Solutions of a Bethe-Salpeter Equation

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The discussion in the preceding paper of a simplified Bethe-Salpeter equation is continued. Two methods are used to find a complete set of solutions, an integral transform method, and an adaption of Fock's treatment of the hydrogen atom. The degeneracy is found to be the same as that of the nonrelativistic hydrogen. In addition to the solutions which have the expected nonrelativistic limit, a large number of anomalous solutions are obtained. The behavior in the limit in which the mass of one particle becomes infinite is considered, and it is found that in this limit the ladder approximation gives an equation which does not correspond to the motion of a particle in the field of a fixed center of force.

I. INTRODUCTION

THE Bethe-Salpeter equation for two scalar particles has an especially simple form when these particles are assumed to interact through a massless scalar field. In the preceding paper,[1] Wick has given a general discussion of the Bethe-Salpeter equation and has concluded by examining some of the solutions of this particular equation. The study of this equation is continued in this paper, using the methods introduced by Wick, which enable one to obtain a complete set of bound-state solutions. Since this relativistic equation for two bosons can be solved easily, it not only provides an excellent illustration of the general results of Wick, but also gives one an insight into the techniques that one might apply to other equations, which would be more complicated, but also more realistic.

If we use the notation of Wick, the equation we wish to solve is written in momentum space as

$$[(p+i\eta\mu_a)^2+m_a^2][(p-i\eta\mu_b)^2+m_b^2]\phi(p)=\frac{\lambda}{\pi^2}\int\frac{d^4k\phi(k)}{(p-k)^2}. \quad (1)$$

The relative energy has been continued to the imaginary axis, in accordance with Wick's theorem. The energy of a bound state, which is proportional to ϵ, is considered to be given, so that the corresponding interaction constant λ, which is to be determined, is then an eigenvalue of the above equation. We begin by supposing that the two interacting particles have equal masses. The general case can be treated by exactly the same methods, but for simplicity is deferred to the last part of this paper.

Two methods, which supplement each other but are essentially independent, are used in the discussion of Eq. (1), the stereographic projection method of Fock,[2] and the integral transform method already introduced by Wick. When η is set equal to zero, in addition to the masses being equated, Eq. (1) is formally very much like the three-dimensional equation for the nonrelativistic hydrogen atom, as has already been pointed out by Wick. This remark led to the conjecture that the method Fock used to study the hydrogen atom could be applied here as well.[3] Fock's transformation does indeed lead to a great simplification in the form of Eq. (1), and is particularly useful for exhibiting the completeness as well as the degeneracy of its solutions. In the third section of this paper, Wick's integral transformation method, the essence of which is that one writes the solution as a superposition of one-particle propagation functions, is extended so that a complete set of solutions is also obtained by that method, which appears to be more readily applicable to other problems.

II. FOCK'S TRANSFORMATION

If the mass of each particle is set equal to unity, Eq. (1) becomes

$$[(p+i\eta)^2+1][(p-i\eta)^2+1]\phi(p)=\frac{\lambda}{\pi^2}\int\frac{d^4k\phi(k)}{(p-k)^2}. \quad (2)$$

By following the method of Fock and Lévy, the four-dimensional momentum space is mapped upon the surface of a five-dimensional sphere by a stereographic projection. A polar coordinate system is first chosen in momentum space, with polar angles β, θ, ϕ; β being the angle between the four vector p and the (imaginary) relative time axis. These angles are then used for three of the polar angles in five-dimensional space, the fourth angle ζ being defined by $p_0\tan\frac{1}{2}\zeta=|p|$, where $|p|$ is the magnitude of the momentum four vector and $p_0=(1-\eta^2)^{\frac{1}{2}}$ is the diameter of the five-dimensional sphere. Before continuing, it should be remarked that Wick's analytic continuation theorem plays an essential although somewhat hidden role in this transformation. In fact, it is possible to map the momentum space upon a closed manifold in an invariant manner, only because the momentum space of Eq. (2) has a positive definite metric. Upon making the stereographic mapping, it is found that on the left-hand side of (2),

$$[(p+i\eta)^2+1][(p-i\eta)^2+1]=p_0^2\sec^4\tfrac{1}{2}\zeta[1-\eta^2+\eta^2\sin^2\zeta\cos^2\beta]. \quad (3)$$

[1] G. C. Wick, preceding paper [Phys. Rev. 96, 1124 (1954)].
[2] V. Fock, Z. Physik 98, 145 (1935); M. Lévy, Proc. Roy. Soc. (London) A204, 145 (1950).
[3] I am indebted to Professor Wick for discussions of various aspects of Fock's method.

