Properties of Bethe-Salpeter Wave Functions

G. C. WICK
Carnegie Institute of Technology, Pittsburgh, Pennsylvania
(Received June 30, 1954)

A boundary condition at $t=\pm\infty$ (t being the "relative" time variable) is obtained for the four-dimensional wave function of a two-body system in a bound state. It is shown that this condition implies that the wave function can be continued analytically to complex values of the "relative time" variable; similarly the wave function in momentum space can be continued analytically to complex values of the "relative energy" variable p_0. In particular one is allowed to consider the wave function for purely imaginary values of t, or respectively p_0, i.e., for real values of $x_4=ict$ and $p_4=ip_0$. A wave equation satisfied by this function is obtained by rotation of the integration path in the complex plane of the variable p_0, and it is further shown that the formulation of the eigenvalue problem in terms of this equation presents several advantages in that many of the ordinary mathematical methods become available.

In an especially simple case ("ladder approximation" equation for two spinless particles bound by a scalar field of zero rest mass) an integral representation method is presented which allows one to reduce the problem exactly (and for arbitrary values of the total energy of the bound state) to an eigenvalue problem of the Sturm-Liouville type. A complete set of solutions for this problem is obtained in the subsequent paper by Cutkosky.

1. INTRODUCTION

THE formulation of a completely relativistic wave equation for two-body systems[1] has, in a certain sense, solved a long-standing problem of quantum mechanics. The natural and simple way in which relativistic invariance is achieved is, of course, very real progress, which may lead one to hope that the main features of the equation are more permanent than the solidity of its present field theoretic foundation might suggest. Furthermore, it is hardly necessary to recall that the usefulness of the equation has been amply demonstrated in several high-precision calculations of energy levels.[2]

Nevertheless, it is generally recognized that several serious and valid doubts remain about the significance and the self-consistency of the equation. Some of these doubts, of course, stem from the remaining unresolved convergence questions of renormalized quantum electrodynamics (and other similar theories). It goes without saying, however, that these deeper questions lie entirely beyond the scope of the present investigation[3] The questions and doubts we shall be concerned with[4] arise at a less formidable level; they have to do with the several unfamiliar features of the equation itself.

These are (and the list is probably incomplete):

(a) The appearance of a relative time (or respectively a relative energy) variable, the physical role of which is not entirely clear; in particular, it is admitted that the boundary conditions on the wave function for infinite values of the relative time have not been adequately formulated.

(b) The presence of strong singularities in the interaction kernel, to be avoided by special prescriptions. Standard mathematics has practically nothing to say about integral equations of this type. In particular, the prescriptions referred to imply properties of analyticity, about which one would like to know a lot more.

(c) The absence of a positive-definite norm for the wave function and of any orthogonality theorem.

(d) The fact that when the coupling constant λ is set equal to zero, the equation admits obviously improper solutions. Notwithstanding all that can be said about it, this feature is a little disturbing. It is connected to the other feature that the "order" of the differential operator in the equation is higher than that of the corresponding one-body equation. This leads to the expectation that the equation may have "too many" solutions. On the other hand, circumstance (b) has led some authors to suspect that there are no solutions at all!

(e) Finally, as explained by Goldstein,[4] we are faced with the paradoxical circumstance that, owing to the nonrelativistic perturbation approach employed, the highly successful numerical results obtained do not really offer any direct clue as to the actual properties of the relativistic equation.

The investigation described in the following pages was aimed at throwing some light on these questions. It really consists of two quite different lines of attack. The first of these starts from the remark (Sec. 2) that an additional condition for the Bethe-Salpeter (B-S) wave function follows from its definition[5] supplemented by simple stability requirements. From this, then, some unexpected consequences can be derived about

[1] E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951); J. Schwinger, Proc. Natl. Acad. Sci. **37**, 455 (1951). Other closely related but more general relativistic schemes recently developed by various authors will not be discussed here.

[2] E. E. Salpeter, Phys. Rev. **87**, 328 (1952); R. Karplus and A. Klein, Phys. Rev. **87**, 848 (1952).

[3] In particular, expressions such as "the general structure" of the equation, "the analytic properties" of the interaction kernel, etc., will be used on the assumption that such properties may be inferred correctly from truncated expressions of finite order in the coupling constant, for example, from the lowest-order ("ladder") approximation.

