Gravitation and Electromagnetism

SURAJ N. GUPTA
Department of Physics, Purdue University, Lafayette, Indiana
(Received May 10, 1954; revised manuscript received September 10, 1954)

Einstein's theory of gravitation is compared with Maxwell's theory of the electromagnetic field, and some common features of the two theories are pointed out. It is also shown that the well-known peculiarities of Einstein's theory are a necessary consequence of the fact that Einstein's field corresponds to particles of spin 2.

1. INTRODUCTION

ALTHOUGH Einstein's theory of the gravitational field is the most widely accepted theory of gravitation, it is rather disconcerting to note that Einstein's theory appears to be strikingly different from the present theories of the electromagnetic field and the meson fields. In fact, in the formulation of fundamental physical laws we always seek for harmony in nature, and we intuitively expect that there should be some uniformity in the description of various fields in nature. Therefore, Einstein himself and others[1] have tried to construct a unified theory of the gravitational and the electromagnetic fields, while some other authors[2] have tried to find a linear theory of the gravitational field in flat (Minkowskian) space analogous to other existing field theories. All such attempts, however, are still in a speculative stage.

The aim of the present paper is to compare Einstein's theory of the gravitational field with Maxwell's theory of the electromagnetic field, and to show that the two theories have many features in common. We shall also see that the main differences between the two theories can be attributed to the fact that while the Maxwell field corresponds to particles of spin 1, the Einstein field corresponds to particles of spin 2. This shows that Maxwell's theory of electromagnetism and Einstein's theory of gravitation provide us with a fairly uniform description of nature, and therefore there is no *philosophic* necessity for trying to alter any of these theories.

2. REMARKS ON EINSTEIN'S THEORY OF GRAVITATION

Einstein's gravitational field is described by the field equation

$$R^{\mu\nu}-\tfrac{1}{2}Rg^{\mu\nu}=-\tfrac{1}{2}\kappa^2T^{\mu\nu}, \qquad (1)$$

where the symbols $g^{\mu\nu}$, $R^{\mu\nu}$ and R have the usual meaning, κ is a constant, and $T^{\mu\nu}$ is the energy-momentum tensor of the "matter" field, which includes everything except the gravitational field. We shall, however, express the gravitational field equation in a different form, which will be more suitable for the present purpose.

As is well known, the Lagrangian density for Einstein's gravitational field may be taken as

$$\mathfrak{L}=-\kappa^{-2}\mathfrak{g}^{\mu\nu}\left(\begin{Bmatrix}\alpha\\ \mu\beta\end{Bmatrix}\begin{Bmatrix}\beta\\ \nu\alpha\end{Bmatrix}-\begin{Bmatrix}\alpha\\ \mu\nu\end{Bmatrix}\begin{Bmatrix}\beta\\ \alpha\beta\end{Bmatrix}\right), \qquad (2)$$

which gives for the canonical energy-momentum pseudotensor density of the gravitational field

$$\mathfrak{t}_\nu{}^\mu=\frac{\partial\mathfrak{L}}{\partial(\partial\mathfrak{g}^{\alpha\beta}/\partial x^\mu)}\frac{\partial\mathfrak{g}^{\alpha\beta}}{\partial x^\nu}-\delta_\nu{}^\mu\mathfrak{L}. \qquad (3)$$

We can further obtain the symmetrical energy-momentum pseudotensor density of the gravitational field by Belinfante's method.[3] For this, we consider an infinitesimal linear transformation

$$\delta x^\mu=-\epsilon^{\mu\lambda}\delta\omega_{\lambda\nu}x^\nu \quad\text{with}\quad \delta\omega_{\lambda\nu}=-\delta\omega_{\nu\lambda}, \qquad (4)$$

where $\epsilon^{\mu\nu}$ is a set of quantities given by

$$\epsilon^{\mu\nu}=\begin{bmatrix}-1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & +1\end{bmatrix}. \qquad (5)$$