In the integral on the right-hand side,

$$[d^4k]=\tfrac{1}{16}p_0^4 \sec^8\tfrac{1}{2}\zeta' d\Omega_5', \tag{4}$$

where $d\Omega_5'$ is an element of solid angle in five dimensions, and

$$(p-k)^2=\tfrac{1}{2}p_0^2 \sec^2(\tfrac{1}{2}\zeta) \sec^2(\tfrac{1}{2}\zeta')(1-\cos\alpha), \tag{5}$$

where α is the angle between the images of p and k on the sphere. Thus with the definition $H(\zeta,\beta,\theta,\phi) =\sec^6(\tfrac{1}{2}\zeta)\phi(p)$, Eq. (2) becomes:

$$(1-\eta^2+\eta^2 \sin^2\zeta \cos^2\beta)H(\zeta,\beta,\theta,\phi) =\frac{\lambda}{8\pi^2}\int\frac{d\Omega_5' H(\zeta',\beta',\theta',\phi')}{1-\cos\alpha}. \tag{6}$$

If $\eta=0$, Eq. (6) is clearly invariant under all rotations of the five-dimensional sphere. It follows that a complete set of solutions is given by the spherical harmonics in five-dimensions, which can be constructed from the Gegenbauer polynomials.[4] In Appendix A it is shown that when $\eta=0$, the eigenvalues of (6) are $\lambda_N=N(N+1)$, where N is a positive integer. It likewise follows immediately that the degeneracy is that associated with the rotation group in five dimensions, so the Nth eigenvalue is $\tfrac{1}{6}N(N+1)(2N+1)$-fold degenerate.

When ϵ is positive, Eq. (6) is no longer invariant in all rotations of five-dimensional space. However, there is still an axis of symmetry, which can be seen if a set of Cartesian coordinates ξ_α is introduced in the five-dimensional space. For points on the sphere of diameter p_0,

$$\begin{aligned}\xi_5&=\tfrac{1}{2}p_0 \cos\zeta,\\ \xi_4&=\tfrac{1}{2}p_0 \sin\zeta \cos\beta,\\ \xi_3&=\tfrac{1}{2}p_0 \sin\zeta \sin\beta \cos\theta,\end{aligned} \tag{7}$$

and so forth. Since in Eq. (6) the coordinates are contained explicitly only in the combination $\sin\zeta \cos\beta =\cos\gamma$, where γ is the angle with the ξ_4 coordinate axis, all rotations which leave this axis unchanged also leave unchanged the form of Eq. (6). For positive values of ϵ the degeneracy is therefore that associated with the four-dimensional rotation group, which is known to be the degeneracy of the nonrelativistic hydrogen atom.[2] Thus no fine-structure splitting is exhibited by Eq. (2), for any value of the interaction constant.

An interesting equation is obtained if the points on the sphere are mapped onto another flat four-dimensional space, by using a stereographic projection of which ξ_4 is the polar axis, instead of ξ_5. With $q=\tan(\tfrac{1}{2}\gamma)$ and $Q(q)=\cos^6(\tfrac{1}{2}\gamma)H$, Eq. (6) is transformed into

$$[q^4+2(1-2\eta^2)q^2+1]Q(q)=\frac{\lambda}{\pi^2}\int\frac{d^4q' Q(q')}{(q-q')^2}. \tag{8}$$

This equation is of the same form as Eq. (1) at zero energy, except that the difference of the masses is imaginary. From the form of either Eq. (6) or Eq. (8), it is evident that any solution can be written as a product of a four-dimensional spherical harmonic and a function of q^2, this function being the solution of a one-dimensional integral equation. There is therefore evidently a connection between the degeneracy of Eq. (2) and the fact that by using Wick's transform method it is possible to find solutions in terms of a simple one-parameter integral. It will be shown that the method of Wick is indeed exactly equivalent to solving (8) directly, so it is not necessary to pursue further the solution of this equation at this point.

III. THE INTEGRAL TRANSFORMATION METHOD

Wick was able to find a set of solutions to (2) in terms of an integral over an unknown function $g(z)$, which is the solution of a simple integral equation. In order to find a complete set of solutions, a somewhat more complicated ansatz must be used, and a clue as to the appropriate generalization is given by the structure of the spherical harmonics, which are solutions at zero energy. One is thus led to try to find solutions which are linear combinations of functions of the form

$$\phi_n^{lm}(p,z)=\mathcal{Y}_l^m(\mathbf{p})/[p^2+2izp\eta+1-\eta^2]^{n+2}, \tag{9}$$

where $\mathcal{Y}_l^m(\mathbf{p})$ is a solid harmonic function of the three space components of p. In order to determine what linear combinations to take, it is necessary to find out what happens when $\phi_n^{lm}(p,z)$ is inserted into the right-hand side of the integral equation (2).

The integration over the momentum k is done in the usual way, by using Feynman's formula

$$\begin{aligned}\frac{1}{\pi^2}\int\frac{d^4k}{(p-k)^2}\phi_n^{lm}(k,z)&=\frac{n+2}{\pi^2}\int_0^1 u^{n+1}du\int\frac{d^4k\,\mathcal{Y}_l^m(\mathbf{k}+(1-u)\mathbf{p})}{[k^2+u(1-u)p^2+u(1-\eta^2)+z^2u^2\eta^2+2izu(1-u)p\eta]^{n+3}}\\ &=\frac{1}{n+1}\int_0^1(1-u)^l du\,\mathcal{Y}_l^m(\mathbf{p})[(1-u)p^2+1-\eta^2+z^2u\eta^2+2iz(1-u)p\eta]^{-n-1}\\ &=\sum_{k=0}^{n-l-1}\frac{(n-l-1)!(n-k-1)!}{(n+1)!(n-l-k-1)!}\frac{\phi_{n-k-2}^{lm}(p,z)}{(1-\eta^2+\eta^2z^2)^{k+1}}.\end{aligned} \tag{10}$$

[4] E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge University Press, London, 1950), p. 329.

