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Integral Equations for Cascade Showers

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Integral equations of a new type are derived for the lateral and angular distribution functions in a cascade shower. The moment recursion relations of Nordheim are direct consequences of these integral equations. The method of derivation has some intrinsic interest since it involves the formal device of the introduction of negative probabilities for events in which nothing happens. This formal device may have applications in other fields. No solutions of the integral equations derived here are given in this note.

1. INTRODUCTION

THE angular and lateral distribution of the particles in large air showers presents a difficult theoretical problem, which is so far only incompletely solved. The present note is intended to indicate a possible line of attack, not a solution.

Rather than restricting ourselves from the start to the (usually considered) electron-photon cascade initiated by a single electron or photon, we shall derive integral equations for a more general type of shower which we shall call a type A shower. Type A showers may contain any number of components (particles of different types) which are genetically related in any way. The crucial assumptions are: (1) Only one kind of particle (particles of type 1, say) suffers appreciable angular deviation in passing through the medium. (2) The production processes involved in the cascade do not lead to appreciable angular deviations between the directions of the primary and produced particles.

These conditions are satisfied to a good approximation in electron-photon cascades. Photons are deviated in direction only by Compton scattering, which is of negligible importance in air showers, and the production processes (pair creation and bremsstrahlung) involve angular deviations much smaller than the angular deviations due to Coulomb scattering of the electrons by the nuclei in the medium, for electrons of energies less than 10^{12} ev. Higher-energy electrons are on the average so close to the axis of the shower that their angular and lateral deviations are of little experimental importance.

These conditions are also satisfied in the multiple scattering of a single particle without cascade multiplication. Thus the equations we obtain will be equally applicable to this trivial kind of "cascade." Since the equations for multiple scattering are simple and well-known, the equations derived here can be tested on that example.

We shall consider the projected angular and lateral distributions, i.e., the distributions projected onto a plane containing the shower axis (the line defined by the direction of the initial particle of the shower). Furthermore, we are interested in the distribution functions for particles of "type 1" only, i.e., for those particles which themselves suffer angular deviations during flight. Since these are commonly charged particles which are most easily detected, this is a reasonable restriction.

The next restriction is apparently serious: we require that the particle starting the shower itself be of type 1. However, it is easily seen that a shower started by some other type of particle can be thought of as a superposition of showers generated by type 1 particles, as far as the angular and lateral development is concerned. We follow through the initial development of the cascade until the *first* particle of type 1 is created; this particle is still produced along the shower axis, and moving along it. Hence the solution of the angular or lateral problem for showers initiated by type 1 particles forms a Green's function for showers initiated by particles of other types. This point is developed in more detail in the accompanying note.[1]

We now define the conditional probability:

$Q(E_0,E,t;\theta)d\theta$ = probability of finding a particle at a projected angle with the shower axis between θ and $\theta+d\theta$, *given that* the shower was started by a particle of type 1 with energy E_0, and that the observed particle is of type 1, has energy E, and is observed at a depth t from the origin of the shower. (1.1)

It is understood that there may be any number of other particles in the shower at depth t, particles of various types, energies, and angles. On the other hand, for an infinitesimal angle interval $d\theta$, the probability of finding more than one particle within this interval is of order $(d\theta)^2$ and can be ignored.

We define a similar conditional probability for the projected lateral distribution:

$P(E_0,E,t;x)dx$ = probability of finding a particle at a projected distance from the shower axis between x and $x+dx$, *given that* the shower was started by a particle of type 1 with energy E_0, and that the observed particle is of type 1, has energy E, and is observed at a depth t from the origin of the shower. (1.2)

[1] B. Chartres and H. Messel, following paper [Phys. Rev. 96, 1651 (1954)].

As a result of these definitions, we have the normalization conditions,

$$\int_{-\pi}^{+\pi} Q(E_0,E,t;\,\theta)d\theta=\int_{-\infty}^{\infty} P(E_0,E,t;\,x)dx=1. \quad (1.3)$$

If the cascade is predominantly forward, $Q(E_0,E,t;\theta)$ approaches zero rapidly for values of θ larger than some $\theta_{\max}(E)$, and the first integral in (1.3) may be extended formally to cover the range from $-\infty$ to $+\infty$. We shall use the "small angle approximation" throughout this paper. The methods developed here are *not* restricted to this approximation, however.

