shield, i.e., we set A_{nj}, $B_{nj}=0$ for propagating modes. It is then necessary only to consider the power radiated into the attenuated modes. As mentioned previously, $G_n^{(\infty)}$ contributes nothing for attenuated modes and only F_n enters. Thus we have

$$P_n\lesssim P_n^{(\infty)}+P_n^{(\mathrm{att})},\quad (10)$$

where $P_n^{(\infty)}$ is given by (7) and where

$$P_n^{(\mathrm{att})}=\mathrm{Re}\left\{4in\omega e^2\int_{-\pi}^{\pi}d(\varphi-\varphi')F_n(\mathrm{orbit})\times[1-\beta^2\cos(\varphi-\varphi')]e^{-im(\varphi-\varphi')}\right\}.\quad (11)$$

In order to evaluate Eq. (11), we must estimate the remaining A_{nj} and B_{nj}. The easiest way to do this is as follows. In any high-energy synchrotron, the length R_2-R_1 and the plate separation a are very small compared to the orbit radius R. Thus the cylindrical waves behave very much like plane waves, i.e., the Bessel functions can be replaced by their asymptotic values. To this approximation, a typical attenuated mode of G_n of Eq. (8) has the form of attenuated plane waves emitted by the source plus waves reflected at the boundaries with amplitudes expressible in terms of a complex reflection coefficient of order of magnitude unity. The power transmitted in the attenuated modes is easily calculated in terms of such reflection coefficients and an "upper limit" estimated by choosing the phase of these reflection coefficients properly while setting their magnitudes equal to unity. The simple result is then the following:

$$P_n^{(\mathrm{att})}\simeq(n\omega e^2/R)(4\pi R/a)\sum_{\substack{j=1,3,\cdots\\ j>na/\pi R}}(2/\pi)(a/j\pi R)\times[e^{-2j\pi(R-R_1)/a}+e^{-2j\pi(R_2-R)/a}].$$

In obtaining this result, multiple reflections have been neglected and the argument $\gamma_{nj}R$ of the Bessel functions in Eq. (9) has been approximated by $i(j\pi R/a)$, both being permissible for highly attenuated modes. The various terms which appear are then easily identified. The factor $(n\omega e^2/R)(4\pi R/a)$ is the same normalization factor as in Eq. (7); the factor $(2/\pi)(a/j\pi R)$ arises from the asymptotic expansion of the Bessel functions, while the first exponential gives the attenuation of a wave of unit amplitude originating at the source and then being reflected back to it by the surface at R_1 and similarly for the second exponential term with reflection at R_2. In any event, if we now introduce the dimensionless parameters,

$$\delta_1=(R-R_1)/a,\quad \delta_2=(R_2-R)/a,\quad (12)$$

we obtain finally

$$P_n^{(\mathrm{att})}\simeq(8n\omega e^2/\pi R)\sum_{\substack{j=1,3,\cdots\\ j<na/\pi R}}(1/j)(e^{-2j\pi\delta_1}+e^{-2j\pi\delta_2}).\quad (13)$$

The procedure described above is admittedly crude, but in fact we have obtained the same result by a much more careful treatment in which the shield was regarded as a section of a radial transmission line with proper care being given to asymptotic representations of the Bessel functions in the various domains of order and argument which occur. We shall not reproduce that treatment here except to say that it shows that Eq. (13) is adequate provided $\delta \lesssim 2$ and $(R/a) \gtrsim 20$. This restriction on δ is relaxed somewhat if R/a is larger, as might be expected for actual synchrotrons. For example, if $R/a \gtrsim 100$, then $\delta \lesssim 5$ is suitable. However, as we shall see, the shield behaves very much as if it were infinite when δ appreciably exceeds 2 and hence this is not a serious restriction.

