New Formulation of a General Three-Dimensional Cascade Theory

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(Received September 13, 1954)

A new simple and straightforward formulation of a general three-dimensional cascade is given. The resultant integral equations include those derived by Blatt as a special case. The equations have a number of interesting features which distinguish them from those commonly used in cascade theory. They also lead to a simple moment recursion relation which is valid for all energies and which includes various processes such as Compton effect and ionization loss. The paper greatly simplifies previous work on the subject.

1. INTRODUCTION

An integral equation for the projected lateral distribution function of a cascade shower developing in a uniform medium has recently been obtained by Blatt.[1] This equation is unique in that the angular distribution does not appear in it; neither do the details of the cascade process, such as the bremsstrahlung and pair production cross sections. The only information necessary about the cascade is the average numbers; it is this which takes into account the details of the cascade process. As a consequence a moment recurrence formula can be written down for the lateral distribution which does not include the angular and mixed moments. It should be noted that in all previous work equations for the angular distribution could be written down which are independent of the lateral distribution, but not vice versa.

In his derivation Blatt makes the following assumptions: (1) Only one type of particle involved in the cascade suffers appreciable angular deviations. (2) The multiplicative processes involved in the cascade do not lead to appreciable angular deviations. In other words, the cascading process and the lateral spreading of the shower are due to different types of collisions.

In addition the following two assumptions, which are not necessary for the validity of the final equation, were made to simplify the argument: (*a*) The medium is of uniform composition. (*b*) The cascade is initiated by a particle of the type that suffers angular deflections. His derivation is presented in a general form without reference to the particular nature of the cascade involved, but *it should be noted that the stringent requirements of assumptions (1) and (2) effectively limit the application of Blatt's equation to the electron-photon cascade.*

It is the purpose of this paper to present an alternative, straightforward derivation which is so general in its application that it enables us to dispense completely with all four of the above-mentioned assumptions. This means that we have here a powerful method of tackling the general problem of the lateral spread of the mixed atmospheric cascade,[2,3] with electron-photon showers as a special case. In addition, the equations take into account such processes as the Compton effect, ionization loss, etc. This leads to simple moment recursion relation in which all these processes are taken into account in the average numbers. This vastly simplifies all previous work on the subject.

The general equations are derived using the so-called "first collision" or "regeneration point" method developed in recent work on cascade fluctuation theory.[4,5]

2. DEFINITIONS

Let $p_{i,j}(E_0,t_0;E,r,\theta,t)dEdrd\theta$ be the average number of particles of type $j(j=1,2,\cdots,n)$ of energy in the range E to $E+dE$, at a distance from the shower axis in the range r to $r+dr$, and travelling in a direction making an angle θ to $\theta+d\theta$ to the shower axis at depth t in a cascade initiated at depth t_0 by a particle of type i of energy E_0 travelling along the shower axis. The fact that we allow the cascade to be initiated at depth t_0 makes allowance for the inhomogeneity of the atmosphere. Let $\pi_{i,j}(E_0,t_0;E,t)dE$ be the corresponding average number integrated over all θ and r. We assume that the $\pi_{i,j}$ are known functions; that is, we shall assume that the problem of the longitudinal development of *whatever cascade* (including ionization loss, etc.) is under consideration has been solved.

Let $x_{i,j}(E_0,t;E,\theta)dEd\theta$ be the probability that an i particle of energy E_0 suffers a collision at depth t from which arise any number of any type of secondary particles, but one of them is a j particle of energy E travelling in a direction at an angle θ to the primary i particle.

The probability $x_{i,j}$ is completely general and can include all known cascade phenomena. It may for instance include terms representing collisions which are not contributory to the development of the cascade, that is collisions producing only one secondary which can be identified with the primary. The effect of such

* Also supported by the Nuclear Research Foundation within the University of Sydney.

[1] J. M. Blatt, preceding paper [Phys. Rev. 96, 1644 (1954)].

[2] H. Messel and R. B. Potts, Nuovo cimento **10**, 754 (1953).

[3] H. Messel and H. S. Green, Proc. Phys. Soc. (London) **A66**, 1009 (1953).

[4] L. Janossy, Proc. Phys. Soc. (London) **A63**, 241 (1950).

[5] H. Messel and R. B. Potts, Proc. Phys. Soc. (London) **A65**, 473 and 854 (1952).

a collision is to reduce the energy of the primary particle and/or deviate its direction of motion. There are two examples which are worth noting.