In performing the above computations, it has been supposed that $n>l$. Now, following Wick, this is divided by $[(p+i\eta)^2+1][(p-i\eta)^2+1]$ and the result expressed by another use of Feynman's integral formula

$$\pi^{-2}[(p+i\eta)^2+1]^{-1}[(p-i\eta)^2+1]^{-1}\int\frac{d^4k}{(p-k)^2}\phi_n{}^{lm}(p,z)$$
$$=\frac{1}{2}\sum_{k=0}^{n-l-1}\frac{(n-l-1)!(n-k+1)!}{(n+1)!(n-k-l-1)!}\int_{-1}^{1}dt$$
$$\times\int_0^1\frac{x(1-x)^{n-k-1}dx}{(1-\eta^2+\eta^2z^2)^{k+1}}\phi_{n-k}{}^{lm}(p,\,xt+(1-x)z). \quad (11)$$

It is thus evident that a function of the form

$$\phi_n{}^{lm}(p)=\sum_{k=0}^{n-l-1}\int_{-1}^{1}g_{nl}{}^k(z)\phi_{n-k}{}^{lm}(p,z)dz \quad (12)$$

will be a solution.

The functions $g_{nl}{}^k(z)$ are determined by inserting this into the integral equation (2). Making use of (11), one finds that

$$\sum_{k=0}^{n-l-1}\int_{-1}^{1}g_{nl}{}^k(z)\phi_{n-k}{}^{lm}(p,z)dz$$
$$=\tfrac{1}{2}\lambda\sum_{k=0}^{n-l-1}\sum_{k'=0}^{k}\left\{\frac{(n-k+1)!(n-k'-l-1)!}{(n-k'+1)!(n-k-l-1)!}\right.$$
$$\times\int_{-1}^{1}g_{nl}{}^k(z)dz\int_{-1}^{1}dt\int_0^1\frac{x(1-x)^{n-k-1}dx}{(1-\eta^2+\eta^2z^2)^{k-k'+1}}$$
$$\left.\times\phi_{n-k}{}^{lm}(p,\,xt+(1-x)z)\right\}. \quad (13)$$

Since the $\phi_n{}^{lm}(p,z)$ are a linearly independent set of functions in momentum space, Eq. (13) requires that the $g_{nl}{}^k(z)$ be solutions of the following set of integral equations:

$$g_{nl}{}^k(z)=\tfrac{1}{2}\lambda\sum_{k'=0}^{k}\left\{\frac{(n-k+1)!(n-k'-l-1)!}{(n-k'+1)!(n-k-l-1)!}\right.$$
$$\times\int_0^1 dt\int_0^1 x(1-x)^{n-k-1}dx$$
$$\left.\times\int_{-1}^{1}\frac{\delta(z-[xt+(1-x)\zeta])}{(1-\eta^2+\eta^2\zeta^2)^{k-k'+1}}g_{nl}{}^{k'}(\zeta)d\zeta\right\}. \quad (14)$$

Upon examining this set of integral equations, it is seen that $g_{nl}{}^0(z)$ is the solution of a single homogeneous equation, from which equation λ is determined. For $n>l+1$, the additional functions $g_{nl}{}^k(z)$ are the solutions of inhomogeneous integral equations which involve the eigenfunction $g_{nl}{}^0(z)$. The homogeneous equation does not depend upon the orbital angular momentum l, so that neither $g_{nl}{}^0(z)$ nor λ depends on l. This degeneracy, which the discussion of the previous section has already shown to exist, is again seen to be exactly the same as that of the nonrelativistic hydrogen atom.

The integral equation for $g_{nl}{}^0(z)$, which is the only function of practical importance, is found, after integrating over x and t and also dropping the indices 0 and l, to be

$$g_n(z)=\frac{\lambda}{2n}\int_{-1}^{1}[R(z,\zeta)]^n\frac{g_n(\zeta)d\zeta}{1-\eta^2+\eta^2\zeta^2}, \quad (15)$$

where $R(z,\zeta)$ is the same function introduced by Wick

$$R(z,\zeta)=(1\pm z)/(1\pm\zeta),\quad \text{for}\quad \zeta\gtrless z. \quad (16)$$

Differentiating Eq. (15), one finds that[5]

$$g_n''(z)+2(n-1)z(1-z^2)^{-1}g_n'(z)-n(n-1)(1-z^2)^{-1}g_n(z)$$
$$+\lambda(1-z^2)^{-1}(1-\eta^2+\eta^2z^2)^{-1}g_n(z)=0. \quad (17)$$

The boundary conditions are $g_n(\pm1)=0$.

This equation has an infinite number of solutions, with discrete eigenvalues λ_κ, the index κ denoting the number of zeros of $g_n(z)$ within the interval $(-1, 1)$. For odd values of κ, the g_n are odd functions of z; therefore the corresponding solutions of the Bethe-Salpeter equation are odd functions of the relative time. In Appendices A and B, the solutions of Eq. (17) are discussed in the two limits $\epsilon=0$ and $\epsilon=1$. At zero energy, where $\lambda_\kappa=(n+\kappa)(n+\kappa+1)$, a one-one correspondance can easily be made between the spherical harmonics of five dimensions and the solutions obtained by the method of this section, thus showing that the set of functions of the form (12) provides a complete set of solutions. As $\eta^2\to1$, only those eigenvalues λ_κ for which $\kappa=0$ approach the values obtained from the Schrödinger equation for the hydrogen atom, as $\lambda_\kappa\to\frac{1}{4}$ if κ is not zero.[6] In Fig. 1, curves for λ *versus* η^2 are shown for some of the lowest-lying states. The exact solutions for $\epsilon=0$ and $\epsilon=1$ have been supplemented by some numerical calculations for intermediate points.

