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Scattering of Light of Very Low Frequency by Systems of Spin $\frac{1}{2}$*

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It is shown that the first two terms in the expansion of the scattering amplitude of light by a system of spin $\frac{1}{2}$ in powers of the frequency can be simply expressed in terms of the macroscopic properties of the system. The first term is the well known Thomson amplitude, and depends only on the total charge and mass. The second term is found to depend only on the charge, mass, and magnetic moment of the system.

I. INTRODUCTION

It is well known that the scattering of zero-energy photons by a charged system is given by the Thomson formula and hence is independent of the structure of the system. In this paper it will be shown that not only the zero-frequency limit of the scattering amplitude but also its first derivative with respect to the photon frequency is determined by the static properties of the system (charge, mass, and magnetic moment), at least for a system of spin $\frac{1}{2}$. Unlike the Thomson limit, which appears as a consequence of gauge invariance in the usual theoretical formulation, the new limit appears to be a consequence of gauge and relativistic invariance.

The derivativation to be given here applies to systems consisting of spin-$\frac{1}{2}$ fermions locally coupled to spin-zero bosons, but could presumably be generalized, if necessary, to almost any gauge and Lorentz invariant system. The proof applies formally to all orders in $e^2/\hbar c$ (as well as to all orders in the meson-nucleon coupling) provided the virtual photons are given a fictitious rest mass λ; it may or may not be correct in the limit $\lambda\rightarrow 0$, depending on whether or not the derivative in question exists. Failure of the derivative to exist would, of course, appear as an infrared divergence. In our derivation such an infrared divergence would be due to the existence of too great a density of excited states of the scattering system whose energy was infinitesimally close to that of the ground state. We shall have to assume the scatterer (proton, neutron, electron, etc.) to have a minimum excitation energy with respect to which the photon frequency can be measured. For protons and neutrons this energy is of course $\mu_\pi c^2$, where μ_π is the π-meson rest mass; the corresponding length with respect to which the photon wavelength must be large is $\hbar/\mu_\pi c$.

Our method consists in showing that if the scattering amplitude be expanded in powers of k (for $\hbar ck\ll\mu_\pi c^2$) the first two terms in the expansion can essentially be expressed in terms of matrix elements of the charge density, $\rho(x)$, rather than in terms of the current density, $\mathbf{j}(x)$. Since $\int\rho(x)d\mathbf{x}=e$, the total charge, whereas $\int\mathbf{j}(x)d\mathbf{x}$ has large off-diagonal components, considerable simplification is achieved: no excited states need be included in the sum over states which determines the scattering amplitude to this order in k. The diagonal elements of ρ, as is well known, can be simply expressed in terms of e, m and μ (the static magnetic moment of the system) for small momentum transfer. The first term in the expansion, of course, turns out to be the Thomson term; the second, which is linear in the spin vector of the system, is somewhat more complicated, except in the forward direction, where it must be a multiple of $i\boldsymbol{\sigma}\cdot(\mathbf{e}'\times\mathbf{e})k$; the multiple turns out to be $2\lambda^2$,[1] where λ is the anomalous part of the magnetic moment.

The total scattering amplitude, correct to order k, is

$$\begin{aligned}H'=(e^2/m)\mathbf{e}\cdot\mathbf{e}'-(ie/m)(2\mu-e/m)k\boldsymbol{\sigma}\cdot(\mathbf{e}'\times\mathbf{e})\\ -2\mu^2i\boldsymbol{\sigma}\cdot[(\mathbf{e}\times\mathbf{k})\times(\mathbf{e}'\times\mathbf{k})]/k-i(e/m)\\ \times\mu[(\mathbf{e}\cdot\mathbf{k}')\mathbf{e}'\cdot(\boldsymbol{\sigma}\times\mathbf{k}')/k-(\mathbf{e}'\cdot\mathbf{k})\mathbf{e}\cdot(\boldsymbol{\sigma}\times\mathbf{k})/k].^2\end{aligned}\quad(1.1)$$

Here $\mathbf{e}$ and $\mathbf{e}'$ are the initial and final polarization vectors of the scattered photon.

II. CALCULATION: GAUGE INVARIANCE

We shall work within the framework of a specific theory: we calculate the scattering of light by a spin-$\frac{1}{2}$ system locally coupled to a scalar (or pseudoscalar) meson field. The method of calculation will make it clear that the result has a much wider range of validity.