We shall also need the longitudinal distribution function for particles of type 1; for our purposes this is also best defined as a conditional probability:

$\pi(E_0,E,t)dE$ = probability of finding a particle of type 1 in the energy range E to $E+de$, *given that* the shower was started by a particle of type 1 with energy E_0, and that the point of observation is at a depth t from the origin of the shower. (1.4)

The distribution functions usually considered in cascade theory are given by

$$q(E_0,E,t,\theta)=\pi(E_0,E,t)Q(E_0,E,t;\,\theta), \quad (1.5)$$

and

$$p(E_0,E,t,x)=\pi(E_0,E,t)P(E_0,E,t;\,x). \quad (1.6)$$

These can also be defined by saying that $q(E_0,E,t,\theta)\times dEd\theta$ equals the average number of particles of type 1, with energies between E and $E+dE$ and angles between θ and $\theta+d\theta$, at depth t from the origin of the shower, given that the shower was started by a particle of type 1 with energy E_0.[2] A similar definition can be used for p.

2. REGENERATION POINT EQUATIONS FOR PARTICLES OF KNOWN HISTORY

Suppose we find a particle of type 1, energy E, and angle between θ and $\theta+d\theta$, at a depth t from the origin of the shower. We can then trace back the past history of this particle. This is shown schematically in Fig. 1. Just before reaching depth t, we have the same particle undergoing angular deflections. We then trace back to the point where this particle was produced, let us say by some particle not of type 1. The producing particle, as well as the particles not of type 1 from which it may have originated, are shown by dotted lines. According to our basic assumptions, the dotted lines are straight lines, i.e., these other particles are not deflected in

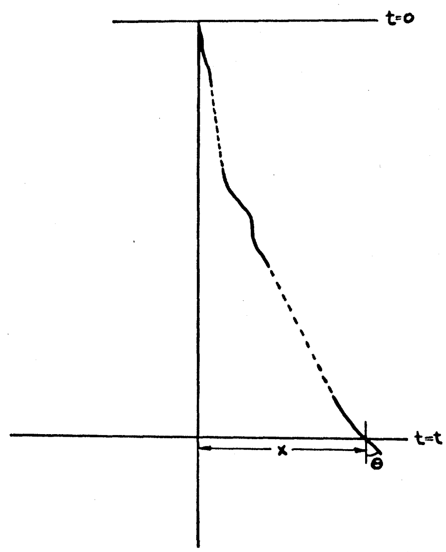

FIG. 1. The ancestry of a particle of type 1, observed at depth t and distance x from the shower axis, can be traced back as indicated schematically in this diagram. The dotted straight portions indicate ancestors of type other than 1, which by assumption do not suffer angular deviations.

passing through the medium, or in transforming into each other. At an even earlier depth, the ancestor of our particle is again a particle of type 1, indicated by the next solid line, and so on, until we reach the originating particle of the shower before the first cascade collision. We specify the "history" of the particle by stating, for each depth t', whether the ancestor at depth t' was a particle of type 1 or of some other type, and if the ancestor was of type 1, stating its energy $E'(t')$. We do *not* include statements about angles or positions in this "history."

The following convention will prove useful: for any value of t' at which the ancestor of our particle was a particle of some other type, we shall say that it can be treated as if it had been a particle of type 1 but energy $E'=0$. The reason why we can use this convention is simply that particles of zero energy (which never occur in the ancestry of a particle of finite energy E) can be assigned any scattering cross section whatever, and we shall assign scattering cross section 0 to particles of energy 0, formally. The advantage of this convention is that now the entire history of our particle is given by a single function $E'(t')$, whoe behavior is shown schematically in Fig. 2. $E'(0)=E_0$, then E' is a monotonically decreasing function of t' (due to collision loss) until we reach the point of the first cascade collision. From then on $E'=0$ until we reach the point in the shower at which the ancestry of our particle again consists of a particle of type 1, and so on. Finally, $E'(t)=E$.

The method of derivation used in this paper is based upon the approach of Nordheim;[3] that is, we

[2] The equations derived here hold for a homogeneous medium, i.e., the relevant cross sections and numbers of scattering centers per unit volume are independent of t. In a nonhomogeneous medium, the derivation goes through with minor changes, which consist mostly in specifying separately the depth of observation t_1 and the depth of the origin of the shower, t_0, not merely their difference $t=t_1-t_0$. We avoid this complication in order to keep the treatment as simple as possible and limited to essentials. The generalization to a nonhomogeneous medium is given in reference 1.