Then, the symmetrical energy-momentum pseudotensor density of the gravitational field will be

$$\theta^{\mu\nu}=\epsilon^{\mu\lambda}\mathfrak{t}_\lambda{}^\nu-\tfrac{1}{2}\frac{\partial}{\partial x^\rho}(\mathfrak{f}^{\mu\nu,\rho}+\mathfrak{f}^{\rho\mu,\nu}+\mathfrak{f}^{\rho\nu,\mu}), \qquad (6)$$

where

$$\mathfrak{f}^{\mu\nu,\rho}\equiv-\frac{\delta\mathfrak{g}^{\alpha\beta}}{\delta\omega_{\mu\nu}}\frac{\partial\mathfrak{L}}{\partial(\partial\mathfrak{g}^{\alpha\beta}/\partial x^\rho)}$$
$$=2\frac{\partial\mathfrak{L}}{\partial(\partial\mathfrak{g}^{\alpha\beta}/\partial x^\rho)}(\epsilon^{\alpha\mu}\mathfrak{g}^{\beta\nu}-\epsilon^{\alpha\nu}\mathfrak{g}^{\beta\mu}). \qquad (7)$$

Adding the symmetrical energy-momentum tensor density of the matter field $\epsilon^{\mu\lambda}\mathfrak{T}_\lambda{}^\nu$ to (6), we obtain for the total energy-momentum pseudotensor density $\Theta^{\mu\nu}$ of the system

$$\Theta^{\mu\nu}=\epsilon^{\mu\lambda}(\mathfrak{T}_\lambda{}^\nu+\mathfrak{t}_\lambda{}^\nu)-\tfrac{1}{2}\frac{\partial}{\partial x^\rho}(\mathfrak{f}^{\mu\nu,\rho}+\mathfrak{f}^{\rho\mu,\nu}+\mathfrak{f}^{\rho\nu,\mu}). \qquad (8)$$

[1] For an account of the unified field theories, see P. G. Bergmann, *Introduction to the Theory of Relativity* (Prentice-Hall, Inc., New York, 1942), and A. Einstein, *The Meaning of Relativity* (Princeton University Press, Princeton, 1950).

[2] F. J. Belinfante and J. C. Swihart, Phys. Rev. **90**, 357 (1953) and **91**, 500 (1953); G. D. Birkhoff, Proc. Natl. Acad. Sci. **29**, 231 (1943).

[3] F. J. Belinfante, Physica **6**, 887 (1939).

Substituting (7) in (8), we get after some simplification[4]

$$\frac{\partial^2}{\partial x^\alpha \partial x^\beta}(\mathfrak{g}^{\mu\nu}\epsilon^{\alpha\beta}-\mathfrak{g}^{\mu\alpha}\epsilon^{\nu\beta}+\mathfrak{g}^{\alpha\beta}\epsilon^{\mu\nu}-\mathfrak{g}^{\alpha\nu}\epsilon^{\mu\beta})=\kappa^2\Theta^{\mu\nu}, \quad (9)$$

which is an alternative form of Einstein's gravitational field equation. It is interesting to note that the left-hand side of (9) is linear in $\mathfrak{g}^{\mu\nu}$, which is due to the fact that all the nonlinear terms of the field equation are contained within the quantity $\Theta^{\mu\nu}$ on the right-hand side of (9).

Since neither $\epsilon^{\alpha\beta}$ nor $\Theta^{\mu\nu}$ transforms as a tensor under arbitrary coordinate transformations, the field Eq. (9) is not manifestly covariant. Nevertheless, Einstein's field equation can be expressed in the form (9) in any arbitrary frame of reference. Moreover, if we confine ourselves only to those frames of reference in which the coordinate condition

$$\partial\mathfrak{g}^{\mu\nu}/\partial x^\nu=0 \quad (10)$$

is satisfied, then (9) reduces to

$$\epsilon^{\alpha\beta}\partial^2\mathfrak{g}^{\mu\nu}/\partial x^\alpha\partial x^\beta=\kappa^2\Theta^{\mu\nu}. \quad (11)$$

Further, we can regard the flat space as the zeroth order approximation to the Riemannian space. It can then be shown[4] that the field quantities, occurring in Einstein's theory, can be expressed as infinite series in the flat space. Therefore, keeping Einstein's theory mathematically unchanged, we can pass over from the Riemannian space to the flat space. After passing over to the flat space, the general covariance of the theory is no longer apparent, but the theory still remains manifestly Lorentz covariant. In this way Einstein's theory can also be regarded as a theory of gravitation in flat space with a Lagrangian density containing an infinite number of terms.