COHERENT RADIATION

Having obtained expressions for the power radiated in the nth harmonic by a single electron for each of the three cases, we now desire expressions for the power radiated by, say, N electrons distributed in a specified way around the circular orbit.[1–3] In particular, suppose the kth electron to have the angular coordinate $\varphi_k + \omega t$ at time t. In the Fourier decomposition of the fields, the contribution of each electron thus contains a phase factor $e^{-in\varphi_k}$ for the nth harmonic. It is then easily established that the power radiated in the nth harmonic by the N electrons is

$$P_n \left| \sum_1^N e^{-in\varphi_k} \right|^2 = NP_n + P_n \sum_{k\neq q}^N \cos n(\varphi_k - \varphi_q). \quad (14)$$

The first term gives just the incoherent power loss. Since the spectrum of this radiation is mostly in the visible or ultraviolet region, it is of course unaffected by the presence of the shields, i.e., when summed over n, it gives the usual results[1–4] in all cases and we shall not discuss it further.

Our interest is in the second term, representing the coherent radiation, which we express as

$$N(N+1)P_n f_n \simeq N^2 P_n f_n, \quad (15)$$

where the form factor f_n is

$$f_n = [1/N(N+1)] \sum_{k\neq q} \cos n(\varphi_k - \varphi_q). \quad (16)$$

Assuming that the electrons are symmetrically distributed about the same mean angle, say zero, and that each electron is independent, we then have at once

$$f_n = \left(\int \cos n\varphi \cdot S(\varphi) d\varphi \right)^2,$$

where $S(\varphi)d\varphi$ is the probability that a given electron is found in the angular interval between φ and $\varphi + d\varphi$. For example, if the electrons are uniformly distributed over an angular interval α, then

$$S(\varphi) = 1/\alpha, \quad -\alpha/2 \leq \varphi \leq \alpha/2$$
$$= 0, \quad \text{otherwise}$$

and

$$f_n = [\sin(\tfrac{1}{2}n\alpha)/(\tfrac{1}{2}n\alpha)]^2. \quad (17)$$

As a second example, if the electrons are distributed according to a Gaussian law, then

$$S(\varphi) = (1/\alpha\sqrt{\pi}) \exp(-\varphi^2/\alpha^2)$$

and

$$f_n = \exp[-(n\alpha/2)^2]. \quad (18)$$

In any event, the total coherent radiation is obtained by summing Eq. (15) over all harmonics and we then have, for the three cases under consideration:

I. No Shielding

$$P_{\text{coh}}^{(0)} = N^2 \sum P_n^{(0)} f_n. \quad (19)$$

II. Infinite Parallel Plate Shields

$$P_{\text{coh}}^{(\infty)} = N^2 \sum P_n^{(\infty)} f_n. \quad (20)$$

III. Finite Parallel Plate Shields

$$P_{\text{coh}} = P_{\text{coh}}^{(\infty)} + P_{\text{coh}}^{(\text{att})}, \quad (21)$$

$$P_{\text{coh}}^{(\text{att})} = N^2 \sum P_n^{(\text{att})} f_n. \quad (22)$$

Using the fact that $P_n^{(0)} \sim n^{\frac{1}{3}}$, Eq. (19) has been evaluated for a uniform distribution by Schwinger[2] and for a Gaussian distribution by Schiff,[1] with the results

$$P_{\text{coh}}^{(0)} = (N^2\omega e^2/R)(\sqrt{3}/\alpha)^{4/3} \text{ (uniform)} \quad (23)$$

and

$$P_{\text{coh}}^{(0)} = (N^2\omega e^2/R)(\sqrt{3}/\alpha)^{4/3} \times (4/\pi\sqrt{3}) 2^{\frac{1}{3}} [\Gamma(2/3)]^2 \text{ (Gaussian).}$$

It is seen that the results are not terribly sensitive to the detailed character of the form factor, and henceforth we shall consider only the uniform distribution. For this distribution, Schwinger[2] has also evaluated Eq. (20), but only under the assumption that the size of the bunch is at least of the order of the plate separation (i.e., that $R\alpha \gtrsim a$), with the result[6]

$$P_{\text{coh}}^{(\infty)} = (N^2\omega e^2/R)(\sqrt{3}a/2R\alpha^2). \quad (24)$$

This restriction on the size of the bunch is not as serious as it seems at first glance, since for $R\alpha$ much less than a the shielding effects become very small and hence are not of significance. Additionally, examination of Schwinger's derivation leads one to the conclusion that Eq. (24) represents essentially an upper limit to the coherent radiation loss as α becomes smaller than a/R. Presumably, therefore, Eq. (24) can be safely used until α becomes small enough that the result is numerically equal to that of Eq. (23), which is the result in the absence of shielding.[7]

[6] See the Appendix for details.