The method outlined above was satisfactory as a way of obtaining the eigenvalues λ, but when $n>l+1$ the construction of the Bethe-Salpeter amplitude is rather cumbersome, as first a set of inhomogeneous equations must be solved successively. However, this can be circumvented if use is also made of the method of Sec. II. When $l=n-1$, the sum in Eq. (12) reduces to a single term. This integral formula for the amplitude function is transformed into

$$Q_n{}^{n-1,m}(q)=p_0{}^{-n+1}\int_{-1}^{1}\frac{\mathcal{Y}_{n-1}{}^m(\mathbf{q})g_n(z)dz}{[1+q^2+iz\epsilon p_0{}^{-1}(1-q^2)]^{n+2}}, \quad (18)$$

[5] The inhomogeneous integral equations can be replaced by second order differential equations similar to (17).

[6] In the previous paper (see reference 1), Wick has treated in detail the case $n=1$. I am grateful to Professor Wick for discussing this limit with me.

when the momentum space is mapped onto the queer space introduced at the end of Sec. II. This amplitude is degenerate with all the amplitudes obtained by rotations in the q space, which can be constructed with the four-dimensional spherical harmonics:

$$Q_n^{lm}(q) = p_0^{1-n}\int_{-1}^{1}\frac{\mathcal{Y}_l^m(\mathbf{q})q^{n-l-1}C_{n-l-1}^{l+1}(q_4/q)g_n(z)dz}{[1+q^2+iz\epsilon p_0^{-1}(1-q^2)]^{n+2}}. \quad (19)$$

Transforming back to ordinary momentum space, one finds

$$\phi_n^{lm}(p) = \int_{-1}^{1}\frac{g_n(z)dz\,\mathcal{Y}_l^m(\mathbf{p})R^{n-l-1}C_{n-l-1}^{l+1}(X/R)}{[p^2+2izp\eta+1-\eta^2]^{n+2}}, \quad (20)$$

where

$$X=1-\eta^2-p^2,\quad R^2=(p^2+1-\eta^2)^2-4p_4^2(1-\eta^2). \quad (21)$$

Returning to (19), with the substitution $z=(p_0\zeta+i\epsilon)/(p_0+i\epsilon\zeta)$, and the dropping of irrelevant factors, this becomes

$$Q_n^{lm}(q) = \int_{-1}^{1}\frac{(p_0+i\epsilon\zeta)^n g_n(z(\zeta))d\zeta\,\mathcal{Y}_l^m(\mathbf{q})q^{n-l-1}C_{n-l-1}^{l+1}(q_4/q)}{[q^2+1-2\eta^2+2ip_0\epsilon\zeta]^{n+2}}. \quad (22)$$

By using the results of the next section it can be shown directly that this is a solution of Eq. (8), which, as has been remarked already, is of the same type as (1) but with complex masses, thus demonstrating the equivalence of the two methods we have used to discuss the Bethe-Salpeter equation (2).

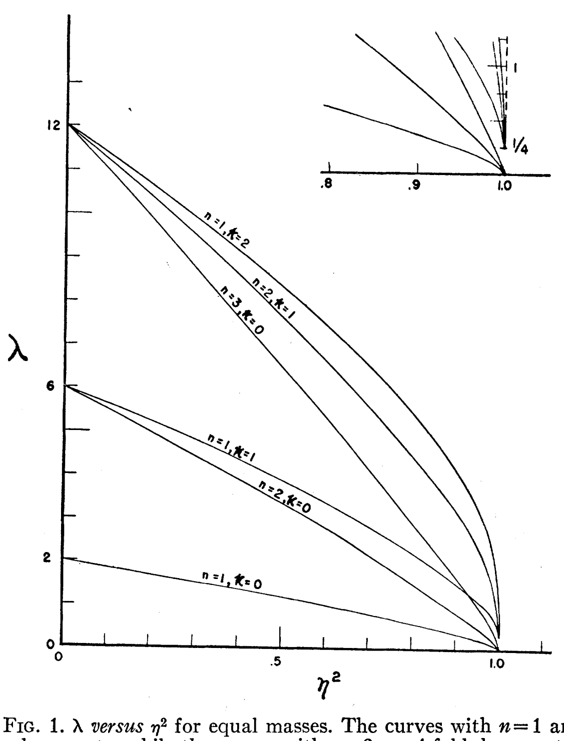

FIG. 1. λ *versus* η^2 for equal masses. The curves with $n=1$ are nondegenerate, while the curves with $n=2$ are 4-fold degenerate, and the curve with $n=3$ is 9-fold degenerate. The inset shows the region near $\eta^2=1$ in greater detail.