[3] L. W. Nordheim, Z. Physik **133**, 94 (1952).

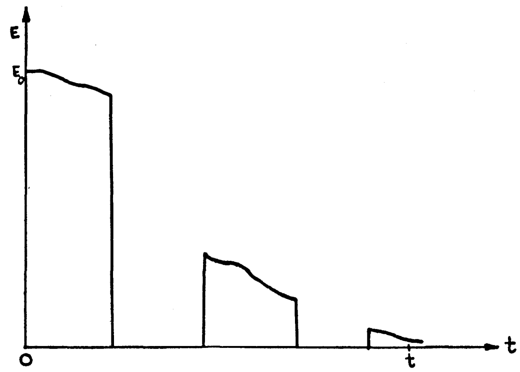

FIG. 2. The energetic history of a particle observed at depth t is shown schematically in this figure. Whenever the ancestor is itself a particle of type 1, its energy decreases monotonically (but not necessarily continuously) with increasing t; whenever the ancestor is a particle of some other type, the energy is formally set equal to zero.

first derive equations for particles of *given* energetic history, and then average over all possible energetic histories of the particles under consideration.

We shall need the probability for angular deviations (in projected angle) between α and $\alpha+d\alpha$ for particles of type 1 and energy E, when traversing an infinitesimal layer dt of the medium. Under the assumption of a homogeneous medium, this probability is independent of t (see reference 2). We define

$\sigma(E,\alpha)d\alpha dt$ = probability of a deviation in projected angle, of amount between α and $\alpha+d\alpha$ for a particle of type 1 and energy E traversing a layer dt of the medium. (2.1)

According to our previous convention, we define

$$\sigma(0,\alpha)=0. \tag{2.2}$$

We also introduce the total deviation probability,

$$\mu(E)=\int_{-\infty}^{\infty}\sigma(E,\alpha)d\alpha. \tag{2.3}$$

We should point out that the probability of a definite deviations in *projected* angle can be defined only in the small-angle approximation (otherwise this probability depends upon the projection angle in the other plane containing the shower axis); hence it is consistent to use infinite limits of integration in Eq. (2.3).

We shall now concentrate our attention on the *first* scattering collision in the history of our particle. Until this first scattering collision, all the ancestors of the particle were moving along the shower axis. We emphasize that we consider only scattering collisions, *not* collisions involved in the cascade process (since by our fundamental assumption (2) the latter collisions do not lead to angular deviations). We shall need the "survival probability,"

$g(t)$ = probability that no scattering collision occurred between $t=0$ and $t=t$. (2.4)

In the special case $\mu(E)$ independent of E we have $g(t)=\exp(-\mu t)$; this special case is of no use to us because over part of the range 0 to t we had particles of type other than 1, for which $E=0$ (by convention) and $\mu(E)=0$ [by fundamental assumption (1)]. The generalization to arbitrary $\mu(E)$ is

$$g(t)=\exp\left\{-\int_0^t \mu[E'(t')]dt'\right\}. \tag{2.5}$$

We then determine the probability that the first collision occurred in the interval t', $t'+dt'$, and deviated the particle by an angle between α and $\alpha+d\alpha$. This probability is

$$g(t')\sigma[E'(t'),\alpha]d\alpha dt'.$$

By assumption we are looking at particles which make an angle θ with the shower axis at depth t. Hence they must have been deviated through an angle $\theta-\alpha$ in the remaining distance $t-t'$. The probability for this is $Q(E', E, t-t'; \theta-\alpha)$. Hence we get a contribution to the final probability $Q(E_0,E,t;\theta)$ equal to

$$g(t')\sigma(E',\alpha)Q(E', E, t-t'; \theta-\alpha)dt'd\alpha.$$

The occurrence of the first scattering collision at depth t', with a scattering angle α, is a unique event, and the various choices of t' and α are mutually exclusive. Hence the probabilities can be added up to yield a total probability. This total, however, is still not equal to $Q(E_0,E,t;\theta)$ because we have ignored the possibility that no scattering collision at all occurred in the range 0 to t. This latter possibility contributes an amount $g(t)\delta(\theta)$, where δ is Dirac's delta function. We thus get the regeneration point equation

$$Q(E_0,E,t;\theta)=\int_0^t dt'\int_{-\infty}^{\infty}d\alpha g(t')\sigma(E',\alpha)\times Q(E', E, t-t'; \theta-\alpha)+g(t)\delta(\theta). \tag{2.6}$$

In this equation, we must replace E', wherever it occurs, by the (presumed known) value $E'(t')$ at the depth in question. We observe that the cascade history of the particle *before* the first scattering collision (at depths $t''<t'$) enters implicitly through the survival probability $g(t')$ [Eq. (2.5)].

