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Tensor Virial Equations*

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The tensor virial equations for an aggregate of particles are developed. They are tensor equations of second rank, and on contraction yield the familiar virial equation. To illustrate their application, the diffusion of molecules through a gas is considered and a kinetic derivation of the Navier-Stokes equations is worked out. The latter may here be carried out in more general terms than is usual, the generalization leading to results that are of interest in turbulence theory. In anisotropic systems the use of the tensor in place of the scalar virial becomes imperative. An astrophysical example of this (anisotropic star cluster) is given in outline.

1. INTRODUCTION

THE use of the virial theorem in kinetic theory is well known. It seems to have escaped general attention that the usual virial equation is the contraction of a second-order tensor equation which is essentially the equation of motion of the moment-of-inertia tensor of an aggregate of particles. Lord Rayleigh,[1] who wrote down the off-diagonal terms of the tensor equations, remarked that they should have application to problems in viscosity. Moreover, one readily finds that the diagonal terms apply to anisotropic systems in the same way in which the usual scalar virial theorem applies to isotropic systems.

Consider a system of particles in a space with Cartesian coordinates x_i. A particle of mass m on which there is a net force F_j has the equation of motion[2]

$$F_j = m d^2x_j/dt^2. \tag{1}$$

We multiply by x_i and rearrange to obtain

$$\frac{d}{dt}\left(mx_i\frac{dx_j}{dt}\right) = m\frac{dx_i}{dt}\frac{dx_j}{dt} + x_iF_j. \tag{2}$$

Summing over all the particles in the system and defining the tensors

$$J_{ij} = \sum mx_i\frac{dx_j}{dt},\quad 2T_{ij} = \sum m\frac{dx_i}{dt}\frac{dx_j}{dt},\quad \Phi_{ij} = \sum x_iF_j, \tag{3}$$

we obtain the *tensor virial equation:*

$$dJ_{ij}/dt = 2T_{ij} + \Phi_{ij}. \tag{4}$$

To interpret T_{ij} and Φ_{ij} we note that the total kinetic energy, T, of the system is just the spur of T_{ij}. Accordingly we shall call T_{ij} the *kinetic tensor*. If F_i is expressible in terms of a potential energy V which is a homogeneous function of degree n of the x_i, then the spur of Φ_{ij} is

$$\Phi = \Phi_{ii} = -\sum x_i(\partial V/\partial x_i) = -nV, \tag{5}$$

by Euler's theorem. For electrostatic and gravitational interactions, $n=-1$. We shall call Φ_{ij} the *potential tensor*.

It is of interest to separate (4) into its symmetric and antisymmetric parts. The moment of inertia tensor I_{ij} of the system is the symmetric part of J_{ij}. Thus,

$$\frac{d}{dt}I_{ij} = \tfrac{1}{2}(J_{ij}+J_{ji}) = \frac{d}{dt}\sum mx_ix_j. \tag{6}$$

We denote the antisymmetric part of J_{ij} by K_{ij} so that

$$2K_{ij} = J_{ij} - J_{ji} = \sum m(x_idx_j/dt - x_jdx_i/dt). \tag{7}$$

$2K_{ij}$ is obviously the angular momentum of the system. The kinetic tensor, T_{ij}, is clearly symmetrical. If, finally, we split the potential tensor, Φ_{ij}, into its symmetrical and antisymmetrical parts,

$$2M_{ij} = \Phi_{ij} + \Phi_{ji},\quad 2N_{ij} = \Phi_{ij} - \Phi_{ji}, \tag{8}$$

Eq. (4) decomposes into the two equations:

$$d^2I_{ij}/dt^2 = 2T_{ij} + M_{ij},\quad dK_{ij}/dt = N_{ij}. \tag{9}$$

The first equation is the equation of motion for the moment of inertia tensor; the second gives us the rate of change of the angular momentum. We shall concern ourselves primarily with the equation for I_{ij}; its contraction gives the familiar scalar virial equation.

2. DIFFUSION

Consider a space filled with a homogeneous distribution of particles. At time t_0 we mark every particle within the rectangle with sides $x_i = \pm a_i(t_0)$. We ask how these marked particles will spread out with time.