IV. UNEQUAL MASSES

The methods of Secs. II and III can both be used to study the more general Eq. (1), which is rewritten with the notation $m_a=1+\Delta$, $m_b=1-\Delta$:

$$F_+F_-\phi = \frac{\lambda}{\pi^2}\int\frac{\phi(k)d^4k}{(p-k)^2}, \quad (23\text{-a})$$

$$F_\pm = p^2 \pm 2i(1\pm\Delta)p\eta+(1-\eta^2)(1\pm\Delta)^2. \quad (23\text{-b})$$

First consider the stereographic projection method of Fock and Lévy, in which $p=p_0\tan\frac{1}{2}\zeta$, with $p_0^2=(1-\eta^2)(1-\Delta^2)$. The right-hand side of (23-a) is transformed in the same way as before, but on the left-hand side,

$$F_\pm = \sec^2(\tfrac{1}{2}\zeta)[p_0^2\pm i\eta p_0(1\pm\Delta)\sin\zeta\cos\beta \pm\Delta(1\pm\Delta)(1-\eta^2)(1+\cos\zeta)]. \quad (24)$$

Now let $\cos\gamma=\cos\alpha\cos\zeta+\sin\alpha\sin\zeta\cos\beta$, where α is chosen such that

$$\tan\alpha = i\eta\Delta^{-1}(1-\Delta^2)^{\frac{1}{2}}(1-\eta^2)^{-\frac{1}{2}}. \quad (25)$$

Then,

$$F_+F_- = p_0^2\sec^4\tfrac{1}{2}\zeta[(1-\eta^2)+(\eta^2-\Delta^2)\cos^2\gamma]. \quad (26)$$

The equation which results is formally very much like (6). There is now an imaginary symmetry axis, rotated from the ξ_5 axis through the angle α in the $\xi_5-\xi_4$ plane. The degeneracy is thus not destroyed by taking the masses to be different.

A four-dimensional equation with complete rotational invariance can again be obtained by making a stereographic mapping of which the symmetry axis is the pole. With the definition, as in Sec. II, $q=\tan(\frac{1}{2}\gamma)$ and $(q^2+1)^3Q(q)=(p^2+p_0^2)^3\phi$, an equation very similar to (8) is obtained

$$(1-\Delta^2)\left[q^4+2\left(1-2\frac{\eta^2-\Delta^2}{1-\Delta^2}\right)q^2+1\right]Q(q) = \frac{\lambda}{\pi^2}\int\frac{d^4q'Q(q')}{(q-q')^2}. \quad (27)$$

The q_μ are at first supposed to vary over complicated contours in their respective complex planes; however, as these contours can be deformed back to the real axes without any difficulty, use of Eq. (27) is quite rigorous. The form of the left-hand side indicates that $\lambda/(1-\Delta^2)$ is a function only of $(\eta^2-\Delta^2)/(1-\Delta^2)$, a relation which was of considerable use in constructing Fig. 2. Furthermore, the solutions of (27), as functions of q, depend only on $(\eta^2-\Delta^2)/(1-\Delta^2)$, which implies a corresponding, but more complicated, relationship between the solutions of (23) for different values of Δ.

The foregoing discussion makes it clear that the integral transform method used in the previous section can be applied to Eq. (23) with very little change. The observation that

$$F_+^{-1}F_-^{-1}=\tfrac{1}{2}\int_{-1}^{1}dz[p^2+2i(z+\Delta)p\eta+(1-\eta^2)(1+\Delta^2+2z\Delta)]^{-2} \quad (28)$$

gives the hint that the definition (9) should be generalized to

$$\phi_n^{lm}(p,z,\Delta)=\frac{\mathcal{Y}_l^m(\mathbf{p})}{[p^2+2i(z+\Delta)p\eta+(1-\eta^2)(1+\Delta^2+2z\Delta)]^{n+2}}. \quad (29)$$

The calculation leading to Eqs. (14), (15), and (17) can be carried out, the appropriate changes being made, exactly as before. One ends with integral and differential equations for the new function $g_n(z,\Delta)$ which differ only little from (15) and (17)

$$g_n(z,\Delta)=\frac{\lambda}{2n}\int_{-1}^{1}\frac{[R(z,\zeta)]^n g_n(\zeta,\Delta)d\zeta}{[(1-\eta^2)(1+\Delta^2+2\Delta\zeta)+\eta^2(\Delta+\zeta)^2]}, \quad (30)$$

$$\begin{aligned}g_n''(z,\Delta)+2(n-1)z(1-z^2)^{-1}g_n'(z,\Delta)\\ -n(n-1)(1-z^2)^{-1}g_n(z,\Delta)\\ +\lambda(1-z^2)^{-1}[(1+\Delta^2+2z\Delta)(1-\eta^2)\\ +\eta^2(z+\Delta)^2]^{-1}g_n(z,\Delta)=0.\end{aligned} \quad (31)$$

The relationship between the solutions of Eqs. (23) and (2), which was proved above by using the Fock projection, can be demonstrated more directly in the integral transform method.[7] We let $\bar{z}=(z-\Delta)/(1-\Delta z)$ and $g_n(z,\Delta)=(1-\Delta\bar{z})^{-n}g_n(\bar{z})$. After noting that $R(z,\zeta)=(1-\Delta\bar{\zeta})(1-\Delta\bar{z})^{-1}R(\bar{z},\bar{\zeta})$ and that

$$(1-\eta^2)(1+\Delta^2+2z\Delta)+\eta^2(z+\Delta)^2=(1-\Delta z)^2\frac{1-\eta^2}{1-\Delta^2}+(z-\Delta)^2\frac{\eta^2-\Delta^2}{1-\Delta^2}, \quad (32)$$

we can transform Eq. (30) into

$$g_n(\bar{z})=\frac{\lambda}{2n(1-\Delta^2)}\int_{-1}^{1}[R(\bar{z},\bar{\zeta})]^n g_n(\bar{\zeta})d\bar{\zeta}\times\left[1-\frac{\eta^2-\Delta^2}{1-\Delta^2}(1-\bar{\zeta}^2)\right]^{-1}. \quad (33)$$

This is identical with Eq. (15), which again shows that $\lambda/(1-\Delta^2)$ depends only upon $(\eta^2-\Delta^2)/(1-\Delta^2)$.