[4] See, especially, J. S. Goldstein, Phys. Rev. **91**, 1516 (1953).

[5] M. Gell-Mann and F. Low, Phys. Rev. **84**, 350 (1951).

the analytic continuation of the wave function to complex values of the relative time (or relative energy) variable. As far as we can tell these properties cannot be obtained from the B-S equation itself. Vice versa, they can be used (Sec. 3) to transform the equation, by rotation of the integration path in the complex plane, to an equation in which $x_4=ix_0$ (respectively $p_4=ip_0$) is real. While the concept of an imaginary relative time variable does not help physical intuition, it has mathematically several advantages. A discussion of the eigenvalue problem in terms of the transformed equation will be given (Sec. 4), and the existence of solutions will be shown to follow, under fairly general assumptions, from considerations similar to those commonly employed in the nonrelativistic case. No claim of completeness or rigor is made for this "proof." Finally in Sec. 5 we shall merely itemize various approximation methods that have been studied, but will be reserved for another publication.

The second line of attack (Sec. 6), which is the subject of a more extensive investigation in the subsequent paper by Cutkosky,[6] is rather different in nature. It is an attempt to make much more specific statements about the exact solutions of the equation, by restricting the character of the equation to an especially simple type. It has not been possible so far to extend this approach to any case of real practical interest. But the fact that in one case, which is not entirely artificial, one can get a complete picture of all the solution (as is shown more completely in the following paper[6]) is not perhaps devoid of general interest. In particular the presence of "abnormal" solutions, which do not possess a nonrelativistic limit, and the circumstances under which they occur may well give a qualitative indication as to properties that will occur also in the cases of real physical interest.

2. THE STABILITY CONDITIONS

The relativistic wave function $\chi(x)$ for a system of two particles, a and b, bound together in a state $|\alpha\rangle$ is defined[5] as the matrix element, between α and the "true" vacuum state $|0\rangle$, of the time ordered product of the Heisenberg field operators ψ_a and ψ_b describing the two kinds of particles. If, for example, the relative time $t=t_a=t_b$ is positive,

$$\chi(x)=e^{-iP\cdot X}\langle 0|\Psi_a(x_a)\Psi_b(x_b)|\alpha\rangle, \tag{1}$$

where $x=x_a-x_b$, $X=(m_ax_a+m_bx_b)/(m_a+m_b)$, and $P\cdot X$ is the four-dimensional scalar product of X with the total momentum P of the system in state α. If for simplicity we assume that the compound system is at rest, then $P=(\mathbf{0}, iE)$, E being the total energy. For a bound state,

$$E=m_a+m_b-B<m_a+m_b. \tag{2}$$

Now the matrix element in (1) can be written

$$\sum_n\langle 0|\Psi_a(x_a)|n\rangle\langle n|\Psi_b(x_b)|\alpha\rangle. \tag{3}$$

The sum extends in principle over all states, but in fact the states n giving a nonzero contribution will belong to a rather special class. Consider for example the case where a and b are an electron and proton, respectively. If Ψ_a and Ψ_b were noninteracting fields, it is obvious that only one-electron states would have to be considered in the sum (3). In the presence of interaction, the states n may also contain photons, electron-positron pairs and proton-antiproton pairs. But at any rate the fundamental integrals of the motion N_a (number of electrons − number of positrons) and N_b (number of protons − number of antiprotons) must have the same values,

$$N_a=1, \quad N_b=0, \tag{4}$$

as the one-electron states. This may be rigorously shown from the commutation properties of N_a and N_b with the field operators, $(N_a+1)\Psi_a=\Psi_aN_a$, etc.

In a similar manner, one can show that the total angular momentum quantum number J for a state n, when measured in a system of reference in which the total momentum $\mathbf{p}$ is zero, must be equal to $\frac{1}{2}$.

Now all states known to us in nature, and satisfying condition (4), also satisfy the inequality,

$$E_n^2-\mathbf{p}^2\geqslant m_a, \tag{5}$$

E_n and $\mathbf{p}$ being the total energy and momentum in the state n. Furthermore, the equality sign holds true only for one-electron states.