The S matrix for scattering of a light quantum from a state $(\mathbf{k},\omega,\mathbf{e})$ to a state $(\mathbf{k}',\omega',\mathbf{e}')$, where $|\mathbf{k}|=\omega$, $|\mathbf{k}'|=\omega'$, $\mathbf{e}\cdot\mathbf{k}=\mathbf{e}'\cdot\mathbf{k}'=0$, is given by

$$\begin{aligned}S=-[2i/(4\omega\omega')^{\frac{1}{2}}]\int\epsilon^2\phi^*(x)\phi(x)e^{i(k-k')x}dx(\mathbf{e}\cdot\mathbf{e}')\\ -[1/(4\omega\omega')^{\frac{1}{2}}]\int P[\mathbf{j}(x)\cdot\mathbf{e},\mathbf{j}(y)\cdot\mathbf{e}']\\ \times e^{ik'y}e^{-ikx}dxdy,\end{aligned}\quad(2.1)$$

where all integrations and coordinates are four dimensional, $\phi(x)$ is the charged meson field operator, P is

[1] We use units in which $\hbar=c=1$, $\epsilon^2/4\pi=1/137$.

[2] The author has been informed that a result essentially equivalent to Eq. (1.1) has been independently obtained by M. Gell-Mann and M. Goldberger using a different method of proof. See M. Gell-Mann and M. Goldberger, Proceedings of the Glasgow Conference on Nuclear Physics, 1954 (unpublished); and Phys. Rev., following paper [Phys. Rev. **96**, 1433 (1954)]. The author would like to thank Dr. Gell-Mann for communicating this result to him.

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Dyson's time-ordering operator, and $\mathbf{j}(x)$ is the current-density operator. In our example

$$j_\mu(x)=\begin{cases}\rho(x)=\epsilon[\phi_1(x)\pi_2(x)-\phi_2(x)\pi_1(x)] \\ \qquad +\epsilon\bar{\psi}(x)\gamma_4\tfrac{1}{2}(1+\tau_3)\psi(x), \\ \mathbf{j}(x)=\epsilon[\nabla\phi_1(x)\phi_2(x)-\nabla\phi_2(x)\phi_1(x)] \\ \qquad +i\epsilon\bar{\psi}(x)\gamma\tfrac{1}{2}(1+\tau_3)\psi(x),\end{cases} \tag{2.2}$$

and

$$\phi=(\phi_1+i\phi_2)/\sqrt{2},\quad \phi^*=(\phi_1-i\phi_2)/\sqrt{2}. \tag{2.3}$$

ϵ is the electronic charge, $\epsilon^2/4\pi=1/137$. The time dependence of all operators is given by the Heisenberg equations of motion for the scattering system. We may rewrite Eq. (2.1) as follows:

$$S=-\epsilon_i'\epsilon_j g_{ij}(4\omega\omega')^{-\frac{1}{2}}, \tag{2.4}$$

where

$$g_{ij}=2i\epsilon^2\int\phi^*(x)\phi(x)e^{i(k-k')x}dx\delta_{ij} + \int P[j_i(x),j_j(y)]e^{iky}e^{-ik'x}dxdy. \tag{2.5}$$

For purposes of orientation we give g_{ij} for Thomson scattering:

$$g_{ij}(\tau)=\delta_{ij}(2\pi)^4 i\delta(\Delta k_\mu+\Delta p_\mu)e^2/m, \tag{2.6}$$

where e and m are the total charge and mass of the scatterer. Δp_μ is the 4-momentum transfer to the scatterer.

The theorem which makes our calculation possible is the following:

$$k_i' g_{ij} k_j=\omega'\omega\int P[\rho(x),\rho(y)]e^{iky}e^{-ik'x}dxdy. \tag{2.7}$$

Equation (2.7) is obviously a consequence of the gauge invariance of the theory. It is valid for any gauge-invariant theory in which the system interacts with a scalar potential $V(x)$ through an interaction Hamiltonian $H_v=\int\rho(x)V(x)d\mathbf{x}$. This can be shown quite simply and generally by considering the scattering of the system by an arbitrary external electromagnetic field and requiring the S matrix to be invariant under a gauge transformation of the external field.