In Fig. 2, λ is plotted against η^2 for $\Delta^2=0.36$, or $m_a=4m_b$. These curves were obtained from the curves

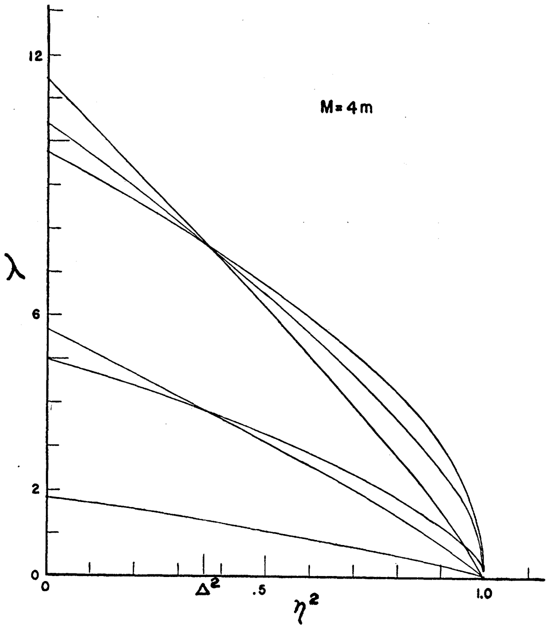

FIG. 2. An example with unequal masses. The quantum numbers are the same as for the corresponding curves of Fig. 1.

of Fig. 1 by using the similarity transformation. Note, in particular, that the curves cross at $\eta^2=\Delta^2$. This corresponds to a critical energy $E_c=m_a-m_b$ for the bound state; if there existed a state of two particles with an energy smaller than E_c, the heavier of the two original particles would be unstable, as it could decay into the bound system and the antiparticle of b. Nevertheless, solutions of Eq. (33) exist and are perfectly well behaved analytically when $\eta^2<\Delta^2$, although it seems that the various solutions can no longer be distinguished in a clear way on the basis of their dependence upon the relative time.[8]

A question of some interest is the behavior of Eq. (23) and of its solutions in the limit $\Delta\to 1$. One may either think of one of the particles becoming infinitely massive, with the binding energy being a fixed multiple of the mass of the lighter particle, or one may suppose that the mass of one particle vanishes, with the binding energy remaining finite. In the first case, the similarity transformation gives the eigenvalues unambiguously, but it is nevertheless worth while to examine the limit directly. We define

$$\begin{aligned}\Delta&=(M-1)/(M+1), & \epsilon&=(M+1-B)/(M+1),\\ \Lambda&=\lambda(1-\Delta^2)^{-1}, & p'&=(1-\Delta)^{-1}p.\end{aligned} \quad (34)$$

Now let $M\to\infty$, with the binding energy B remaining fixed. The limit of Eq. (23) is

$$(p'^2-2ip_4')(2ip_4'+2B)\phi(p')=\frac{\Lambda}{4\pi^2}\int\frac{d^4k'\phi(k')}{(p'-k')^2}. \quad (35)$$

[7] The transformation which follows is due to G. C. Wick.

[8] The ladder approximation is of course complete nonsense when the interaction is this strong.

The mass of one particle is so large that it might be supposed no recoil effects should occur, but in Eq. (35) the relative time still enters in such a fundamental manner that the equation cannot be reduced to a three-dimensional one, unless the binding energy is so small that even the lighter particle moves nonrelativistically.

The solutions of Eq. (35) may be obtained by taking the limit of the solutions of Eq. (23), which have already been discussed

$$g_n\left(\frac{2t}{M+1}-1,\,\Delta\right)\to G_n(t),$$
$$\phi_n{}^{lm}\left(p,\frac{2t}{M+1}-1,\,\Delta\right)\to\frac{\mathcal{Y}_l{}^m(\mathbf{p}')}{[p'^2+2ip_4'(t-1)+2Bt]^{n+2}}. \tag{36}$$

Equation (31) becomes, in the limit

$$0=tG_n''(t)-(n-1)G_n'(t)+\Lambda[(t-1)^2+2Bt]^{-1}G_n(t). \tag{37}$$

The eigenvalue Λ is the same function of $(\frac{1}{2}B)$ that λ is of $(1-\eta^2)$ when $\Delta=0$. In contrast to the three-dimensional theory, from the form of the above equation we see that there is no upper limit on B. The limit of Eq. (37) and of Λ as $B\to\infty$ is discussed in Appendix C.