We use entirely analogous arguments to obtain a regeneration point equation for the lateral distribution function $P(E_0,E,t;x)$. If the first scattering collision occurs at t' and deviates the particle through an angle α, the new "axis" is displaced with respect to the original one at depth t by the amount,

$$x_\alpha(t,t')=\alpha(t-t')+\text{terms of higher order in }\alpha. \tag{2.7}$$

In order that the particle be finally observed at a distance x from the old shower axis, the subsequent deflections (between t' and t) must shift it laterally, with respect to the new axis, by the amount $x-x_\alpha$.

Hence we obtain:

$$P(E_0,E,t;x)=\int_0^t dt'\int_{-\infty}^{\infty} d\alpha g(t')\sigma(E',\alpha)$$
$$\times P[E',E,t-t';x-\alpha(t-t')]+g(t)\delta(x). \quad (2.8)$$

We observe that Eq. (2.8) is an integral equation for the *lateral* distribution function alone; the angular distribution does not occur at all in this equation. This is characteristic for regeneration point equations. The more usual "forward" or "last-collision" equations are derived by considering what happens in the last interval of depth (from $t-dt$ to t) or what happened in the last collision. In either case it is impossible to get an equation for the lateral distribution which does not involve the angular distribution.

Equations (2.7) and (2.8) are *not* the conventional regeneration point equations for the shower. We have concentrated our attention on the first *scattering* collision, whereas the conventional regeneration point method considers the first collision of any kind, scattering or cascade. Conversely, Eqs. (2.7) and (2.8) apply only to particles of known history $E'(t')$, and cannot be used, in this form, for the actual cascade.

3. AN INVARIANCE PROPERTY OF REGENERATION POINT EQUATIONS

The various recursion relations for the moments of the angular and lateral distributions given in the literature[3,4] all have this in common: the lateral and angular moments of all orders can be obtained from a knowledge of the moments of the scattering cross section

$$\sigma_n=\int_{-\infty}^{\infty}\alpha^n\sigma(E,\alpha)d\alpha. \quad (3.1)$$

The remarkable feature of all the moment recursion relations is that *the zeroth moment, i.e., the total cross section* $\sigma_0=\mu$, *never enters into the recursion relations.* To the extent that the moments of the distribution function determine the function uniquely, the angular and lateral distribution of the particles in the cascade do not depend at all on the total cross section for angular deviations.

Let us try to understand this remarkable invariance. Two cross sections $\sigma(E,\alpha)$ and $\sigma'(E,\alpha)$ give the same final distributions provided all the moments agree, $\sigma_n=\sigma_n'$, except for the zeroth moment $\sigma_0\neq\sigma_0'$. Two functions all of whose moments agree except the zeroth moment differ by a multiple of the delta function, hence we conclude that

$$\sigma'(E,\alpha)=\sigma(E,\alpha)+k(E)\delta(\alpha). \quad (3.2)$$

[4] H. S. Green and H. Messel, Phys. Rev. **88**, 331 (1952); L. Landau, J. Phys. (U.S.S.R.) **3**, 237 (1940); L. Eyges and S. Fernbach, Phys. Rev. **82**, 23 (1951); L. Eyges, Phys. Rev. **74**, 1801 (1948).

Here $k(E)$ is an arbitrary function of E. Cross sections σ' and σ lead to the *same* final distribution functions for the cascade.

This is intuitively clear, also, since the term $k(E)\delta(\alpha)$ represents a probability of a scattering event though an angle $\alpha=0$, i.e., a probability of nothing at all happening, experimentally.

Hence Eqs. (2.6) and (2.8) are still correct if we replace σ by σ' everywhere in the equations [including in the definition of the survival probability $g(t)$, Eq. (2.5)]. Since $k(E)$ is an arbitrary function at our disposal, we get an infinite variety of regeneration point equations.

While the invariance under insertion of a probability of nothing happening was shown here for the special case of multiple scattering problems, this is of course a *general invariance property of all regeneration point equations, and last collision equations as well.* We can always introduce a formal probability (per unit path length) of nothing happening at all, and obtain a modified, but still correct, regeneration point equation. We shall use this invariance in the next section.