To prove (2.7) for our special example, we operate on (2.5) with k_i'. Thus:

$$k_i' g_{ij}=2ik_j'\epsilon^2\int\phi^*(x)\phi(x)e^{i(k-k')x}dx + \frac{1}{i}\int P[\partial j_i(x)/\partial x_i,j_j(y)]e^{iky}e^{-ik'x}dxdy. \tag{2.8}$$

The second term on the right may be transformed by using the equation of continuity,

$$\partial j_i(x)/\partial x_i=-\partial\rho(x)/\partial x_0, \tag{2.9}$$

and the relation

$$P\left[\frac{\partial\rho(x)}{\partial x_0},j_j(y)\right]=\frac{\partial}{\partial x_0}P[\rho(x),j_j(y)] -\delta(x_0-y_0)[\rho(x),j_j(y)]. \tag{2.10}$$

We find, after an integration by parts,

$$k_i' g_{ij}=2i\epsilon^2 k_j'\int\phi^*(x)\phi(x)e^{i(k-k')x}dx + \frac{1}{i}\int dxdye^{-ik'x}e^{iky}\delta(x_0-y_0)[\rho(x),j_j(y)] + \omega'\int dxdye^{-ik'x}e^{iky}P[\rho(x),j_j(y)]. \tag{2.11}$$

The first two terms on the right of Eq. (2.11) cancel exactly as a consequence of the definitions (2.2) and (2.3) and the canonical commutation relations. We now multiply the remaining terms in (2.11) by k_j and find easily that

$$\begin{aligned} k_i' g_{ij} k_j &= \omega'\int dxdye^{-ik'x}e^{iky}P\left[\rho(x),-\frac{1}{i}\frac{\partial j_j(y)}{\partial y_j}\right] \\ &= \frac{\omega'}{i}\int dxdye^{-ik'x}e^{iky}P\left[\rho(x),\frac{\partial\rho(y)}{\partial y_0}\right] \\ &= -\frac{\omega'}{i}\int dxdye^{-ik'x}\left[\frac{\partial e^{iky}}{\partial y_0}\right]P[\rho(x),\rho(y)] \\ &= \omega'\omega\int dxdye^{-ik'x}e^{iky}P[\rho(x),\rho(y)], \end{aligned} \tag{2.12}$$

where we have made use of the fact that $\rho(x)$ and $\rho(y)$ commute for $x_0=y_0$.

It follows from (2.12) that if we can obtain enough information about the tensorial character of g_{ij} from other sources, we can use (2.12) to calculate g_{ij} itself. Let us look at the defining equation for g_{ij}, Eq. (2.5).

The first term in (2.5) is a multiple of δ_{ij}, and, since $\phi^*\phi$ is a scalar, must be independent of the spin to order k^2. The second term can be rewritten in terms of a sum over intermediate states with appropriate energy denominators. The contribution $g_{ij}^{(0)}$ to the sum arising from unexcited intermediate states can be explicitly calculated to the required order in k. We shall perform this calculation in Sec. IV. It turns out that $k_i' g_{ij}^{(0)} k_j=0$ so that $g_{ij}^{(0)}$ will give no contribution to (2.12).

In the sum over excited states, $g_{ij}^{(e)}$, there can be no singularities as $\{k,\omega\}\to 0$, because of the minimum excitation energy μ_π. The only tensors that can enter $g_{ij}^{(e)}$ to order k are thus δ_{ij} and σ_{ij}, where $\sigma_{12}=\sigma_3$, etc.

We may therefore write

$$g_{ij}=g_{ij}^{(0)}+A\delta_{ij}+B\sigma_{ij}. \tag{2.13}$$

Substituting (2.13) into (2.12), we find

$$\mathbf{k}'\cdot\mathbf{k}A+\boldsymbol{\sigma}\cdot(\mathbf{k}'\times\mathbf{k})B=\omega'\omega C,$$

where

$$C=\int dxdye^{-ik'x}e^{iky}P[\rho(x),\rho(y)]. \quad (2.14)$$

We may now think of (2.14) expanded into a sum over states. Those terms arising from excited states will contribute at most of order k^2, since $\int\rho(x)d\mathbf{x}$ is a c-number, the total charge. Thus to our order we may calculate (2.14) using only *unexcited* intermediate states.

