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Vacuum Polarization in Mesonic Atoms

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The effects of vacuum polarization on the energy levels of π and μ mesonic atoms are computed to the lowest order for states $1s(Z\leqslant 12)$, $2p(Z\leqslant 30)$, $3d(Z\leqslant 82)$, $4f(Z\leqslant 82)$. If one ignores the finite size of the nucleus, these results may be written explicitly in a closed form. The effect of the finite size of the nucleus is estimated and shown to be <8 percent for all states considered. Since the virtual pairs are described here by plane waves, the fractional error in these results is of order $(Z\alpha)^2$.

THE radiative corrections to an energy level of an ordinary atom are composed of two separate effects, of which by far the more important is the raising of an s level due to the modification of the electron self-energy in that state. In addition, however, there is a small negative contribution arising from the polarization of the vacuum by the field of the nucleus. The first of these is, in first approximation, inversely proportional to the square of the electron mass, and in a mesonic atom it would therefore contain the square of the meson mass in the denominator. On the other hand, the electrostatic potential is modified by vacuum polarization in a manner quite independent of the mass of the orbital particle. It has therefore been pointed out[1] that for a mesonic atom the correction due to vacuum polarization is of order 10^3 of that due to self-energy effects. Indeed, this fact causes vacuum polarization to be the dominant feature of the electrodynamic radiative corrections to any process, such as p-p scattering, meson-nucleon scattering, involving the interaction of two-systems, neither of which is an electron. It also follows that all mesonic atom levels are lowered by this effect and that, because of the much smaller size of these atoms, states of orbital angular momentum $\neq 0$ are also appreciably affected. The magnitude of the correction is, of course, greatly enhanced by the closeness of nucleus and meson in the mesonic atom, and, owing to the rapidly increasing accuracy with which it is becoming possible to measure mesonic atom energy levels,[2] it becomes important to compute the correction as accurately as possible.

To terms of first order in α, an external current density j_μ which is of the form $\exp(iqx)$ but which does not have enough energy to produce real pairs ($q^2 > -4\mu^2$, $\mu = mc/\hbar$) induces a vacuum current, the physically significant part of which is

$$\delta j_\mu = \frac{\alpha}{4\pi}\int_0^1 z(1-z)^{-\frac{1}{2}} \ln\left(1+\frac{zq^2}{4\mu^2}\right) dz\, j_\mu. \tag{1}$$

For a point nucleus, the corresponding fractional change in potential then becomes

$$\begin{aligned}\frac{\delta\varphi}{\varphi} &= \frac{\alpha}{2\pi}\int_0^1 z(1-z)^{-\frac{1}{2}} dz \int_{2\mu r z^{-\frac{1}{2}}}^{\infty} \frac{e^{-x}}{x} dx \\ &= \frac{\alpha}{3\pi}\int_1^\infty e^{-2\mu r x}(x^2-1)^{\frac{1}{2}}(2x^2+1)x^{-4} dx. \end{aligned} \tag{2}$$

This ratio is graphed in Fig. 1. For $\mu r \gg 1$ it becomes

$$\frac{\delta\varphi}{\varphi} \sim \frac{\alpha}{4\sqrt{\pi}} \frac{e^{-2\mu r}}{(\mu r)^{\frac{3}{2}}},$$

while for $\mu r \ll 1$ it tends to

$$\delta\varphi/\varphi \sim -(2\alpha/3\pi)\ln(4.08\mu r).$$

In the latter limit the corresponding induced charge density is

$$\delta\rho \sim (-2\alpha/3\pi)(Ze/4\pi r^3).$$

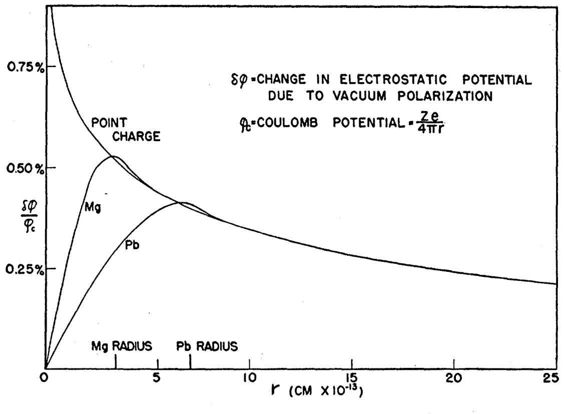

FIG. 1. The radial dependence of the change $\delta\varphi$ in electrostatic potential due to vacuum polarization for (a) a point charge, (b) Pb nucleus, (c) Mg nucleus, assuming a uniform charge distribution for cases (b) and (c). φ_c = Coulomb potential $= Ze/4\pi r$.