4. AVERAGING OVER PAST HISTORIES

Relation (2.6) is valid for particles of known history $E'(t')$. In order to obtain a result for the actual cascade, we must average over all possible past histories $E'(t')$, each weighted according to its probability of occurrence in this particular cascade. This is at first sight a hopeless task unless much more is known about the details of the cascade process than what we have assumed to be known. The survival probability $g(t)$ is particularly awkward to handle in such an average.

We now employ the invariance property pointed out in the preceding section to eliminate, formally, the survival probability $g(t)$ from the equations. To do this, we choose the arbitrary function $k(E)$ in Eq. (3.2) to be

$$k(E)=-\mu(E), \quad (4.1)$$

i.e., $k(E)$ is the negative of the total scattering cross section. Insertion in Eq. (3.2) gives the modified cross section,

$$\sigma'(E,\alpha)=\sigma(E,\alpha)-\mu(E)\delta(\alpha), \quad (4.2)$$

and hence the modified total cross section,

$$\mu'(E)=\int_{-\infty}^{\infty}\sigma'(E,\alpha)d\alpha=\mu(E)-\mu(E)=0. \quad (4.3)$$

The total cross section μ' vanishes as a result of the cancellation of the total probability of actual scattering events against the (assumed negative) probability of nothing happening at all. We emphasize that this negative probability of nothing happening is a purely formal device which is justified by the (physically immediate) invariance of the equations under the substitution (3.2) for all positive functions $k(E)$, and hence by analytic continuation also for negative functions $k(E)$.

Equation (2.5) shows that the modified survival probability $g'(t)=1$, identically, so that Eqs. (2.6) and (2.8) can be transformed into

$$Q(E_0,E,t;\theta)=\int_0^t dt'\int_{-\infty}^{\infty} d\alpha[\sigma(E',\alpha)-\mu(E')\delta(\alpha)]\times Q(E',E,t-t';\theta-\alpha)+\delta(\theta), \tag{4.4}$$

$$P(E_0,E,t;x)=\int_0^t dt'\int_{-\infty}^{\infty} d\alpha[\sigma(E',\alpha)-\mu(E')\delta(\alpha)]\times P[E',E,t-t';x-\alpha(t-t')]+\delta(x). \tag{4.5}$$

We now observe that we can carry out the averaging over past histories in a very direct fashion. Let us first keep E' fixed, and average over histories $E''(t'')$ in the range $t'<t''\leqslant t$. This averaging procedure affects only the term $Q(E',E,t-t';\theta-\alpha)$ in Eq. (4.4) [and the corresponding term in Eq. (4.5)], and it leads simply to the distribution function $Q(E',E,t-t';\theta-\alpha)$ for a *cascade* started by a particle of type 1, of energy E', and propagating through a layer of thickness $t-t'$. Next we average over past histories $E''(t'')$ in the range $0\leqslant t''<t'$. This average is particularly simple since according to the structure of Eq. (4.4) the history before the first "scattering" collision (at t') has no influence whatever. [In the original Eq. (2.6) the corresponding history did have an influence, since it affected the survival probability $g(t')$.]

Next we must average over all possible choices of E' at depth t'. We need the conditional probability $W(E_0,E,t;E',t')dE'$ of finding the ancestor of our particle at depth t' to be a particle of type 1 in the energy range E', $E'+dE'$, given that the initial particle is of type 1 with energy E_0 and the final particle is of type 1 with energy E at depth t. According to the definition of $\pi(E_0,E,t)$, formula (1.4), W is given by[5]

$$W(E_0,E,t;E',t')dE'=\frac{\pi(E_0,E',t')\pi(E',E,t-t')}{\pi(E_0,E,t)}dE'. \tag{4.6}$$

The integrand in Eq. (4.4) must be multiplied by this W, and we must integrate over all possible E' for a given t', i.e., over the range $E<E'<E_0$.

Finally we integrate over all possible t' as indicated in Eq. (4.4). No special weighting of different values of t' is necessary because we have arranged Eq. (4.4) in such a way that the formal survival probability $g'(t')$ is unity, hence the first formal scattering collision (which may be either an actual scattering collision or a scattering through an angle zero) is as likely to occur at one value of t' as at any other.