1. The angular deviations of electrons in an electron-photon cascade by Rutherford scattering. If the scattering cross section is taken as $\sigma_i(E,t;\theta)d\theta$ this can be included in our $x_{i,j}$ as

$$\delta_{i,j}\delta(E_0-E)\sigma_i(E,t;\theta). \tag{2.1}$$

2. The continuous loss of energy of a charged particle through ionization of the medium. If the energy lost is β units of energy per unit distance travelled, this effect can be included by incorporating the following term in $x_{i,j}(E_0,t_0;E,\theta)$:

$$\delta_{i,j}\delta(\theta)\ \operatorname*{Lim}_{\epsilon\to 0}\beta/\epsilon\cdot\delta(E_0-E-\epsilon). \qquad (2.2)^{6,7}$$

Define the cross section $w_{i,j}(E_0,t;E)dE$ by the equation,

$$w_{i,j}(E_0,t;E)=\int_{-\infty}^{\infty} x_{i,j}(E_0,t;E,\theta)d\theta. \tag{2.3}$$

Then $w_{i,j}$ is the analog of $x_{i,j}$ in the theory of the purely longitudinal development of the cascade.

Let $\alpha_i(E_0,t)$ be the total probability that an i particle of energy E_0 will suffer a collision of any type in travelling unit distance at depth t. Then in general,

$$\alpha_i(E_0,t)<\sum_{j=1}^{n}\int_0^{E_0} w_{i,j}(E_0,t;E)dE. \tag{2.4}$$

The right-hand side of (2.4) is the average number of secondary particles produced by an i particle of energy E_0 in travelling unit distance, while $\alpha_i(E_0,t)$ is the average number of collisions suffered by that particle in the same distance. Events of the type defined in (2.1) and (2.2) contribute equal amounts to both sides of (2.4).

The functions defined above can conveniently be grouped in the following matrices:

$$\begin{gathered}
\mathbf{P}=(p_{i,j}),\quad \boldsymbol{\pi}=(\pi_{i,j}),\quad \mathbf{X}=(x_{i,j}),\quad \mathbf{W}=(w_{i,j}),\\
\mathbf{A}=(a_{i,j}),\quad \text{where}\quad a_{i,j}=\delta_{i,j}\alpha_i,\\
\mathbf{Y}(E_0,t;E,\theta)=\mathbf{X}(E_0,t;E,\theta)-\delta(E-E_0)\delta(\theta)\mathbf{A}(E_0,t),\\
\mathbf{V}(E_0,t;E)=\int_{-\infty}^{\infty} d\theta\mathbf{Y}(E_0,t;E,\theta)\\
=\mathbf{W}(E_0,t;E)-\delta(E-E_0)\mathbf{A}(E_0,t).
\end{gathered} \tag{2.5}$$

We note that $\mathbf{P}$ and $\boldsymbol{\pi}$ satisfy the initial condition,

$$\begin{aligned}
\mathbf{P}(E_0,t_0;E,r,\theta,t_0)&=\delta(r)\delta(\theta)\delta(E_0-E)\mathbf{I},\\
\boldsymbol{\pi}(E_0,t_0;E,t_0)&=\delta(E_0-E)\mathbf{I},
\end{aligned} \tag{2.6}$$

where $\mathbf{I}$ is the unit matrix.

3. THE FIRST COLLISION EQUATION

Consider a cascade initiated by a particle of type i which is known to have the energy E_0 at depth t_0 in the medium. The probability $g_i(E_0,t')$ that this particle will reach a depth t' without suffering any type of collision, and therefore still having its original energy E_0, is given by

$$g_i(E_0,t')=\exp\left[-\int_{t_0}^{t'}\alpha_i(E_0,\tau)d\tau\right]. \tag{3.1}$$

A collision giving rise to a k particle of energy E' travelling at an angle θ' to the initiating particle may then occur. The probability of this is

$$x_{i,k}(E_0,t';E',\theta')dE'd\theta'dt'. \tag{3.2}$$

The cascade initiated by this secondary particle then has, for its distribution function at depth t,

$$P_{k,j}(E',t';E,r-\theta'(t-t'),\theta-\theta',t). \tag{3.3}$$

The contribution to the total cascade arising from the probability of this particular event occurring first is therefore given by the product of expressions (3.1), (3.2), and (3.3). By now summing over the contributions of all possible first collisions we should recover the distribution function of the total cascade. This procedure yields the "first collision integral equation":

$$\begin{aligned}
&p_{i,j}(E_0,t_0;E,r,\theta,t)\\
&\quad=\delta(E-E_0)\delta(r)\delta(\theta)\delta_{i,j}g_i(E_0,t)+\int_{t_0}^{t}dt'g_i(E_0,t')\\
&\qquad\times\sum_{k=1}^{n}\int_E^{E_0}dE'\int_{-\infty}^{\infty}d\theta' x_{i,k}(E_0,t';E',\theta')\\
&\qquad\qquad\times p_{k,j}(E',t';E,r-\theta'(t-t'),\theta-\theta',t).
\end{aligned} \tag{3.4}$$