The tensor virial equations do not determine the form of the spatial distribution: If we assume a distribution described by certain characteristic lengths, then from (9) we can compute how these lengths vary with time. Thus our solution has the disadvantage of being an approximate one but the related advantage that the computation will be easy.

Let us characterize the distribution of the particles at time t by three moments, say $\frac{1}{2}a_i(t)$. At time t_0 the a_i represent the initial rectangular step functions; as time goes on, the step functions will spread out into

* This work was in part supported by the Office of Naval Research.
[1] Lord Rayleigh, Phil. Mag. 50, 210 (1900). This reference was pointed out to us by Professor Chandrasekhar.
[2] We confine ourselves here to the Newtonian formalism. The Lagrangian generalization is straightforward.

Gaussians; $\frac{1}{2}a_i(t)$ will then approximately represent the standard deviation. The tensor virial equations reduce to

$$\frac{d}{dt}\left[a_i(t)\frac{d}{dt}a_j(t)\right]=0. \tag{10}$$

The solution of (10) is

$$a_i(t)=a_i(t_0)(t/t_0)^{\frac{1}{2}},\ da_i(t)/dt=a_i(t_0)/(2t_0)(t_0/t)^{\frac{1}{2}}. \tag{11}$$

To evaluate the integration constant t_0, we note that at time t_0 the expansion of the rectangular region of marked particles is brought about by only those particles within a distance L of each face, where L is the mean free path; of all the particles in the region, only a fraction $L/a_i(t)$ are contained in a slab of thickness L and normal to the i direction. Half of these particles are moving outward across the face with velocity $\langle v_i\rangle_{\text{Av}}$. Thus, $a_i(t)$ must at time $t=t_0$ increase at the rate,

$$da_i(t)/dt=\tfrac{1}{2}\langle v_i\rangle_{\text{Av}}L/a_i(t). \tag{12}$$

Comparing (12) with (11), we find that

$$t_0=[a_i(t_0)]^2/(L\langle v_i\rangle_{\text{Av}}), \tag{13}$$

and (11) becomes

$$a_i(t)=(L\langle v_i\rangle_{\text{Av}}t)^{\frac{1}{2}},\ da_i(t)/dt=[L\langle v_i\rangle_{\text{Av}}/(4t)]^{\frac{1}{2}}. \tag{14}$$

3. MOTION OF A FLUID

Let us next use the tensor virial equation to derive the equation of motion of a *finite* region of fluid. We let the boundaries of the region move with the fluid. If y_i is the center of mass of the region, we set for each particle

$$x_i=y_i+\xi_i. \tag{15}$$

Defining w_i as the velocity of the center of mass, we write the velocity v_i of a particle as

$$v_i=w_i+u_i,\quad w_i=dy_i/dt,\quad u_i=d\xi_i/dt. \tag{16}$$

Obviously,

$$\sum m\xi_i=\sum mu_i=0. \tag{17}$$

We shall refer to u_i as the *local velocity field* and w_i as the *translocal field*; the local field is the portion of the velocity composed of fluctuations of smaller scale than the region. Finally, defining G_i as the average force on the region so that

$$G_i=mdw_i/dt, \tag{18}$$

we write

$$F_i=G_i+f_i. \tag{19}$$

With these definitions we may rewrite (3) as

$$J_{ij}=My_iw_j+\sum m\xi_iu_j,\quad 2T_{ij}=Mw_iw_j+\sum mu_iu_j,\quad \Phi_{ij}=y_i\sum G_j+\sum \xi_if_j, \tag{20}$$

where M is the total mass of the region. After some rearranging of terms (4) may be rewritten as

$$M\frac{dw_j}{dt}=\sum G_j+\frac{1}{y_i}\left[\sum mu_iu_j+\sum \xi_if_j-\frac{d}{dt}\sum m\xi_iu_j\right]. \tag{21}$$

However, since for the center of mass

$$Mdw_j/dt=\sum G_j, \tag{22}$$

Eq. (21) reduces to

$$\frac{d}{dt}\sum m\xi_iu_j=\sum mu_iu_j+\sum \xi_if_j. \tag{23}$$

We see that *the tensor virial equation referred to the center of mass of a region is invariant with respect to translocal accelerations.*