The result is an integral equation for the probability function $Q(E_0,E,t;\theta)$ for the *cascade*, i.e., if one takes into account all possible past histories $E'(t')$ of the particles under consideration.

$$Q(E_0,E,t;\theta)=\int_0^t dt'\int_E^{E_0} dE'\int_{-\infty}^{\infty} d\alpha W(E_0,E,t;E',t')\times[\sigma(E',\alpha)-\mu(E')\delta(\alpha)]\times Q(E',E,t-t';\theta-\alpha)+\delta(\theta). \tag{4.7}$$

This equation can be rewritten in terms of the more usual distribution function $q(E_0,E,t,\theta)$ given by Eq. (1.5). We multiply both sides of Eq. (4.7) by $\pi(E_0,E,t)$ to get

$$q(E_0,E,t,\theta)=\int_0^t dt'\int_E^{E_0} dE'\int_{-\infty}^{\infty} d\alpha\, \pi(E_0,E',t')\times[\sigma(E',\alpha)-\mu(E')\delta(\alpha)]\, q(E',E,t-t',\theta-\alpha)+\pi(E_0,E,t)\delta(\theta). \tag{4.8}$$

By an entirely analogous argument we obtain an integral equation for the lateral distribution function p, Eq. (1.6):

$$p(E_0,E,t,x)=\int_0^t dt'\int_E^{E_0} dE'\int_{-\infty}^{\infty} d\alpha\, \pi(E_0,E',t')\times[\sigma(E',\alpha)-\mu(E')\delta(\alpha)]\, p[E',E,t-t',x-\alpha(t-t')]+\pi(E_0,E,t)\delta(x). \tag{4.9}$$

Equations (4.8) and (4.9) are the main results of this note. We now point out some features of these equations. First of all, the equations have been derived without making any specific assumptions about the genetic relations in the cascade. The basic assumptions are merely that particles of type other than 1 are not scattered by the medium, and that no appreciable angular deviation occurs in the various production processes which give rise to the cascade. The details of the cascade process, including even the average numbers of particles of type other than 1 at various depths, are not needed in Eqs. (4.8) and (4.9). The only necessary knowledge about the cascade is the longitudinal distribution function $\pi(E_0,E,t)$, i.e., the *average* number of particles of type 1 and energy E at depth t in a cascade started by a particle of type 1 and energy E_0 at depth $t=0$.

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[5] This formula was first given by Nordheim, reference 4, Eq. (13). The factor dt' in Nordheim's formula 13 is misleading, however, and should be dropped. We would also like to point out that this is the only point at our derivation where we make essential use of the fact that the shower was started by a particle of type 1, and that the observed particle, at depth t, is also of type 1. Suppose that the initial particle is of type i and the observed particle is of type j; we then need the conditional probability $W_{ij}(E_0,E,t;E',t')$ of finding a particle of type 1 (because these are the only particles that scatter) at depth t' with energy in E', $E'+dE'$; this is given by

$$W_{ij}(E_0,E,t;E',t')dE'=\frac{\pi_{i1}(E_0,E',t')\pi_{1j}(E',E,t-t')}{\pi_{ij}(E_0,E,t)}dE', \tag{4.6a}$$

where $\pi_{ij}(E_0,E,t)$ is the probability of finding a particle of type j and energy E at depth t in a shower started by a particle of type i and energy E_0. With this modification and suitable addition of subscripts to the functions Q and P we get Eq. (5.2) of reference 1; however, this method does not give in any direct way the more general equations in Eq. (3.10) of reference 1.

In particular, we can specialize to simple multiple scattering without cascade multiplication or energy loss by setting

$$\pi(E_0,E,t)=\delta(E_0-E)$$
$$\text{(multiple scattering without energy loss).} \quad (4.10)$$

It is shown in the appendix that the equations of Scott and Snyder,[6] for example, are consistent with Eqs. (4.8) and (4.9) if Eq. (4.10) is substituted for $\pi(E_0,E,t)$. We also observe Eqs. (4.8) and (4.9) determine the respective distribution functions $q(t,\theta)$ and $p(t,x)$ *uniquely*.

To get multiple scattering with definite energy loss, we first determine the energy $E(t)$ at depth t and then set

$$\pi(E_0,E,t)=\delta[E-E(t)]$$
$$\text{(multiple scattering with unique energy loss).} \quad (4.11)$$

If there is straggling in the energy loss, the function $\pi(E_0,E,t)$ must be determined from the longitudinal equations.