The first term on the right-hand side of (3.4) covers the possibility of no collision at all occurring. By using the matrix notation (2.5) the first collision equation becomes

$$\begin{aligned}
&\mathbf{P}(E_0,t_0;E,r,\theta,t)\\
&\quad=\delta(E_0-E)\delta(r)\delta(\theta)\exp\left[-\int_{t_0}^{t}\mathbf{A}(E_0,\tau)d\tau\right]\\
&\qquad+\int_{t_0}^{t}dt'\exp\left[-\int_{t_0}^{t'}\mathbf{A}(E_0,\tau)d\tau\right]\\
&\qquad\cdot\int_E^{E_0}dE'\int_{-\infty}^{\infty}d\theta'\mathbf{X}(E_0,t';E',\theta')\\
&\qquad\qquad\times\mathbf{P}(E',t';E,r-\theta'(t-t'),\theta-\theta',t).
\end{aligned} \tag{3.5}$$

[6] Ionization loss is usually treated as a continuous deterministic process (see reference 7) whereas a cross section like $x_{i,j}$ defines a statistical process with resultant fluctuations. By defining ionization loss as in (2.2) we are expressing a deterministic process as a limit of a statistical one.

[7] H. J. Bhabha and S. K. Chakrabarty, Proc. Roy. Soc. (London) **A181**, 267 (1943).

Differentiating Eq. (3.5) with respect to t_0 gives the first collision integro-differential equation,

$$\frac{\partial}{\partial t_0}\mathbf{P}(E_0,t_0;E,r,\theta,t)=-\int_E^{E_0}dE'\int_{-\infty}^{\infty}d\theta'\mathbf{Y}(E_0,t_0;E',\theta')\times\mathbf{P}(E',t_0;E,r-\theta'(t-t_0),\theta-\theta',t).\quad(3.6)$$

Integrating the above equation through all r and θ yields the equivalent equation for the average numbers solution $\boldsymbol{\pi}$,

$$\frac{\partial}{\partial t_0}\boldsymbol{\pi}(E_0,t_0;E,t)+\int_E^{E_0}dE'\mathbf{V}(E_0,t_0;E')\times\boldsymbol{\pi}(E',t_0;E,t)=0.\quad(3.7)$$

Equation (3.6) can be put in a form closely resembling (3.7) by the addition to both sides of the expression

$$\int_E^{E_0}dE'\int_{-\infty}^{\infty}d\theta'\mathbf{Y}(E_0,t_0;E',\theta')\mathbf{P}(E',t_0;E,r,\theta,t),\quad(3.8)$$

which on the left-hand side is put in the form

$$\int_E^{E_0}dE'\mathbf{V}(E_0,t_0;E')\mathbf{P}(E',t_0;E,r,\theta,t),\quad(3.8\text{a})$$

and on the right-hand side in the form

$$\int_E^{E_0}dE'\int_{-\infty}^{\infty}d\theta'\mathbf{V}(E_0,t_0;E')\delta(\theta')\times\mathbf{P}(E',t_0;E,r-\theta'(t-t_0),\theta-\theta',t).\quad(3.8\text{b})$$

Equation (3.6) then reads

$$\begin{aligned}\frac{\partial}{\partial t_0}\mathbf{P}(E_0,t_0;E,r,\theta,t)&+\int_E^{E_0}dE'\mathbf{V}(E_0,t_0;E')\mathbf{P}(E',t_0;E,t)\\&=-\int_E^{E_0}dE'\int_{-\infty}^{\infty}d\theta'[\mathbf{Y}(E_0,t_0;E',\theta')-\delta(\theta')\mathbf{V}(E_0,t_0;E')]\times\mathbf{P}(E',t_0;E,r-\theta'(t-t_0),\theta-\theta',t)\\&=-\int_E^{E_0}dE'\int_{-\infty}^{\infty}d\theta'[\mathbf{X}(E_0,t_0,E',\theta')-\delta(\theta')\mathbf{W}(E_0,t_0;E')]\times\mathbf{P}(E',t_0;E,r-\theta'(t-t_0),\theta-\theta',t)\end{aligned}\quad(3.9)$$