When $\eta^2=0$, Eq. (23) has a particularly interesting limiting form. With $p=(1-\Delta)p'$, as before, Eq. (19) becomes

$$\left[1+\left(\frac{1-\Delta}{1+\Delta}\right)^2p'^2\right](1+p'^2)\phi(p')=\frac{\lambda}{\pi^2(1+\Delta)^2}\int\frac{d^4k'\phi(k')}{(p'-k')^2}. \tag{38}$$

If one takes the limit $\Delta\to1$, the ensuing equation is the same as the Bethe-Salpeter equation for two Dirac particles in a singlet state at zero energy, which has been found by Goldstein[9] to have a rather singular behavior. However, if $\Delta\neq1$, there is another factor of p'^2 on the left-hand side of (38), which provides a kind of "cutoff," at large momenta. The solutions are then given by the methods already discussed; we must solve the equation

$$(1-z^2)g_n''(z,\Delta)+2(n-1)zg_n'(z,\Delta)-n(n-1)g_n(z,\Delta)+\lambda(1+\Delta^2+2\Delta z)^{-1}g_n(z,\Delta)=0, \tag{39}$$

which is just a special case of (31). As has been remarked before, this equation has a set of eigenfunctions with discrete eigenvalues, the κth eigenfunction having κ zeros within the interval $(-1,1)$. The corresponding Bethe-Salpeter amplitude has κ nodal hyperspheres in configuration space. In Appendix C it is shown that as $\Delta\to1$, all the eigenvalues for a given n tend to exactly the same limit: $\lambda_\kappa\to\lambda_0=n^2$. For $n=1$, this is the same as the value Goldstein found for the symmetric solutions of the limit of Eq. (38). In configuration space, the nodal hyperspheres of the amplitudes all shrink into the origin, and at finite distances the amplitudes for given n, l, and m all approach the same limiting function.

In summary, we have found a complete set of bound-state solutions for a simplified Bethe-Salpeter equation. These solutions have been discussed for all values of the bound state energy and for various values of the masses. The simple form of the solutions of this equation, as well as the degeneracy of the eigenvalues, which is the same as the degeneracy in the nonrelativistic hydrogen atom, are shown to be due to the complete separability of the equation in a transformed coordinate system. Although this unexpected symmetry is not a general feature of the Bethe-Salpeter equation, one is nevertheless able to learn a great deal from this example. It was found that for large values of the interaction constant, the equation studied has abnormal solutions, which do not correspond to any familiar physical situation. The limiting form of the solutions when one of the masses becomes infinite does not correspond to the motion of a particle in the field of a fixed center of force, and at zero energy sheds some light upon the pathological character of the two-fermion equation.

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APPENDIX A

In the text following Eq. (6), it was pointed out that at zero energy the solutions of this equation were spherical harmonics. It suffices to consider the Gegenbauer polynomials $C_{N-1}{}^{\frac{3}{2}}(\cos\zeta)$ as the spherical harmonics constructed from one of these with the use of the addition theorem are of course degenerate with it. When $\zeta=0$, Eq. (6) becomes

$$C_{N-1}{}^{\frac{3}{2}}(1)=\frac{\lambda_N}{8\pi^2}\int\frac{C_{N-1}{}^{\frac{3}{2}}(\cos\zeta')\,\sin^3\zeta'd\zeta'd\Omega_4'}{1-\cos\zeta'}. \tag{A-1}$$

This integral can be evaluated with the aid of the generating function for the Gegenbauer polynomials

$$\sum_0^\infty r^nC_n{}^\nu(x)=(1-2rx+r^2)^{-\nu}. \tag{A-2}$$

Therefore

$$\sum_1^\infty\lambda_N{}^{-1}r^{N-1}C_{N-1}{}^{\frac{3}{2}}(1)=\tfrac{1}{4}\int_0^\pi\frac{\sin^3\zeta'd\zeta'}{(1-\cos\zeta')(1-2r\cos\zeta'+r^2)^{\frac{3}{2}}},$$
$$=\tfrac{1}{2}(1-r)^{-1}. \tag{A-3}$$

From (A-2) it is found that $C_{N-1}{}^{\frac{3}{2}}(1)=\frac{1}{2}N(N+1)$; thus $\lambda_N=N(N+1)$.

[9] J. S. Goldstein, Phys. Rev. 91, 1516 (1953).

The eigenvalues at zero energy can also be obtained from Eq. (16) of the text.[10] Let $g_n(z)=(1-z^2)^n f_n(z)$. The equation satisfied by $f_n(z)$ is

$$(1-z^2)f_n''(z)-2(n+1)f_n'(z)+[\lambda-n(n+1)]f(z)=0, \quad \text{(A-4)}$$

which is the differential equation of the Gegenbauer polynomials. It is thus found that

$$g_n(z)=(1-z^2)^n C_\kappa^{n+\frac{1}{2}}(z),$$

with $\lambda=(n+\kappa)(n+\kappa+1)=N(N+1)$.