Now f_i in the purely hydrodynamic case results only from collisional forces. Thus, it cancels out over the interior of the region where both members of each pair of colliding particles enter into the summation; only the collisions at the surface of the region contribute. It becomes convenient to introduce[3] the usual stress tensor σ_{ij} interpreted as the force per unit area in the i direction across an element of area normal to the j direction. We choose our signs so that σ_{ij} represents the force exerted by the matter on the positive side of the area on the matter on the negative side. This is the customary definition in elasticity[4] and electrodynamics. Remembering that all forces other than collisional are assumed to be zero, we may express $\sum \xi_if_j$ as the surface integral $\int dS_k\xi_i\sigma_{jk}$. Replacing the summation by a volume integral in the other terms of (23) and using Gauss's theorem, we obtain

$$\frac{d}{dt}\int dV\rho\xi_iu_j=\int dV\rho u_iu_j+\int dV\sigma_{ij}+\int dV\xi_i\frac{\partial}{\partial\xi_k}\sigma^{jk}, \tag{24}$$

where summation convention is used with respect to k.

Let us now introduce the assumption that

$$\int dV\rho u_iu_j\gg\frac{d}{dt}\int dV\rho\xi_iu_j. \tag{25}$$

[3] The introduction of a stress tensor and the associated processes and parameters such as integration, differentiation, pressure, viscosity, etc. require a limited form of continuity in our hitherto unrestricted system. Our notion of infinitesimal becomes that of the physical infinitesimal, *viz.*, that the smallest elements of volume dV which we consider must be sufficiently large so as to contain many particles. If n represents the number of particles in dV, then the fluctuation of n is of the order of $\sqrt{n}$. We must require that

$$\sqrt{n}\ll n$$

in order that our averaging process over dV have a smoothing effect. To be treated as an infinitesimal, dV, of course, must be of smaller scale than the phenomena in which we are interested. The alternative to these restrictions on dV is to consider an ensemble of systems, so that the average may be carried out over the given dV in all the member systems rather than just in a single system.

[4] A. Sommerfield, *The Mechanics of Deformable Bodies* (Academic Press, New York, 1950), p. 61.

Physically, this condition implies that the acceleration of the rate of distortion of the region is small over periods of time comparable to the time of local circulation, ξ_i/u_j. This requires that the local forces $\partial\sigma_{ik}/\partial\xi_k$ do not vary too rapidly over the region, in which case the last term of (24) is also small, and (24) reduces to

$$0=\int dV\rho u_i u_j+\int dV\sigma_{ij}. \qquad (26)$$

This is satisfied if

$$\sigma_{ij}=-\rho u_i u_j, \qquad (27)$$

and we see that σ_{ij} is the familiar Reynolds tensor.

It is of interest to note that if (26) were rigorously true, rather than only in the approximation of (25), then (24) would reduce to

$$\frac{d}{dt}\int dV\rho\xi_i u_j=\int dV\xi_i f_j. \qquad (28)$$

Let us investigate under what circumstances this relation is valid.

The antisymmetric part of (28) is the angular momentum equation,

$$\frac{d}{dt}\int dV\rho(\xi_i u_j-\xi_j u_i)=\int dV(\xi_i f_j-\xi_j f_i),$$

and is rigorously true. The symmetric part, on the other hand is

$$\frac{d}{dt}\int dV\rho(\xi_i u_j+\xi_j u_i)=\int dV(\xi_i f_j+\xi_j f_i). \qquad (29)$$

Assuming an incompressible flow, we may write (29) as

$$\int dV\rho\left(2u_i u_j+\xi_i\frac{du_j}{dt}+\xi_j\frac{du_i}{dt}\right)=\int dV(\xi_i f_j+\xi_j f_i).$$

Inasmuch as

$$\rho du_k/dt=f_k,$$

we may reduce the relation to

$$\int dV\rho u_i u_j=0, \qquad (30)$$

which implies that u_i and u_j are uncorrelated over the region. In (32) we shall assume that u_i and u_j are correlated by an amount proportional to $\partial w_i/\partial y_i$ times the mean free path. Thus, (30) leads one to assume that this product be small, which is equivalent to (25).