[7] Actually one can do considerably better than this as follows: As the bunch size α decreases toward zero, for fixed plate separation a, the shielding effect becomes negligible. In other words,

Finally, we calculate the correction term for finite shields from Eq. (22), using Eq. (17) and Eq. (13):

$$P_{\text{coh}}^{(\text{att})}=(8N^2\omega e^2/\pi R)\sum_{j=1,3,\cdots}(1/j)$$
$$\times(e^{-2j\pi\delta_1}+e^{-2j\pi\delta_2})\sum_{n=1}^{j\pi R/a}n\left(\frac{\sin\frac{1}{2}n\alpha}{\frac{1}{2}n\alpha}\right)^2.$$

For $\delta\geq\frac{1}{2}$, only the $j=1$ term contributes significantly, as is easily verified, so that, using

$$\sum_1^{\pi R/a}(1/n)\sin^2(\tfrac{1}{2}n\alpha)$$
$$\simeq\int_0^{\pi R\alpha/a}(1/x)\sin^2(x/2)dx=S(\pi R\alpha/a)\quad(25)$$

we obtain

$$P_{\text{coh}}^{(\text{att})}\simeq(N^2\omega e^2/R)(32/\pi\alpha^2)$$
$$\times[e^{-2\pi\delta_1}+e^{-2\pi\delta_2}]S(\pi R\alpha/a).\quad(26)$$

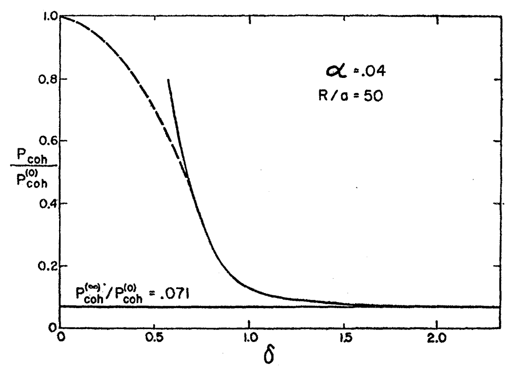

FIG. 1. Coherent power loss, relative to the loss in the absence of shielding, *vs* plate width in units of the plate separation.

This result is valid under the conditions $R/a\gtrsim20$ and $\frac{1}{2}\leq\delta\leq2$. Although this may seem to be a small domain of validity, it actually covers the most important region. The quantity $S(\pi R\alpha/a)$ is easily expressed in terms of known functions;[8] viz.,

$$S(y)=\tfrac{1}{2}[C+\log y-\text{Ci}(y)],\quad(27)$$

where $C=0.577\cdots=$ Euler's constant, and $\text{Ci}(y)$ is the cosine integral; so that the final result is extremely simple.

As an example, in Fig. 1, we present a plot of the coherent power loss, relative to the loss in the absence of shielding, against plate width for the special case $\delta_1=\delta_2=\delta/2$ and for $\alpha=0.04$ and $R/a=50$. Although only a portion of the curve can be calculated [using (21), (23), and (26)] we do know the limit points when $\delta=0$ (no shielding) and $\delta=\infty$ (infinite parallel plate shielding) and hence the remainder of the curve can be sketched in without serious error. The dotted portion in the figure has been so sketched, while the solid portion has been calculated according to the above.

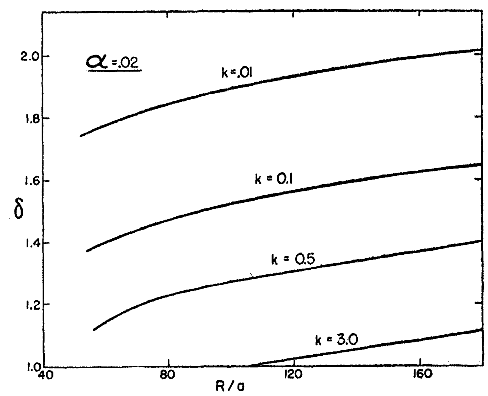

FIG. 2. Plate width *vs* orbit radius, both in units of the plate separation, for constant ratio of coherent power loss to that for infinite parallel plate shields and for the electrons bunched uniformly over an angular interval 0.02 radian.