APPENDIX B

We shall now consider the differential Eq. (17) in the limit $\eta^2\to 1$,

$$(1-z^2)g_n''+2(n-1)zg_n'-n(n-1)g_n+\lambda(1-\eta^2+\eta^2z^2)^{-1}g_n=0. \quad (17)$$

If, as $\eta^2\to 1$, $g_n(z)$ approaches a limit uniformly, we may replace $(1-\eta^2+\eta^2z^2)^{-1}$ by $\pi(1-\eta^2)^{-\frac{1}{2}}\delta(z)$ in the original integral equation (15), from which we find $g_n(z)\sim(1-|z|)^n$ and $2-2\epsilon=B\sim(\lambda\pi)^2/8n^2$, where B is the binding energy. This is the solution for $\kappa=0$. By inserting the asymptotic form of $g_n(z)$ into (20), we find the asymptotic form of the amplitude:

$$\phi(p)\sim(p^2+1-\eta^2)^{-n}[(p+i\eta)^2+1]^{-1}[(p-i\eta)^2+1]^{-1}\times R^{n-l-1}C_{n-l-1}^{l+1}(X/R)\mathcal{Y}_l^m(\mathbf{p}). \quad \text{(B-1)}$$

In (B-1), we see that the important values of p_4 are of the order $1-\eta^2$, while $|\mathbf{p}|\sim(1-\eta^2)^{\frac{1}{2}}$. Thus,

$$\phi(p)\sim\pi\delta(p_4)(\mathbf{p}^2+1-\eta^2)^{-l-2}C_{n-l-1}^{l+1}\times\left(\frac{1-\mathbf{p}^2}{1+\mathbf{p}^2}\right)\mathcal{Y}_l^m(\mathbf{p}). \quad \text{(B-2)}$$

This will be recognized as the general form of the solutions of Schrödinger's equation for the hydrogen atom, so that in the limit $\lambda\to 0$, not only the right energy but also the correct nonrelativistic wave function is obtained. Equation (B-1) actually gives a rather good approximation to $\phi(p)$ for all values of η^2.

To obtain the limit for $\kappa\geq 1$, we set $\eta^2=1$ in (17),

$$(1-z^2)g_n''+2(n-1)zg_n'-n(n-1)g_n+\lambda z^{-2}g_n=0. \quad \text{(B-3)}$$

[10] This was first done by G. C. Wick, for the case $n=1$.

The solution of the above equation which satisfies the boundary conditions at $z=\pm 1$ is

$$g_n(z)=(1-z^2)^n z^{\frac{1}{2}+\rho}F[\tfrac{1}{4}(2n+3+2\rho),\tfrac{1}{4}(2n+1+2\rho);n+1;1-z^2], \quad \text{(B-4)}$$

where ρ has been defined by Wick's Eq. (62). To examine the behavior of $g_n(z)$ at $z=0$, we let $x=z(1-\eta^2)^{-\frac{1}{2}}$, whence

$$d^2g_n/dx^2+\lambda(1+x^2)^{-1}g_n=0, \quad \text{(B-5)}$$

which is identical with Wick's Eq. (61).

Continuing Wick's analysis, we find again,

$$\lambda_\kappa\sim\tfrac{1}{4}+\pi^2(\kappa-1)^2[\ln(1-\eta^2)]^{-2}. \quad \text{(B-6)}$$

Note that this is independent of n.

APPENDIX C

The limit $\Delta\to 1$ is found from Eq. (39) if $\eta^2=0$, while for $\eta^2\sim 1$ Eq. (37) must be used. By letting $\Delta\to 1$ in Eq. (23) a connection between these two extreme values is provided

$$(1-\eta^2)q^2Q(q)=\frac{\lambda}{4\pi^2}\int\frac{d^4q'Q(q')}{(q-q')^2}; \quad \text{(C-1)}$$

therefore, $\lambda=\lambda_0(1-\eta^2)$. From the definitions (30), it follows that when B is very large, $\Lambda\sim\frac{1}{2}B\lambda_0$.

From Eq. (35) it is found that

$$(1-z^2)g_n''(z,1)+2(n-1)zg_n'(z,1)-n(n-1)g_n(z,1)+\tfrac{1}{2}\lambda_0(1+z)^{-1}g_n(z,1)=0. \quad \text{(C-2)}$$

The indices at $z=-1$ are $\frac{1}{2}n\pm\frac{1}{2}(n^2-\lambda_0)^{\frac{1}{2}}$. In order to examine the boundary condition at $z=-1$, we take the limit of (35) with $z+1=\frac{1}{2}(1-\Delta)^2u$, which gives

$$ud^2g_n/du^2-(n-1)dg_n/du+\lambda_0g_n/4(1+u)=0. \quad \text{(C-3)}$$

The indices of this at $u=\infty$ are also $\frac{1}{2}n\pm\frac{1}{2}(n^2-\lambda_0)^{\frac{1}{2}}$. Equations (C-2) and (C-3) can be solved in terms of hypergeometric functions; using the same method that was used to study the limit $\eta^2\to 1$, we find that $\lambda_0\to n^2$ for all values of κ.

Now consider Eq. (37) with $2Bt=u$ when B is large,

$$ud^2G_n/du^2-(n-1)dG/du+\Lambda G_n/2B(1+u)=0. \quad \text{(C-4)}$$

If $\Lambda=\frac{1}{2}B\lambda_0$, Eqs. (C-3) and (C-4) are identical.

If we take the limit with $t=2Bz$, we find

$$d^2G_n/dz^2-(n-1)z^{-1}dG_n/dz+\Lambda G_n/2Bz^2(1+z)=0. \quad \text{(C-5)}$$

Matching the solutions of (C-4) and (C-5) we find $\Lambda\sim\frac{1}{2}Bn^2$, which is consistent with the result inferred from Eq. (C-1).