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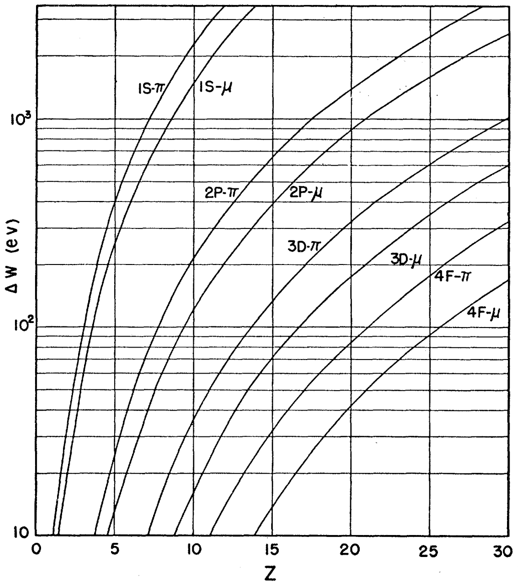

FIG. 2. The Z dependence of level shifts in π and μ mesonic atoms due to vacuum polarization for $1s$, $2p$, $3d$, and $4f$ levels, $Z \leqslant 30$.

The energy shift of a particular mesonic atom level is then given by

$$\Delta W = -e \int \psi^* \psi \delta\varphi d^3r, \tag{3}$$

which becomes, for a state for which $l = n-1$ (Coulomb wave functions),

$$\frac{\Delta W}{Mc^2} = \frac{-\alpha^3}{3\pi}\left(\frac{Z}{n}\right)^2 \int_\epsilon^{1+\epsilon} dx \frac{(x-\epsilon)^{2n-1}}{x^{2n}} \times (2+\epsilon^2-2\epsilon x+x^2)(1-\epsilon^2+2\epsilon x-x^2)^{\frac{1}{2}}, \tag{4}$$

where $\epsilon = nm/ZM\alpha$. (M = meson mass, m = electron mass). For $1s$ states this yields

$$\Delta W = (-\alpha^3/3\pi)MZ^2\{-11/3+3\pi\epsilon/2-4\epsilon^2+2\pi\epsilon^3 + (2-\epsilon^2-4\epsilon^4)(1-\epsilon^2)^{-\frac{1}{2}} \ln[\epsilon^{-1}(1+(1-\epsilon^2)^{\frac{1}{2}})]\}.$$

Similar expressions in closed form may be derived for $2p$, $3d$, $4f$ states and the values to which they lead are shown in Figs. 2 and 3 for π and μ mesonic atoms.

The above results would be valid for a point nucleus, and within the approximations of Eqs. (1) and (3) they are exact. To take account of the finite size of the nucleus, we may consider the latter to be a uniformly charged sphere of radius $R = 1.2 \times 10^{-13} A^{\frac{1}{3}}$ cm:

$$\rho = \frac{Ze}{(2\pi)^3} \frac{3}{R^3} \int (\sin qR - qR \cos qR) q^{-3} e^{i\mathbf{q}\cdot\mathbf{r}} d^3q.$$

The energy shift is then modified by the consequent changes in both $\delta\varphi$ and ψ. Comparing $\delta\varphi$ to the potential φ_c of a point nucleus of charge Ze, we have, instead of (2), for $r \leqslant R$:

$$\frac{\delta\varphi}{\varphi_c} = \frac{\alpha}{\pi\eta^3} \int_1^\infty dx e^{-\eta x}(x^2-1)^{\frac{1}{2}}(2x^2+1)x^{-7} \times [\sigma\eta x e^{\eta x} - (1+\eta x)\sinh(\sigma\eta x)] \tag{2'}$$