In Figs. 2, 3, and 4, we present the results in convenient form by introducing the parameter $k(\delta,\alpha,R/a)$ defined by

$$P_{\text{coh}}=[1+k(\delta_1,\alpha,R/a)+k(\delta_2,\alpha,R/a)]P_{\text{coh}}^{(\infty)},\quad(28)$$

where $P_{\text{coh}}^{(\infty)}$ is given by Eq. (23). Values of δ *vs* R/a for constant k have been plotted for $\alpha=0.02$, 0.04, and 0.06. These curves enable one to estimate the width of shielding required to reduce the coherent radiation loss to a given amount in units of the loss for infinite shields. As an example, given $R/a=60$ and $\alpha=0.02$, it might be desirable to know the plate widths necessary to reduce the coherent power to twice $P_{\text{coh}}^{(\infty)}$. Selecting $\delta_1=\delta_2$ so that $k(\delta_1,\alpha,R/a)=k(\delta_2,\alpha,R/a)=0.5$, we find from the curves

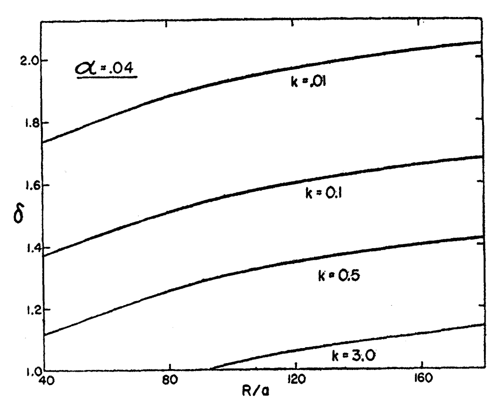

FIG. 3. Plate width *vs* orbit radius, both in units of the plate separation, for constant ratio of coherent power loss to that for infinite parallel plate shields and for the electrons bunched uniformly over an angular interval 0.04 radian.

regarded as a function of α, $P_{\text{coh}}^{(\infty)}$ approaches $P_{\text{coh}}^{(0)}$ as α approaches zero. Knowing that the result (24), valid for large α, is essentially an upper limit for small α, one can then easily sketch in the entire curve to reasonable accuracy.

[8] See, for example, E. Jahnke and F. Emde, *Tables of Functions* (Dover Publications, New York, 1943), pp. 2–6.

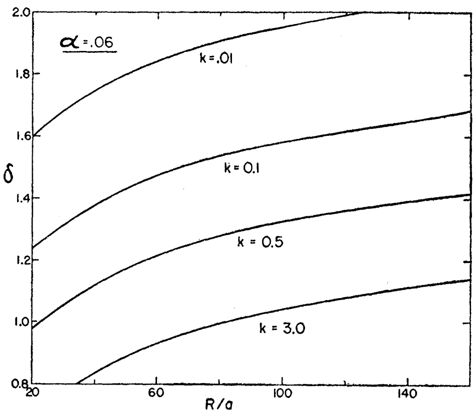

FIG. 4. Plate width *vs* orbit radius, both in units of the plate separation, for constant ratio of coherent power loss to that for infinite parallel plate shields and for the electrons bunched uniformly over an angular interval 0.06 radian.

of Fig. 2 that $\delta_1=\delta_2=1.15$; and hence, from (12), $R-R_1=R_2-R=1.15a$.

ACKNOWLEDGMENT

We should like to express our appreciation to L. Jackson Laslett for calling this problem to our attention and for providing us with some of his own unpublished material as well as the material of reference 2.