By treating the term on the right-hand side of (3.9) as a known inhomogeneity, a comparison of (3.9) and (3.7) yields the following general integral equation for $\mathbf{P}$:

$$\begin{aligned}\mathbf{P}(E_0,t_0;E,r,\theta,t)&=\delta(\theta)\delta(r)\boldsymbol{\pi}(E_0,t_0;E,t)+\int_{t_0}^{t}dt'\int_E^{E_0}dE'\times\int_E^{E'}dE''\int_{-\infty}^{\infty}d\theta'\boldsymbol{\pi}(E_0,t_0;E',t')\\&\times[\mathbf{X}(E',t';E'',\theta')-\delta(\theta')\mathbf{W}(E',t';E'')]\times\mathbf{P}(E'',t';E,r-\theta'(t-t'),\theta-\theta',t).\end{aligned}\quad(3.10)$$

4. MOMENT RECURRENCE RELATIONS

The integral equation (3.10) enables us to write down very simply the recurrence relations for the moments, angular, radial and mixed if desired, of the lateral distribution functions $\mathbf{P}$. We are interested in the nth radial moment defined by

$$\mathbf{R}_n(E_0,t_0;E,t)=\int_{-\infty}^{\infty}d\theta\int_{-\infty}^{\infty}r^ndr\mathbf{P}(E_0,t_0;E,r,\theta,t),\quad(4.1)$$

and the mth angular moment,

$$\mathbf{Q}_m(E_0,t_0;E,t)=\int_{-\infty}^{\infty}\theta^md\theta\int_{-\infty}^{\infty}dr\mathbf{P}(E_0,t_0;E,r,\theta,t).\quad(4.2)$$

We obtain the solutions:

$$\mathbf{R}_n(E_0,t_0;E,t)=\sum_{p=1}^{n}\binom{n}{p}\int_{t_0}^{t}dt'(t-t')^p\int_E^{E_0}dE'\int_E^{E'}dE''\times\boldsymbol{\pi}(E_0,t_0;E',t')\mathbf{X}_p(E',t';E'')\mathbf{R}_{n-p}(E'',t';E,t),\quad(4.3)$$

$$\mathbf{Q}_m(E_0,t_0;E,t)=\sum_{q=1}^{m}\binom{m}{q}\int_{t_0}^{t}dt'\int_E^{E_0}dE'\int_E^{E'}dE''\boldsymbol{\pi}(E_0,t_0;E',t')\times\mathbf{X}_q(E',t';E'')\mathbf{Q}_{m-q}(E'',t';E,t),\quad(4.4)$$

where

$$\mathbf{X}_n(E,t;E')=\int_{-\infty}^{\infty}d\theta\theta^n\mathbf{X}(E,t;E',\theta).\quad(4.5)$$

5. ELECTRON-PHOTON SHOWERS

In the electron-photon cascade it is assumed that the multiplicative processes, *viz.*, bremsstrahlung, pair production and Compton scattering, do not contribute to the lateral spread of the shower. Hence their differential cross sections have the form,

$$x_{i,j}(E_0,t;E,\theta)=\delta(\theta)w_{i,j}(E_0,t,E)_{i,j=1,2},\quad(5.1)$$

and will therefore not appear in Eq. (3.10). If expression (2.1) is used for the Coulomb scattering of the electrons Eq. (3.10) becomes

$$p_{i,j}(E_0,t_0;E,r,\theta,t)=\delta(\theta)\delta(r)\pi_{i,j}(E_0,t_0;E,t)$$
$$+\int_{t_0}^{t}dt'\int_{E}^{E_0}dE'\int_{-\infty}^{\infty}d\theta'\pi_{i,1}(E_0,t_0;E',t')$$
$$\times[\sigma_1(E',t';\theta')-\delta(\theta')\mu_1(E',t')]$$
$$\times p_{1,j}(E',t';E,r-\theta'(t-t'),\theta-\theta',t). \quad (5.2)$$

This is also a generalization of Blatt's equation[1] in that it also treats the case of an incident photon ($i=2$). It should be noted that for this case it yields an immediate solution for $p_{2,j}$ in terms of the equivalent expression for a primary electron.