We proceed to rewrite Eq. (2.14) as a sum over states (suppressing all excited states):

$$\begin{aligned}\langle\mathbf{p}_f|P[\rho(x),\rho(y)]|\mathbf{p}_i\rangle &= e^{iE_fx_0}e^{-iE_iy_0}\langle\mathbf{p}_f|\rho(\mathbf{x})e^{-iH(x_0-y_0)}\rho(\mathbf{y})|\mathbf{p}_i\rangle \quad (x_0>y_0)\\ &= e^{iE_fy_0}e^{-iE_ix_0}\langle\mathbf{p}_f|\rho(\mathbf{y})e^{-iH(y_0-x_0)}\rho(\mathbf{x})|\mathbf{p}_i\rangle \quad (x_0<y_0),\end{aligned}$$

so that

$$\begin{aligned}C&=\int_{-\infty}^{\infty}dx_0e^{i(\omega'+E_f)x_0}\left\langle\mathbf{p}_f\left|\int\rho(\mathbf{x})e^{-i\mathbf{k}'\cdot\mathbf{x}}d\mathbf{x}e^{-iHx_0}\right.\right.\\
&=\int_{-\infty}^{x_0}dy_0e^{i(H-E_i-\omega)y_0}\int\rho(\mathbf{y})e^{i\mathbf{k}\cdot\mathbf{y}}d\mathbf{y}\left|\mathbf{p}\right\rangle\\
&=\int_{-\infty}^{\infty}dy_0e^{i(E_f-\omega)y_0}\left\langle\mathbf{p}_f\left|\int\rho(\mathbf{y})e^{i\mathbf{k}\cdot\mathbf{y}}d\mathbf{y}e^{-iHy_0}\right.\right.\\
&\times\int_{-\infty}^{y_0}dx_0e^{i(H-E_i+\omega')x_0}\int\rho(\mathbf{x})e^{-i\mathbf{k}'\cdot\mathbf{x}}d\mathbf{x}\left|\mathbf{p}_i\right\rangle\\
&=\frac{2\pi}{i}\delta(\omega'+E_f-\omega-E_i)\left[\left\langle\mathbf{p}_f\left|\int\rho(\mathbf{x})e^{-i\mathbf{k}'\cdot\mathbf{x}}d\mathbf{x}\right.\right.\right.\\
&\times\frac{1}{H-E_i-\omega}\int\rho(\mathbf{y})e^{i\mathbf{k}\cdot\mathbf{y}}d\mathbf{y}\left|\mathbf{p}_i\right\rangle+\left\langle\mathbf{p}_f\left|\int\rho(\mathbf{y})\right.\right.\\
&\times e^{i\mathbf{k}\cdot\mathbf{y}}d\mathbf{y}\frac{1}{H-E_i+\omega'}\int\rho(\mathbf{x})e^{-i\mathbf{k}'\cdot\mathbf{x}}d\mathbf{x}\left|\mathbf{p}_i\right\rangle\Bigg]\\
&=\frac{(2\pi)^4}{i}\delta(\Delta p_\mu+\Delta k_\mu)\sum_{\boldsymbol{\sigma}}\left[\frac{\langle\mathbf{k}-\mathbf{k}'|\rho|\mathbf{k},\boldsymbol{\sigma}\rangle\langle\mathbf{k},\boldsymbol{\sigma}|\rho|0\rangle}{E(k)-E(0)-\omega}\right.\\
&\qquad\left.+\frac{\langle\mathbf{k}-\mathbf{k}'|\rho|-\mathbf{k}',\boldsymbol{\sigma}\rangle\langle-\mathbf{k}',\boldsymbol{\sigma}|\rho|0\rangle}{E(k')-E(0)+\omega'}\right], \quad (2.15)\end{aligned}$$

where we have made use of momentum conservation and set $\mathbf{p}_i=0$ in arriving at (2.15).

We may now find immediately the Thomson limit from (2.15). Since $\int\rho(x)d\mathbf{x}=e$, $\langle\mathbf{k}|\rho|0\rangle\approx\langle0|\rho|0\rangle=e+\sim k^2$, so that as $k\rightarrow0$ we have for the bracketed quantity

$$\begin{aligned}&e^2\left[\frac{1}{E(k)-E(0)-\omega}+\frac{1}{E(k')-E(0)+\omega'}\right]\\
&=e^2[E(k')-E(0)+\omega'+E(k)\cdot E(0)-\omega]\\
&\quad\times[E(k)-E(0)-\omega]^{-1}[E(k')-E(0)+\omega]^{-1}\\
&\approx-\frac{e^2}{\omega\omega'}[E(k')-E(0)+E(k)-E(0)+E(0)-E(k'-k)]\\
&\cong-\frac{e^2}{\omega\omega'}\left[\frac{k'^2}{2m}+\frac{k^2}{2m}-\frac{(\mathbf{k}-\mathbf{k}')^2}{2m}\right]=-\frac{e^2}{m}\frac{\mathbf{k}\cdot\mathbf{k}'}{\omega\omega'}, \quad (2.16)\end{aligned}$$

whence the constant A in (2.14) is given by $A=-(2\pi)^4 i\delta(k_\mu'-k_\mu+\Delta p_\mu)e^2/m$ which leads trivially to the Thomson formula (2.6).