APPENDIX

Because of the unavailability of reference 2, we give an outline of Schwinger's derivation of Eq. (24). Taking the indicated real part of (7) we have

$$P_n^{(\infty)}=(n\omega e^2/R)(4\pi R/a)\times\sum_{\substack{j=1,3\cdots\\ j<na\beta/\pi R}}\left[-J_n^2+\tfrac{1}{2}\beta^2(J_{n-1}^2+J_{n+1}^2)\right]$$

$$=(n\omega e^2/R)(4\pi R/a)\times\sum_{\substack{j=1,3\cdots\\ j<na\beta/\pi R}}\left[\beta^2 J_n'^2+\frac{(j\pi R/a)^2}{(n\beta)^2-(j\pi R/a)^2}J_n^2\right],\quad\text{(A1)}$$

where the argument of the Bessel Functions is

$$\gamma_{nj}R=\left[(n\beta)^2-(j\pi R/a)^2\right]^{\frac{1}{2}}.$$

Since $\pi R/a\gg1$, the harmonics involved in the radiation are sufficiently high so that approximation formulas for Bessel functions of large order are applicable and we write,[9] placing $\beta=1$,

$$J_n([n^2-(j\pi R/a)^2]^{\frac{1}{2}})=\frac{1}{\sqrt{3}\pi}\left(\frac{j\pi R/a}{n}\right)K_{\frac{1}{3}}[(j\pi R/a)^3(1/3n^2)],$$

$$J_n'([n^2-(j\pi R/a)^2]^{\frac{1}{2}})=\frac{1}{\sqrt{3}\pi}\left(\frac{j\pi R/a}{n}\right)^2K_{2/3}[(j\pi R/a)^3(1/3n^2)].\quad\text{(A2)}$$

Recognizing that the contributions to $P_n^{(\infty)}$ are negligible unless $n\gtrsim(j\pi R/a)(j\pi R/a)^{\frac{1}{2}}$, so that n must exceed $j\pi R/a$ by a rather large factor, we simplify the second term of (A-1) accordingly and obtain

$$P_n^{(\infty)}=(\omega e^2/R)(4R/3\pi a)\sum_{\substack{j=1,3,\cdots\\ \gamma_j<n}}(\gamma_j^4/n^3)\times[K_{\frac{1}{3}}^2(\gamma_j^3/3n^2)+K_{2/3}^2(\gamma_j^3/3n^2)],$$

where

$$\gamma_j=j\pi R/a.$$

The total coherent power is then given by

$$P_{\text{coh}}^{(\infty)}=(N^2\omega e^2/R)(4R/3\pi a)\times\sum_{n=1}\left(\frac{\sin\frac{1}{2}n\alpha}{\frac{1}{2}n\alpha}\right)^2\sum_{\substack{j=1,3,\cdots\\ \gamma_j<n}}(\gamma_j^4/n^3)\times[K_{\frac{1}{3}}^2(\gamma_j^3/3n^2)+K_{2/3}^2(\gamma_j^3/3n^2)].$$

Replacing the sum over n by an integral, and introducing $x=\gamma_j^3/3n^2$, we then have

$$P_{\text{coh}}^{(\infty)}=(N^2\omega e^2/R)(12R/\pi a\alpha^2)\times\sum_{j=1,3,\cdots}(1/\gamma_j^2)\int_0^\infty\sin^2([\gamma_j^3/3x]^{\frac{1}{2}}\alpha/2)\times[K_{\frac{1}{3}}^2(x)+K_{2/3}^2(x)]x\,dx,\quad\text{(A3)}$$

where the correct upper limit of the integral, $\gamma_j/3$, which is large compared to unity, has been replaced by infinity. Now the main contribution to the integral comes for values of x in the interval $0\leq x\lesssim 1$. In this interval the argument of the $\sin^2$ term in the integral is at least of order $\alpha\gamma_j\sqrt{\gamma_j}\simeq(jR\alpha/a)(jR/a)^{\frac{1}{2}}\gg R\alpha/a$. Hence, if $R\alpha/a$ is at least of order unity, the $\sin^2$ term can be replaced by its average value $\frac{1}{2}$, the known integrals[8] and sums performed and the result of Eq. (24) follows.

Without this restriction on $R\alpha/a$, the evaluation of Eq. (A3) seems possible only numerically. However, we remark that if α is small enough that the argument of the $\sin^2$ term is rather small over the important range, then this term is considerably less than its average value and hence, as indicated in the text, one errs only on the conservative side in extending Schwinger's result. Needless to say, an absolute upper limit is obtained by replacing the $\sin^2$ term by unity, thus giving twice the result of Eq. (24), but this seems unnecessarily conservative.

[9] G. N. Watson, *Bessel Functions* (The MacMillan Company, New York, 1945), p. 248.