In cascade theory, the longitudinal distribution func- $\pi(E_0,E,t)$ is well-known for electron-photon cascades (or at least its Mellin transform is well-known). In cascade theory it has been customary until recently to use the Landau approximation of pure multiple scattering, i.e., to neglect single and plural scattering contributions. This amounts to an assumption about the scattering cross section $\sigma(E,\alpha)$, namely

$$\sigma(E,\alpha)=(E_s/2E)^2\delta''(\alpha), \quad (4.12)$$

where $E_s=21$ Mev is a characteristic energy for multiple scattering and the second derivative of the delta function has to be understood in the usual way, in terms of integrations by parts. The resulting equation for the angular distribution has been given in the literature[7] and has proved to be useful. The corresponding equation for the lateral distribution in the Landau approximation had been surmised from the Nordheim moment recursion relation[8] but had not been derived directly. It has not proved useful so far in cascade theory, since it is very difficult to obtain explicit solutions of that equation.

The derivation of Eqs. (4.8) and (4.9) given here has made use of the absolute minimum amount of information about the cascade, namely that information which eventually enters the final equations, i.e., the function $\pi(E_0,E,t)$. It is of course possible to derive these equations in other ways. One way, suggested by H. Messel, consists in writing down the conventional regeneration point (first collision) equations for the cascade in question, and afterwards using the (presumed known) solutions of the purely longitudinal problem to eliminate explicit reference to the cross sections involved in the cascade multiplication. This program is carried through in the accompanying paper.[1]

From integral equations such as Eqs. (4.8) and (4.9) it is usually possible to derive recursion relations for the moments of the distribution functions. This case is no exception. We define moments by

$$q_n(E_0,E,t)=\int_{-\infty}^{\infty}\theta^n q(E_0,E,t,\theta)d\theta, \quad (4.13)$$

$$p_n(E_0,E,t)=\int_{-\infty}^{\infty}x^n p(E_0,E,t,x)dx, \quad (4.14)$$

and observe that

$$q_0(E_0,E,t)=p_0(E_0,E,t)=\pi(E_0,E,t). \quad (4.15)$$

We multiply both sides of Eq. (4.8) by θ^n and integrate over all values of θ. We notice that odd powers of θ give zero by symmetry; we also write $\theta^n=(\theta-\alpha+\alpha)^n$ and use the binomial theorem; this gives

$$q_{2n}(E_0,E,t)=\sum_{k=1}^{n}\binom{2n}{2k}\int_0^t dt'\int_E^{E_0}dE'\pi(E_0,E',t')$$
$$\times\sigma_{2k}(E')q_{2n-2k}(E',E,t-t')+\delta_{n,0}\pi(E_0,E,t). \quad (4.16)$$

Here σ_{2k} is defined by Eq. (3.1), and $\delta_{n,0}$ equals 1 if $n=0$, and equals 0 otherwise. The Nordheim recursion relation for the angular distribution moments (reference 3) is a special case of Eq. (4.16) obtained by the substitution (4.12), i.e., $\sigma_2(E)=\frac{1}{2}(E_s/E)^2$ and all other moments of σ vanish.

An entirely similar derivation gives the lateral moment recursion relation,

$$p_{2n}(E_0,E,t)=\sum_{k=1}^{n}\binom{2n}{2k}\int_0^t dt'\int_E^{E_0}dE'\pi(E_0,E',t')$$
$$\times\sigma_{2k}(E')(t-t')^{2k}p_{2n-2k}(E',E,t-t')$$
$$+\delta_{n,0}\pi(E_0,E,t). \quad (4.17)$$

Again the Nordheim moment recursion relation is a special case of Eq. (4.17). It should be pointed out that these moment recursion relations can be obtained directly by an extension of Nordheim's derivation, without going via the integral equations (4.8) and (4.9), and hence also without the introduction of negative probabilities. This explicit generalization of Nordheim's proof has been performed by the author, but it is too complicated to warrant repeating here.