It should be noted that $\pi_{i,1}$ may contain such processes as ionization loss, Compton effect, etc., and hence (5.2) is valid for any energy range for which we have the average numbers. It is obvious that this leads to a simple recursion relation for the moments which greatly simplifies previous work. See for instance Chartres and Messel.[8]

One of us (B.A.C.) wishes to thank Commonwealth Scientific and Industrial Research Organization for the grant of a studentship.

[8] B. A. Chartres and H. Messel, Proc. Phys. Soc. (London) **A67**, 158 (1954).

PHYSICAL REVIEW VOLUME 96, NUMBER 6 DECEMBER 15, 1954

Possible Triple-Scattering Experiments*

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(Received September 7, 1954)

Triple-scattering experiments may be used to get additional information about the spin-dependence of the scattering matrix of the second scatterer. In general two new parameters describing the scattering may be determined by means of two distinct experiments, one in which the successive scattering planes are parallel and one in which the successive scattering planes are at right angles to each other. The relation between these parameters and the scattering matrix is given for the cases of protons scattered from a spin-zero target and of proton-proton scattering. For the former case the magnitude of the left-right asymmetry in the third scattering is calculated on the basis of a phenomenological model due to Fermi in the Born approximation. Further experimental possibilities for p-p scattering are discussed.

1. GENERAL FORMULATION

RECENT successful experiments[1] on the double scattering of high-energy protons make it of interest to note that further information may be obtained by means of triple-scattering experiments. Such experiments would be designed to determine how the second scatterer changes the direction and/or magnitude of the polarization of the proton; thus, the first scatterer serves simply as a polarizer and the final scatterer as an analyzer.

To describe the geometry of a triple-scattering experiment we first define for each scattering the unit vector $\mathbf{n}$,

$$\mathbf{n}=(\mathbf{k}\times\mathbf{k}')/|\mathbf{k}\times\mathbf{k}'|, \quad (1.1)$$

where $\mathbf{k}$ and $\mathbf{k}'$ are unit vectors in the incident and outgoing laboratory directions, respectively. The beam incident on the second scatterer is polarized along the direction $\mathbf{n}_1$. For a given scattering angle θ, the second scattering is completely defined by an azimuthal angle φ, here defined by

$$\cos\varphi=\mathbf{n}_1\cdot\mathbf{n}_2 \quad \sin\varphi=\mathbf{n}_1\times\mathbf{n}_2\cdot\mathbf{k}_2. \quad (1.2)$$

In the third scattering a left-right asymmetry is measured relative to a direction $\mathbf{n}_3$; since polarization along the direction of motion cannot be detected, two "settings" of the analyzer are sufficient; that is, two directions for $\mathbf{n}_3$. Therefore we need only consider the cases when $\mathbf{n}_3$ is parallel to $\mathbf{n}_2$ and when $\mathbf{n}_3$ is along the direction

$$\mathbf{s}=\mathbf{n}_2\times\mathbf{k}_2'. \quad (1.3)$$

Thus the third scattering may be chosen to determine either $\langle\boldsymbol{\sigma}\rangle_2\cdot\mathbf{n}_2$ or $\langle\boldsymbol{\sigma}\rangle_2\cdot\mathbf{s}$, where $\langle\boldsymbol{\sigma}\rangle_2$ is the expectation value of the spin vector after the second scattering; the corresponding asymmetries in triple scattering will be designated e_{3n} and e_{3s}.[2] The discussion throughout is nonrelativistic.

* Work supported in part by U. S. Atomic Energy Commission.
[1] Oxley, Cartwright, and Rouvina, Phys. Rev. **93**, 806 (1954); Chamberlain, Segrè, Tripp, Wiegand, and Ypsilantis, Phys. Rev. **93**, 1430 (1954); Marshall, Marshall, and Carvalho, Phys. Rev. **93**, 1431 (1954); J. M. Dickson and D. C. Salter, Nature **173**, 946 (1954); Kane, Stallwood, Sutton, Fields, and Fox, Phys. Rev. **95**, 1694 (1954).

[2] The asymmetry e_{3n} (or e_{3s}) is defined as $[I_3(+)-I_3(-)]/[I_3(+)+I_3(-)]$ where $I_3(\pm)$ refers to scattering such that $\mathbf{n}_3$ is parallel to $\pm\mathbf{n}_2$ (or $\pm\mathbf{s}$).