III. RELATIVISTIC INVARIANCE

We return now to Eq. (2.15), and investigate the properties of $\langle\mathbf{p}_2|\rho|\mathbf{p}_1\rangle$.

The four vector $\langle\mathbf{p}_2|j_\mu|\mathbf{p}_1\rangle$ must have the general form, for a spin-$\frac{1}{2}$ particle,

$$\begin{aligned}\langle\mathbf{p}_2|j_\mu|\mathbf{p}_1\rangle=i\bar{u}(\mathbf{p}_2)[e\gamma_\mu f((\Delta p_\lambda)^2)\\ -\sigma_{\mu\nu}\Delta p_\nu g((\Delta p_\lambda)^2)]u(\mathbf{p}_1), \quad (3.1)\end{aligned}$$

where u and $\bar{u}$ are the Dirac spinors for the momentum states $\mathbf{p}_2$ and $\mathbf{p}_1$ and the γ's and σ's are the well-known Dirac matrices. Here f and g are functions of $(\Delta p_\lambda)^2$ only, and $f(0)=1$, $g(0)=\lambda$, the anomalous magnetic moment of the particle.

Using (3.1), we find

$$\begin{aligned}\langle\mathbf{p}_2|\rho|\mathbf{p}_1\rangle&=(1/i)\langle\mathbf{p}_2|j_4|\mathbf{p}_1\rangle\\ &=\bar{u}(\mathbf{p}_2)[\beta ef-\sigma_{4i}\Delta p_i g]u(\mathbf{p}_1)\\ &=efu^*(\mathbf{p}_2)u(\mathbf{p}_1)+gu^*(\mathbf{p}_2)\beta\boldsymbol{\alpha}\cdot\Delta\mathbf{p}u(\mathbf{p}_1). \quad (3.2)\end{aligned}$$

Let us call the bracketed quantity in Eq. (2.15) Q:

$$Q=\sum_{\boldsymbol{\sigma}}\left[\frac{\langle\mathbf{k}-\mathbf{k}'|\rho|\mathbf{k},\boldsymbol{\sigma}\rangle\langle\mathbf{k},\boldsymbol{\sigma}|\rho|0\rangle}{E(k)-E(0)-\omega}+\frac{\langle\mathbf{k}-\mathbf{k}'|\rho|-\mathbf{k}',\boldsymbol{\sigma}\rangle\langle-\mathbf{k}',\boldsymbol{\sigma}|\rho|0\rangle}{E(k')-E(0)+\omega'}\right] \quad (3.3)$$

We wish to evaluate Q only to order k. Thus, the numerator in (3.3) need be accurate only to order k^2. Since $(\Delta E)^2\approx k^4$, we may set $(\Delta p_\lambda)^2=(\Delta\mathbf{p})^2$, and since the $(\Delta\mathbf{p})^2$ are the same in both terms of (3.3) (although in opposite order), the f and g functions will be common factors to both terms. Their $\Delta\mathbf{p}$ dependence may therefore be neglected in our limit, which considerably simplifies the remaining calculation. We thus set $f=1$,

$g=g(0)=\lambda$, the anomalous part of the magnetic moment. Equation (3.3) for Q becomes, if we insert projection operators for the spin sums over intermediate states:

$$Q=u^*(\mathbf{k}-\mathbf{k}')\left[\frac{[e+\lambda\beta\boldsymbol{\alpha}\cdot(-\mathbf{k}')]}{[E(k)-E(0)-\omega]}\frac{[\boldsymbol{\alpha}\cdot\mathbf{k}+\beta m+E(k)]}{2E(k)}\times(e+\lambda\beta\boldsymbol{\alpha}\cdot\mathbf{k})+\frac{(e+\lambda\beta\boldsymbol{\alpha}\cdot\mathbf{k})}{[E(k')-E(0)+\omega']}\times\frac{[-\boldsymbol{\alpha}\cdot\mathbf{k}'+\beta m+E(k')]}{2E(k')}[e+\lambda\beta\boldsymbol{\alpha}\cdot(-\mathbf{k}')]\right]u(0). \quad (3.4)$$