3. COMPARISON OF EINSTEIN'S THEORY OF GRAVITATION AND MAXWELL'S THEORY OF ELECTROMAGNETISM

We shall now compare some aspects of Einstein's theory of the gravitational field and Maxwell's theory of the electromagnetic field, and show that the two theories have many features in common. In this section and the subsequent ones, we shall follow the usual simplified flat space notation, taking the space-time coordinates as x_1, x_2, x_3 and $x_4=ict$.

Field Equations

According to Maxwell and Lorentz, the electromagnetic field is described by the field equation

$$\Box^2 A_\mu=-(1/c)j_\mu, \quad (12)$$

with the supplementary condition

$$\partial A_\mu/\partial x_\mu=0, \quad (13)$$

where A_μ is the electromagnetic potential, and j_μ is the current four-vector. It is evident that (12) and (13) have a remarkable similarity to (11) and (10). In fact, the only difference between the two sets of equations is that in place of the four-vectors A_μ and j_μ in the electromagnetic field equations, we have the symmetrical quantities $\mathfrak{g}^{\mu\nu}$ and $\Theta^{\mu\nu}$ in the gravitational field equations.

Covariance

The field equation (12) is covariant not only under Lorentz transformations but also under all those gauge transformations which leave the supplementary condition (13) invariant. Similarly, the gravitational equation (11) is covariant not only under Lorentz transformation but also under all those general coordinate transformations which leave the coordinate condition (10) invariant. Thus, the two theories share the common property that they are both covariant under groups of transformations, which are more general than the Lorentz transformations. This property not only enhances the mathematical beauty of Maxwell's and Einstein's theories, but it is also very helpful in the investigation of the interaction of electromagnetic and gravitational fields with other fields.

Quantization

In the quantization of the electromagnetic field the supplementary condition gives rise to certain difficulties,[5] which can be overcome by using a generalized vector space with an indefinite metric.[6] When we try to quantize the gravitational field, similar difficulties appear, and they can again be overcome by means of an indefinite metric.[7] It is then found that the quantized gravitational field corresponds to gravitational quanta or gravitons of spin 2, while it is well known that the quantized electromagnetic field corresponds to photons of spin 1. However, on quantization the electromagnetic and the gravitational fields have several properties in common. Both of these fields correspond to neutral particles of integral spin and vanishing rest-mass, and both photons and gravitons have two states of polarization with their spin axes parallel or antiparallel to their directions of motion.

Although there are several interesting analogies between the electromagnetic and the gravitational fields, there are also some dissimilarities between the two fields, which will be discussed in the next section.

4. PECULIARITIES OF EINSTEIN'S GRAVITATIONAL FIELD

The most striking difference between the gravitational and the electromagnetic fields is that the gravitational field is nonlinear and its Lagrangian density in flat space

[4] For details see S. N. Gupta, Proc. Phys. Soc. (London) **A65**, 608 (1952).

[5] F. J. Belinfante, Phys. Rev. **76**, 226 (1949).

[6] S. N. Gupta, Proc. Phys. Soc. (London) **A63**, 681 (1950).

[7] S. N. Gupta, Proc. Phys. Soc. (London) **A65**, 161, 608 (1952).

consists of an infinite number of terms. We shall now show that these peculiarities of the gravitational field are a necessary consequence of the fact that the gravitational field corresponds to particles of spin 2.

If we assume that the gravitational field corresponds to neutral particles of zero rest-mass and spin 2, its field equation in the absence of interaction will be[8]

$$\Box^2 U_{\mu\nu}=0 \tag{14}$$

with the supplementary condition

$$\partial U_{\mu\nu}/\partial x_\mu=0, \tag{15}$$

where $U_{\mu\nu}$ is a real symmetrical tensor.

In the presence of interaction, the field equation (14) will be modified as

$$\Box^2 U_{\mu\nu}=\kappa\Theta_{\mu\nu}, \tag{16}$$

where κ is a constant, and $\Theta_{\mu\nu}$ is some symmetrical tensor. Then, differentiating (16) with respect to x_μ, and using (15), we get

$$\partial\Theta_{\mu\nu}/\partial x_\mu=0. \tag{17}$$

The only known physical quantity, which is described by a symmetrical tensor of vanishing divergence, is the total energy-momentum tensor of a closed system of fields. This suggests that the quantity $\Theta_{\mu\nu}$ in (16) should represent the total energy-momentum tensor due to the gravitational field as well as other fields. It should be noted that the gravitational radiation has not yet been experimentally observed. Therefore, from the experimental point of view it is conceivable to have a theory of gravitation in which the divergence of the non-gravitational energy-momentum tensor alone vanishes. However, it is mathematically impossible to construct such a theory in a consistent way, except in the trivial case when there is no interaction between the gravitational field and other fields.