$$= \frac{\alpha\sigma}{3\pi} \int_1^\infty dx e^{-\eta x}(x^2-1)^{\frac{1}{2}}(2x^2+1)x^{-4}[\tfrac{1}{2}(3-\sigma^2) + \tfrac{1}{2}(1-\sigma^2)\eta x + \tfrac{1}{8}(1-\tfrac{1}{5}\sigma^4)\eta^2 x^2 + \cdots], \tag{2''}$$

where $\eta = 2\mu R$, $\sigma = r/R$. For $r > 2R$, the corresponding expression is represented by (2) to within 1 percent for $Z = 12$. For $r = R$, a direct comparison between (2) and (2'') is possible, (2'') being approximately 2 percent greater than (2) at this point. A simple approximation to (2''), valid to within 1 percent, may be obtained by expansion and the function is graphed in Fig. 1 for Mg and Pb. From such an expansion it may be shown that the error in ΔW due to the deviation of (2') from (2) is, for example, <5 percent for $Z = 12$, $1s$ state (π), $Z = 30$, $2p$ state (π), and $Z = 82$, $3d$ state (π), and <2 percent for a $Z = 82$, $4f$ state (π). Since this error in ΔW for a μ state is smaller than for a π state with the same n and Z, this error will be <5 percent for all n and Z values of Figs. 2 and 3.

The effect on ΔW of the deviation of the wave functions ψ from the hydrogen wave functions used in deriving (4) may be estimated from elementary per-

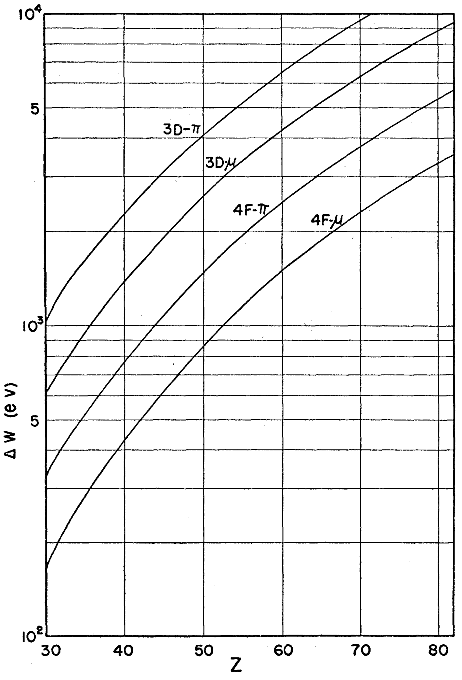

FIG. 3. The Z dependence of level shifts in π and μ mesonic atoms due to vacuum polarization for $3d$ and $4f$ levels, and $30 \leqslant Z \leqslant 82$.

turbation theory. Here again the states most greatly affected are the 1s states and for these the error is less than 3 percent for $Z=12(\pi)$, and less than 2 percent for $Z=12(\mu)$.

It should be noted that our results for the μ states, 2p and 2s in carbon, are in agreement with the values published by Pomeranchuk.[1] However, since no account is taken of the effect of the Coulomb field on the motion of the virtual electron-positron pairs, the values given here includes a fractional error of order $(Z\alpha)^2$.

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Quantization of Multiple Frequency Linear Systems

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The commutation relations and constants of motion for a wave equation whose frequencies have a common degeneracy are discussed. Although the quantum relations separate into those of simple subsystems, the corresponding decomposition holds only for a restricted class of constants of motion, excluding the energy.

A DETAILED investigation of the quantum mechanics of linear systems[1] has brought to light some points of interest relating to the multiple frequency systems first considered by Pais and Uhlenbeck.[2]