All the terms of the form $\beta\boldsymbol{\alpha}\cdot\mathbf{k}$ or $\boldsymbol{\alpha}\cdot\mathbf{k}$ in (3.4) will be of order k^2/ω, so that they may be evaluated to their lowest nonvanishing order. The only term we have to watch is the one with every $\boldsymbol{\alpha}\cdot\mathbf{k}$ left out. Since $\beta u(0)=u(0)$, and since $k'^2-k^2\approx k^3/m$, this term is

$$(u^*(\mathbf{k}-\mathbf{k}'),u(0))e^2\frac{[m+E(k)]}{2E(k)}\left[\frac{1}{E(k)-E(0)+\omega}+\frac{1}{E(k')-E(0)+\omega'}\right]=-\frac{e^2}{m}\frac{\mathbf{k}\cdot\mathbf{k}'}{\omega\omega'[1+\sim k^2]}\cong-\frac{e^2}{m}\frac{\mathbf{k}\cdot\mathbf{k}'}{\omega'\omega}.$$

In evaluating the remaining terms in (3.4) we may set $E(k')=E(k)=m$, $\omega'=\omega$ everywhere. Since $u(p)\cong(1+\boldsymbol{\alpha}\cdot\mathbf{p}/2m)u(0)$, we have

$$\begin{aligned}Q&=-\frac{e^2}{m}\left(\frac{\mathbf{k}'\cdot\mathbf{k}}{\omega'\omega}\right)+\frac{1}{\omega}u_f^*(0)\left(1+\frac{\boldsymbol{\alpha}\cdot(\mathbf{k}-\mathbf{k}')}{2m}\right)\\&\quad\times\left[(e+\beta\boldsymbol{\alpha}\cdot\mathbf{k})\left(\frac{-\boldsymbol{\alpha}\cdot\mathbf{k}'+m+\beta m}{2m}\right)(e-\lambda\beta\boldsymbol{\alpha}\cdot\mathbf{k}')\right.\\&\quad\left.-(e-\lambda\beta\boldsymbol{\alpha}\cdot\mathbf{k}')\left(\frac{\boldsymbol{\alpha}\cdot\mathbf{k}+m+\beta m}{2m}\right)(e+\lambda\beta\boldsymbol{\alpha}\cdot\mathbf{k})\right]u_i(0)\\&=-\frac{e^2}{m}\left(\frac{\mathbf{k}\cdot\mathbf{k}'}{\omega'\omega}\right)+i\left(\frac{\boldsymbol{\sigma}\cdot(\mathbf{k}'\times\mathbf{k})}{\omega}\right)\left(\frac{e^2}{2m^2}+\frac{2\lambda e}{m}\right)\\&=-\frac{e^2}{m}\left(\frac{\mathbf{k}\cdot\mathbf{k}'}{\omega'\omega}\right)+i\left(\frac{\boldsymbol{\sigma}\cdot(\mathbf{k}'\times\mathbf{k})}{\omega}\right)\frac{e}{m}\left(2\mu-\frac{e}{2m}\right).\end{aligned}\quad(3.5)$$

Here μ is the total magnetic moment. Equation (2.45) now becomes

$$C=\frac{(2\pi)^4}{i}\delta(\Delta p_\mu+\Delta k_\mu)\times\left[-\frac{e^2}{m}\frac{\mathbf{k}\cdot\mathbf{k}'}{\omega\omega'}+i\boldsymbol{\sigma}\cdot(\mathbf{k}'\times\mathbf{k})\frac{e}{m\omega}\left(2\mu-\frac{e}{2m}\right)\right],\quad(3.6)$$

and (2.14) informs us that

$$A=-\frac{(2\pi)^4}{i}\delta(\Delta p_\mu+\Delta k_\mu)\frac{e^2}{m},\quad(3.7)$$

and

$$B=\frac{(2\pi)^4}{i}\delta(\Delta p_\mu+\Delta k_\mu)i\frac{\omega e}{m}\left(2\mu-\frac{e}{2m}\right).\quad(3.8)$$

The off-diagonal contribution to the scattering is thus given, correct to order (k,ω), by

$$g_{ij}-g_{ij}^{(0)}=\frac{(2\pi)^4}{i}\delta(\Delta p_\mu+\Delta k_\mu)\times\left[-\frac{e^2}{m}\delta_{ij}+i\omega\frac{e}{m}\left(2\mu-\frac{e}{2m}\right)\sigma_{ij}\right].\quad(3.9)$$

Note that this term is exactly zero for a neutron.