Given that the hydrodynamic stresses are of the form (27), we return to (22) and write the equation of motion for the center of mass of the region as

$$\frac{dw_i}{dt}=-\frac{1}{M}\int dS_k\rho u_i u_k=-\frac{1}{M}\int dV\frac{\partial}{\partial\xi_k}(\rho u_i u_k). \qquad (31)$$

To write (31) in the conventional form of hydrodynamics, we introduce the usual argument that u_i and $u_i(i\neq j)$ will be correlated only if there is translocal shearing present: If u_j transports a particle an average distance L_j across a velocity gradient $\partial w_i/\partial y_j$ without collision, the particle will have a velocity $-L_j\partial w_i/\partial y_j$ in the i direction relative to the general velocity field; L_j is essentially the mean free path, or mixing length. We construct the relation, for $\alpha\neq\beta$,

$$\langle\rho u_\alpha u_\beta\rangle\equiv\frac{1}{V}\int dV\rho u_\alpha u_\beta=-a\langle\rho u_\alpha L_\alpha\rangle\frac{\partial w_\beta}{\partial y_\alpha}-b\langle\rho u_\beta L_\beta\rangle\frac{\partial w_\alpha}{\partial y_\beta},$$

(where we indicate by Greek indices that we do not observe summation convention); a and b are numerical constants. More generally,

$$\langle\rho u_\alpha u_\beta\rangle=p(\alpha)\delta_{\alpha\beta}-a\mu(\alpha)\frac{\partial w_\beta}{\partial y_\alpha}-b\mu(\beta)\frac{\partial w_\alpha}{\partial y_\beta}, \qquad (32)$$

where $\delta_{\alpha\beta}$ is the Kronecker delta and $\mu(\alpha)$ is the viscosity,

$$\mu(\alpha)=\langle\rho u_\alpha L_\alpha\rangle.$$

$p(\alpha)$ is taken to be the average over the region of the portion of $\rho u_\alpha u_\alpha$ which is independent of the translocal shear; $p(\alpha)$ is the pressure and is independent of α if the local velocity field is statistically isotropic.

Now,

$$\int dV\frac{\partial}{\partial\xi_k}(\rho u_i u_k)=V\frac{\partial}{\partial y_k}\langle\rho u_i u_k\rangle, \qquad (33)$$

where V is the volume of the region. Thus (28) may be rewritten as

$$\langle\rho\rangle\frac{dw_\alpha}{dt}=-\frac{\partial}{\partial y_\alpha}p(\alpha)+a\frac{\partial}{\partial y_k}\mu(\alpha)\frac{\partial w_k}{\partial y_\alpha}+b\frac{\partial}{\partial y_k}\mu(k)\frac{\partial w_\alpha}{\partial y_k}, \qquad (34)$$

where $\langle\rho\rangle=M/V$. (34) is the equation for the translocal field in the general case that the local velocity field is anisotropic.

Assuming isotropy so that $p(\alpha)$ and $\mu(\alpha)$ are independent of α, assuming that $\mu(\alpha)$ is independent of y_i, and assuming that an isotropic dilatation of the region produces no dissipation of energy, we obtain the familiar Navier-Stokes equations for a compressible fluid.

We note how the tensor virial equations with the assumption (24) lead quite naturally to the Reynolds stress tensor. Equation (31) indicates that the translocal velocity field depends only on the statistical properties of the local field. Introducing Prandtl's mixing length ideas, we find that the statistical properties of the local motions appear as an "eddy" viscosity, as was assumed by Heisenberg[5] in his theory of isotropic turbulence.

[5] W. Heisenberg, Z. Physik **124** 628 (1948).

4. ISOLATED SYSTEMS

The tensor virial equation is of particular interest in problems involving an anisotropic isolated system of particles; for instance, an interstellar gas cloud or a cluster of stars. One postulates some definite spatial distribution for the system; the differential Eq. (9) can then be used to investigate how the scale of such a distribution varies with time.