The quantization formalism employed by the writer[3] for E.B. systems is briefly as follows: Suppose the linear system to be given by $\Sigma M_{\alpha\beta}\psi_\beta(x)=0$, or $M\psi=0$, where $M=(M_{\alpha\beta})$ is a self-adjoint matric differential operator; x stands for all n independent variables one of which, x_0, is timelike in the sense that there is a complete set of solutions ψ which vanish strongly at ∞ of any surface $x_0=$ constant, and M is of finite degree d in $\partial_0=\partial/\partial x_0$. If we now introduce the expression

$$\mathcal{T}(\omega,\psi)=i/\hbar\int\omega^T[M,\,Z(x_0-x_0')]\psi dx^n, \tag{1}$$

where $Z(y)$ is the step function $[1+\mathrm{sgn}(y)]/2$ and ω^T is the transpose of ω, the commutation relations may be written in the form,

$$\mathcal{T}(\omega,\psi)\cdot F\psi(x')=F\omega(x'), \tag{2}$$

for any vector $\omega=(\omega_\alpha(x))$ and any linear operator F of degree $<d$ in ∂_0; here $A\cdot B$ stands for $[A,B]$. (Actually, it suffices to consider only ω for which $M\omega=0$; in such a case, $\mathcal{T}(\omega,\psi)$ is independent of x_0 and F may be taken as arbitrary.) Furthermore, if the system is invariant under the operator T ($T\psi$ is a solution whenever ψ is), then if $\mathcal{T}(\psi,T\psi)+\mathcal{T}(T\psi,\psi)=0$ as well, the constant of motion corresponding to T is given by

$$Q_T=\tfrac{1}{2}i\hbar\mathcal{T}(\psi,T\psi), \tag{3}$$

which transforms correctly and also satisfies

$$[F\psi,Q_T]=i\hbar FT\psi. \tag{4}$$

Now consider the multiple frequency linear system $\mathrm{N}^r\psi=0$, where N is self-adjoint, of degree s in ∂_0; since an operator F of degree $<d=rs$ in ∂_0 can generally be written as $F=\sum_{b=0}{}^{r}\mathrm{H}_b\mathrm{N}^b$, where H_b is of ∂_0-degree $<s$, (1) and (2) are equivalent to

$$(i/\hbar)\int\omega^T[\mathrm{N}^r,\,Z(x_0-x_0')]\psi dx^n\cdot\mathrm{HN}^b\psi(x')=\mathrm{HN}^b\omega(x'),$$

or to

$$(i/\hbar)\sum_{k=0}^{r-1}\int(\mathrm{N}^k\omega)^T[\mathrm{N},\,Z(x_0-x_0')]\mathrm{N}^{r-k-1}\psi dx^n\cdot\mathrm{HN}^b\psi(x')=\mathrm{HN}^b\omega(x'), \tag{5}$$

where H is of ∂_0-degree $<s$. The kth integral in (5) depends only on $\mathrm{N}^k\omega$, $\partial_0\mathrm{N}^k\omega$, $\cdots$, $\partial_0{}^{s-1}\mathrm{N}^k\omega$, evaluated at $x_0=x_0'$; since these functions may be chosen independently, (5) decomposes into

$$(i/\hbar)\int(\mathrm{N}^a\omega)^T[\mathrm{N},\,Z(x_0-x_0')]\mathrm{N}^{r-a-1}\psi dx^n\cdot\mathrm{HN}^b\psi(x')=\delta_{a,\,b}\mathrm{HN}^b\omega(x'). \tag{6}$$

According to (6), only $\mathrm{KN}^a\psi\cdot\mathrm{HN}^{r-1-a}\psi\neq0$ for K, H of degree $<s$ in ∂_0; thus the commutation relations decompose in a pairwise manner, with an additional single set if r is odd. Moreover, we obtain from (6), on redefining ω, the relations,

$$\left.\begin{aligned}&(/i\hbar)\int\omega^T[\mathrm{N},\,Z(x_0-x_0')]\mathrm{N}^{r-1-a}\psi dx^n\cdot\mathrm{HN}^a\psi(x')=\mathrm{H}\omega(x'),\\&(i/\hbar)\int(\mathrm{N}\omega)^T[\mathrm{N},\,Z(x_0-x_0')]\mathrm{N}^a\psi dx^n\cdot\mathrm{HN}^{r-1-a}\psi(x')=\mathrm{HN}\omega(x'),\end{aligned}\right\} \tag{7}$$

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[1] J. K. Percus, Columbia University Dissertation, 1954 (unpublished).

[2] A. Pais and G. E. Uhlenbeck, Phys. Rev. **79**, 145 (1950).

[3] The formalism used here may be shown to be equivalent to those of other authors when they are applicable to a given linear system; a more detailed discussion of this point will appear in a subsequent paper.