In the next section we shall calculate $g_{ij}^{(0)}$.

IV. DIAGONAL MAGNETIC SCATTERING

In complete analogy with the derivativation of Eq. (2.15) for the quantity C defined by Eq. (2.14), we may derive an expression for $g_{ij}^{(0)}$ directly from the defining Eq. (2.5). We find

$$g_{ij}^{(0)}=\frac{(2\pi)^4}{i}\delta(\Delta p_\mu+\Delta k_\mu)\sum_{\boldsymbol{\sigma}}\left[\frac{\langle\mathbf{k}-\mathbf{k}'|j_i|\mathbf{k}\rangle\langle\mathbf{k}|j_j|0\rangle}{E(k)-E(0)-\omega}+\frac{\langle\mathbf{k}-\mathbf{k}'|j_j|-\mathbf{k}'\rangle\langle-k'|j_i|0\rangle}{E(k')-E(0)+\omega'}\right],\quad(4.1)$$

with

$$\langle\mathbf{p}_2|j_i|\mathbf{p}_1\rangle=i\bar{u}(\mathbf{p}_2)[e\gamma_i f((\Delta p_\lambda)^2)-\sigma_{i\nu}\Delta p_\nu g((\Delta p_\lambda)^2)]u(\mathbf{p}_1).\quad(4.2)$$

Since γ_i and $\sigma_{ij}\Delta p_j$ vanish with k, we may disregard $\sigma_{i4}\Delta E$ and set $f((\Delta p_\lambda)^2)=1$, $g((\Delta p_\lambda)^2)=\lambda$. With these approximations

$$\begin{aligned}\langle\mathbf{p}_2|\mathbf{j}|\mathbf{p}_1\rangle&=u^*(\mathbf{p}_2)[e\boldsymbol{\alpha}-i\lambda\Delta\mathbf{p}\times\boldsymbol{\sigma}]u(\mathbf{p}_1)\\&\cong(e/2m)(\mathbf{p}_2+\mathbf{p}_1)+i(e/2m+\lambda)\boldsymbol{\sigma}\times\Delta\mathbf{p}\\&=(e/2m)(\mathbf{p}_2+\mathbf{p}_1)+i\mu\boldsymbol{\sigma}\times\Delta\mathbf{p},\end{aligned}\quad(4.3)$$

which is, of course, the current of a nonrelativistic particle interacting with a magnetic field according to the Hamiltonian

$$H_{\mathrm{N.R.}}'=-(e/2m)(\mathbf{p}\cdot\mathbf{A}+\mathbf{A}\cdot\mathbf{p})-\mu\boldsymbol{\sigma}\cdot\mathbf{H}.\quad(4.4)$$

We may now verify the statement made in Sec. II that $k_i'g_{ij}^{(0)}k_j=0$. This is obvious for all those terms involving the spin, since $\Delta\mathbf{p}$ is in each case the $\mathbf{k}$ with which an inner product is being taken. For the spin-independent terms, we have

$$g_{ij}^{(0)}|_{\text{no spin}}=\frac{(2\pi)^4}{i}\delta(\Delta p_\mu+\Delta k_\mu)\left(\frac{e}{2m}\right)^2\frac{2}{\omega}\times(k_i'k_j'-k_ik_j),\quad(4.5)$$

which is manifestly orthogonal to $k_i'k_j$. Furthermore, since $\mathbf{e}\cdot\mathbf{k}=\mathbf{e}'\cdot\mathbf{k}'=0$, the spin-independent terms make no contribution to the scattering matrix.

The contribution to the scattering matrix of $g_{ij}^{(0)}$ is thus

$$S^{(0)}=-\frac{1}{(4\omega'\omega)^{\frac{1}{2}}}\frac{(2\pi)^4}{i}\delta(\Delta p_\mu+\Delta k_\mu)$$

$$=\left[\left\{\frac{e}{m}\mathbf{e}'\cdot\mathbf{k}+i\mathbf{e}'\cdot[\boldsymbol{\sigma}\times(-\mathbf{k}')]\mu\right\}\frac{i\mathbf{e}\cdot(\boldsymbol{\sigma}\times\mathbf{k})\mu}{-\omega}+\left\{-\frac{e}{m}\mathbf{e}\cdot\mathbf{k}'+i\mathbf{e}\cdot(\boldsymbol{\sigma}\times\mathbf{k})\mu\right\}\frac{i\mathbf{e}'\cdot[\boldsymbol{\sigma}\times(-\mathbf{k}')\mu]}{\omega}\right]$$