Now, we can write the total energy-momentum tensor $\Theta_{\mu\nu}$ as

$$\Theta_{\mu\nu}=t_{\mu\nu}+T_{\mu\nu}, \tag{18}$$

where $t_{\mu\nu}$ is the energy-momentum tensor of the gravitational field, and $T_{\mu\nu}$ is the energy-momentum tensor of all other fields. Hence, in the absence of all other fields, a pure gravitational field of spin 2 will be described by the field equation

$$\Box^2 U_{\mu\nu}=\kappa t_{\mu\nu}. \tag{19}$$

According to the present ideas of the field theory, the field Eq. (19) must be derived from a Lagrangian density by means of a variational principle. We shall, therefore, try to find the required Lagrangian density for the pure gravitational field by successive approximations.

[8] M. Fierz and W. Pauli, Proc. Roy. Soc. (London) **A173**, 211 (1939).

Let us first choose the Lagrangian density as

$$L=-\tfrac{1}{2}\frac{\partial U_{\mu\nu}}{\partial x_\lambda}\frac{\partial U_{\mu\nu}}{\partial x_\lambda}, \tag{20}$$

which gives the field equation

$$\Box^2 U_{\mu\nu}=0 \tag{21}$$

and the energy-momentum tensor

$$t_{\mu\nu}=\frac{\partial U_{\lambda\rho}}{\partial x_\mu}\frac{\partial U_{\lambda\rho}}{\partial x_\nu}-\tfrac{1}{2}\delta_{\mu\nu}\frac{\partial U_{\lambda\rho}}{\partial x_\sigma}\frac{\partial U_{\lambda\rho}}{\partial x_\sigma}. \tag{22}$$

We now have to modify (20) in such a way that in the resulting equation the quantity (22) appears on the right-hand side of (21). For this it is evident that we must take a Lagrangian density of the form

$$L'=-\tfrac{1}{2}\frac{\partial U_{\mu\nu}}{\partial x_\lambda}\frac{\partial U_{\mu\nu}}{\partial x_\lambda}+f_3, \tag{23}$$

where f_3 consists of one or more terms such that each term is a product of *three* factors, each factor being either $U_{\mu\nu}$ or its derivative. However, then the energy-momentum tensor becomes

$$t_{\mu\nu}'=\left(\frac{\partial U_{\lambda\rho}}{\partial x_\mu}\frac{\partial U_{\lambda\rho}}{\partial x_\nu}-\tfrac{1}{2}\delta_{\mu\nu}\frac{\partial U_{\lambda\rho}}{\partial x_\sigma}\frac{\partial U_{\lambda\rho}}{\partial x_\sigma}\right)+g_3, \tag{24}$$

where g_3 also consists of one or more terms such that each term is a product of three factors, each factor being either $U_{\mu\nu}$ or its derivative. Again, we have to modify (23) in such a way that in the resulting field equation the quantity (24) appears on the right-hand side of (21). For this we must choose a Lagrangian density of the form

$$L''=-\tfrac{1}{2}\frac{\partial U_{\mu\nu}}{\partial x_\lambda}\frac{\partial U_{\mu\nu}}{\partial x_\lambda}+f_3+f_4, \tag{25}$$

where f_4 consists of one or more terms such that each term is a product of *four* factors, each factor being either $U_{\mu\nu}$ or its derivative. In this way it follows that in order to obtain the field equation (19) we shall have to introduce an infinite number of terms in the Lagrangian density.

Hence, not only do Maxwell's theory of the electromagnetic field and Einstein's theory of the gravitational field have many similarities, but the dissimilarities between these fields are a necessary consequence of the difference in their spins.

It is a pleasure to thank Professor K. Lark-Horovitz for his interest in this work, and Professor F. J. Belinfante for stimulating discussions. I am also indebted to Professor P. G. Bergmann for a very profitable correspondence.