Consider a star cluster, or another isolated dynamical system, in which the velocities of the individual stars are not statistically isotropic. In such a case one may assume that the particles are distributed uniformly throughout an ellipsoidal region of space. To compare the calculations with an observed system, one interprets the semi-axes of the ellipsoid as representing the scale of the system, say the distance from the center out to where the density has decreased by e. Choosing the coordinate axes along the axes of the ellipsoid, one computes the potential tensor to be[6]

$$\Phi_{\alpha\beta}=\begin{cases}0 & \text{if } \alpha\neq\beta\\ -(3/10)GM^2(a_\alpha)^2N_\alpha & \text{if } \alpha=\beta.\end{cases} \tag{35}$$

M is the total mass of the system; G is the gravitational constant; a_α is the semi-axis in the α direction. We arrange the axes so that $a_1 \geqslant a_2 \geqslant a_3$; the N_α are given by

$$N_1=2[(a_1)^2-(a_3)^2]^{-\frac{3}{2}}k^{-2}[v-E(\omega,k)],$$

$$N_2=2[(a_1)^2-(a_3)^2]^{-\frac{3}{2}}\left\{\frac{E(\omega,k)}{k^2(1-k^2)}-\frac{1}{(1-k^2)}\frac{\operatorname{sn}v\operatorname{cn}v}{dnv}-\frac{v}{k^2}\right\}, \tag{36}$$

$$N_3=2[(a_1)^2-(a_3)^2]^{-\frac{3}{2}}(1-k^2)^{-1}[\operatorname{sn}v\operatorname{dn}v/cnv-E(\omega,k)],$$

where

$$\sin\omega=\operatorname{sn}v=[1-(a_3/a_1)^2]^{\frac{1}{2}},$$

$$\operatorname{cn}v=a_3/a_1,\quad \operatorname{dn}v=a_2/a_3, \tag{37}$$

and

$$k^2=[(a_1)^2-(a_2)^2]/[(a_1)^2-(a_3)^2],\quad v=F(\omega,k). \tag{38}$$

[6] W. D. MacMillan, *The Theory of the Potential* (McGraw-Hill Book Company, Inc., New York, 1930), p. 60.

$F(\omega,k)$ and $E(\omega,k)$ are Legrendre's elliptic integrals of the first and second kind, respectively.

If $\langle(u_\alpha)^2\rangle$ is the mean square velocity in the α direction, the kinetic tensor is given by

$$2T_{\alpha\beta}=\begin{cases}0 & \text{if } \alpha\neq\beta\\ M\langle(u_\alpha)^2\rangle+(1/10)M(da_\alpha/dt)^2 & \text{if } \alpha=\beta.\end{cases} \tag{39}$$

The complexity of the potential tensor prohibits our giving a general integration of (9). Even in the somewhat more simple case when $a_1=a_2$ this does not seem feasible. Thus, we investigate here only the equilibrium configuration. Setting $d^2I_{ij}/dt^2=0$ in (9), we obtain a relation for the mean velocities in terms of the anisotropy:

$$\langle(u_\alpha)^2\rangle=(3/10)GM(a_\alpha)^2N_\alpha. \tag{40}$$

Integration of (9) in the case of spherical symmetry is elementary, and we but briefly indicate the development. We use the scalar virial equation, the contraction of (9). For a homogeneous sphere it is readily shown that

$$\Phi_{ii}=-3GM/(5a),\quad I_{ii}=3Ma^2/10, \tag{41}$$

where a is the radius of the sphere. The radial oscillations contribute $(3/10)M(da/dt)^2$ to the kinetic energy. If the internal motions follow a polytrope law,

$$\langle u^2\rangle=\langle u^2\rangle_0(a/a_0)^{3(A-1)} \tag{42}$$

where A is the effective γ of the motions, then

$$2T_{ii}=\tfrac{3}{5}M(da/dt)^2+M\langle u^2\rangle_0(a_0/a)^{3(A-1)}. \tag{43}$$

Putting (41) and (43) into the scalar virial equation, we obtain the differential equation

$$\frac{d^2a}{dt^2}=\frac{5}{3}\langle u^2\rangle_0\frac{a_0^{3(A-1)}}{a^{3A-2}}-\frac{GM}{a^2}, \tag{44}$$

which is readily integrated to give da/dt and finally a as a function of time.

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