We notice that the recursion relations (4.16) and (4.17) determine all the moments q_{2n} and p_{2n} uniquely, starting only from the longitudinal distribution function π of particles of type 1. This is a way of seeing that the integral equations (4.8) and (4.9) determine the distribution functions uniquely. We conclude that *the*

[6] H. S. Snyder and W. T. Scott, Phys. Rev. **76**, 220 (1949).
[7] M. H. Kalos and J. M. Blatt, Australian J. Phys. (to be published).
[8] Osborne, Nordheim, and Blatt, Proceedings of the Echo Lake Conference on Cosmic Rays, 1949 (unpublished).

angular and lateral distribution functions of type 1 *particles in a cascade satisfying our fundamental conditions of Sec. 1 are determined uniquely by the average longitudinal numbers* $\pi(E_0,E,t)$ *of type 1 particles and by the scattering probability* $\sigma(E,\alpha)$, *independently of the detailed nature of the cascade process.* This is a rather surprising conclusion, since it is quite possible to obtain the same $\pi(E_0,E,t)$ from two fundamentally different cascades. Conversely, this theorem shows that one must be careful in interpreting experimental information on lateral and angular distributions unless the basic nature of the cascade process is well understood from other sources. This may be important to keep in mind when analyzing angular and lateral distribution data on the nucleon cascade inside a heavy nucleus or in the atmosphere.

5. CONCLUSION

Having derived these equations, we now consider possible applications. These fall into two classes: applications of the equations themselves, and applications of the tricks used in the derivation. As regards the equations themselves, we believe that a straightforward attack on the lateral distribution function, using Eq. (4.9) and standard analytic methods, is likely to fail. Equation (4.9) is a rather nasty equation, even in the Landau approximation (4.12), and our attempts to obtain explicit analytic solutions to it have not succeeded so far. However, we feel that a purely numerical attack on Eq. (4.9), using the Monte Carlo method and a fast digital computer, may perhaps be worth while.

The more interesting applications are in the direction of applying the tricks used in the derivation here to other stochastic processes. The introduction of negative probabilities for events in which nothing happens has enabled us to average over one set of stochastic variables while retaining probability statements about another set of stochastic variables, in spite of the fact that the two sets of variables (e.g., energy and angle) are genetically related to each other. There are surely many other fields of physics in which similar "partial probabilities" are desired, but have so far been found only indirectly by first solving the complete stochastic problem and *afterwards* averaging over the unwanted stochastic variables.

APPENDIX. MULTIPLE SCATTERING WITHOUT ENERGY LOSS

We wish to show that Eqs. (4.8) and (4.9) with the substitution (4.10) are correct for multiple scattering without energy loss. Putting Eq. (4.10) into Eq. (4.8), cancelling the common factor $\delta(E_0-E)$, and setting $t-t'=t''$ gives

$$q(t,\theta)=\int_0^t dt''\int_{-\infty}^{\infty} d\alpha[\sigma(\alpha)-\mu\delta(\alpha)] \times q(t'',\theta-\alpha)+\delta(\theta). \quad (A.1)$$

When we differentiate both sides with respect to t, we get directly the usual equation for multiple scattering [for example, Scott and Snyder, reference 6, Eq. (5); their Eq. (5) includes the lateral distribution, and must be integrated over their variable x to obtain the equivalent of Eq. (A.1); also the notation differs somewhat: their η is our θ, their z is proportional to our t, and their $p(\eta)$ is proportional to our $\sigma(\alpha)$].

The equivalence is harder to show for the lateral equation (4.9), since Scott and Snyder do not have such an equation at all. Inserting Eq. (4.10) into Eq. (4.9) gives, again setting $t''=t-t'$,

$$p(t,x)=\int_0^t dt''\int_{-\infty}^{\infty} d\alpha[\sigma(\alpha)-\mu\delta(\alpha)] \times p(t'',x-\alpha t'')+\delta(x). \quad (A.2)$$

We introduce a Fourier transform on x,

$$u(t,k)=\int_{-\infty}^{\infty} dx\,\exp(ikx)p(t,x), \quad (A\ 3)$$

and the notation

$$S(k)=\int_{-\infty}^{\infty} d\alpha\,\exp(ik\alpha)\sigma(\alpha) \quad (A.4)$$

to get

$$u(t,k)=\int_0^t dt''\,[S(kt'')-\mu]\,u(t'',k)+1. \quad (A.5)$$

We differentiate with respect to t,

$$\partial u(t,k)/\partial t=[S(kt)-\mu]u(t,k), \quad (A.6)$$

and observe that Eq. (A.5) implies the initial condition $u(0,k)=1$. The solution can be written best in terms of the function

$$h(t)=\int_0^t [\mu-S(t')]dt', \quad (A.7)$$

and is

$$u(t,k)=\exp\left[\frac{h(0)-h(kt)}{k}\right]. \quad (A.8)$$

This is exactly the solution of Scott and Snyder, except for notation.