$$=-\frac{1}{(4\omega'\omega)^{\frac{1}{2}}}\frac{(2\pi)^4}{i}\delta(\Delta p_\mu+\Delta k_\mu)\times\left\{\frac{2i\mu^2}{\omega}\boldsymbol{\sigma}\cdot[(\mathbf{e}\times\mathbf{k})\times(\mathbf{e}'\times\mathbf{k}')]+\frac{ie\mu}{m\omega}\times[(\mathbf{e}\cdot\mathbf{k}')\mathbf{e}'\cdot(\boldsymbol{\sigma}\times\mathbf{k})-(\mathbf{e}'\cdot\mathbf{k})\mathbf{e}\cdot(\boldsymbol{\sigma}\times\mathbf{k})]\right\}.\quad(4.6)$$

The contribution to the scattering matrix of $g_{ij}-g_{ij}^{(0)}$ is

$$S-S^{(0)}=-\frac{e_i'e_j}{(4\omega'\omega)^{\frac{1}{2}}}\frac{(2\pi)^4}{i}\delta(\Delta p_\mu+\Delta k_\mu)\times\left[-\frac{e^2}{m}\delta_{ij}+i\sigma_{ij}\omega\frac{e}{m}\left(2\mu-\frac{e}{2m}\right)\right],$$

so that in all

$$S=\frac{(2\pi)^4}{i}\frac{1}{(4\omega'\omega)^{\frac{1}{2}}}\delta(\Delta p_\mu+\Delta k_\mu)\times\left[\frac{e^2}{m}\mathbf{e}\cdot\mathbf{e}'-i\frac{e}{m}\omega\boldsymbol{\sigma}\cdot(\mathbf{e}'\times\mathbf{e})\left(2\mu-\frac{e}{2m}\right)-\frac{2i\mu^2}{\omega}\boldsymbol{\sigma}\cdot(\mathbf{e}\times\mathbf{k})\times(\mathbf{e}'\times\mathbf{k}')-\frac{e\mu i}{m\omega}\times[(\mathbf{e}\cdot\mathbf{k}')\mathbf{e}\cdot(\boldsymbol{\sigma}\times\mathbf{k}')-\mathbf{e}'\cdot\mathbf{k}\mathbf{e}\cdot(\boldsymbol{\sigma}\times\mathbf{k})]+\sim k^2\right].\quad(4.7)$$

The bracketed term is what we have called the scattering amplitude, H', in the introduction.

In the forward direction $\mathbf{k}=\mathbf{k}'$, and (4.7) simplifies considerably. We find for this case:

$$H'=(\mathbf{e}\cdot\mathbf{e}')\frac{e^2}{m}-\frac{ie}{m}\left(2\mu-\frac{e}{2m}\right)\omega\boldsymbol{\sigma}\cdot(\mathbf{e}'\times\mathbf{e})+2\mu^2 i(\mathbf{e}\times\mathbf{k})\cdot\mathbf{e}'(\mathbf{k}\cdot\boldsymbol{\sigma})$$

$$=\frac{e^2}{m}\mathbf{e}\cdot\mathbf{e}'+i\omega\boldsymbol{\sigma}\cdot(\mathbf{e}'\times\mathbf{e})\left[2\mu^2-\frac{e}{m}\left(2\mu-\frac{e}{2m}\right)\right]$$

$$=(e^2/m)\mathbf{e}\cdot\mathbf{e}'+i\omega\boldsymbol{\sigma}\cdot(\mathbf{e}'\times\mathbf{e})2(\mu-e/2m)^2$$

$$=(e^2/m)\mathbf{e}\cdot\mathbf{e}'+i\omega\lambda^2\boldsymbol{\sigma}\cdot(\mathbf{e}'\times\mathbf{e}),\quad(4.8)$$

where λ is the anomalous part of the magnetic moment.

In conclusion, we may make a few general remarks. Our results depend primarily on gauge and relativistic invariance, so that they should not be difficult to generalize to cases of higher spin. They are valid for atoms, nuclei, and elementary particles. Unfortunately, experimental verification seems almost out of the question since in every case the coefficient of k (which we have calculated) is anomalously small compared to the coefficient of k^2 (Rayleigh scattering) which is known to be structure-dependent in a nontrivial way.