The inequality (5) means that among all the states having the same values of the fundamental constants of the motion $\mathbf{p}$, N_a, N_b, etc., as a one-electron state, the latter is the state of lowest energy. We shall refer to (5), therefore, as the stability condition for an electron.

In a similar way, when the relative time t is negative, the wave function χ may be shown to depend on the sum

$$\sum_{n'}\langle 0|\Psi_b(x_b)|n'\rangle\langle n'|\Psi_a(x_a)|\alpha\rangle, \tag{3'}$$

in which the contributing states n' must satisfy the condition,

$$N_a=0; \quad N_b=1, \tag{4'}$$

and hence the inequality,

$$E_{n'2}-\mathbf{p}^2\geqslant m_b, \tag{5'}$$

which shall be called the stability condition for a proton.

Summing up, we have three inequalities (2), (5), and (5'), which will form the basis of the following discussion. It should be pointed out that the above considerations can be extended to other systems. If a and b were a neutron and proton, bound together in the ground state α of the deuteron by a meson field, with the customary assumptions, one would then have, as integrals of the motion, the number of nucleons minus antinucleons N and the total electric charge Q. The states n could be shown to have values $N=1$, $Q=0$ and the states n' the values $N=1$, $Q=1$. In a theory

[6] R. Cutkosky, following paper [Phys. Rev. **96**, 1135 (1954)].

which neglects the β-decay interaction, one has the right to regard both neutron and proton as essentially stable particles. If there were states $n(n')$ not satisfying conditions (5) (5′) the neutron (proton) could decay into those states by emission of photons, without violating any of the known conservation theorems. Thus it is extremely reasonable to postulate that these conditions must again be satisfied.

Now going back to (1) and using (3) with the conditions (2) and (5), we see that for $t>0$, and assuming $P=(\mathbf{0},iE)$, $\chi(x)$ is of the form

$$\chi(x)=\int d\mathbf{p}\int_{\omega_{\min}}^{+\infty} d\omega f(\mathbf{p},\omega)\exp(i\mathbf{p}\cdot\mathbf{x}-i\omega t), \quad (6)$$

where

$$\omega_{\min}=B\mu_a+(m_a^2+\mathbf{p}^2)^{\frac{1}{2}}-m_a>B\mu_a>0, \quad (7)$$

with $\mu_a=m_a/(m_a+m_b)$. Thus, when $t>0$, $\chi(x)$ is a superposition of positive frequency terms only.

Similarly, from (2) and (5′) it follows that, when $t<0$, $\chi(x)$ contains negative frequencies only. Thus we find that $\chi(x)$ has properties with which we are familiar in the case of Feynman propagation kernels. There is, of course, an analogy between the definition of these kernels and Eq. (1).

Let us now consider t as a complex variable. Equation (6) shows that $\chi(x)$ can be continued analytically in the lower half-plane, in the region $0\geqslant\arg t>-\pi$. Similarly starting from the negative real axis, $\chi(x)$ can be continued in the upper half-plane, in the region $\pi\geqslant\arg t>0$. There is, of course, no analytic continuation from one half-plane to the other; the two regions touch one another at one point only, $t=0$.

It should be pointed out that the statements just made are not dependent on the assumption that the state α is bound; they follow from well-known properties of the Laplace transform from the mere fact that ω is finite. If, however, $B>0$ and hence $\omega_{\min}>0$, we can further assert that $\chi(x)\to 0$ when t tends to ∞ in any direction in the lower or upper half-plane different from the real axis. This suggests that the eigenvalue problem may take a more familiar and a simpler form if the wave function and the wave equation are considered on the imaginary t axis (i.e., for $x_4=it$ real).

In order to examine this possibility carefully, it is desirable to go over to momentum space. We write $\chi(x)=\chi_1+\chi_2$, where $\chi_1=0$ for $t<0$ and $\chi_2=0$ for $t>0$. Let us calculate the Fourier transform of χ_1.

$$\phi_1(\mathbf{p},p_0)=(2\pi)^{-4}\int d_3x e^{-i\mathbf{p}\cdot\mathbf{x}}\int_0^{+\infty} dt e^{ip_0t}\chi(x). \quad (8)$$

From (6) one easily finds

$$\phi_1(\mathbf{p},p_0)=\frac{1}{2\pi i}\int_{\omega_{\min}}^{+\infty} f(\mathbf{p},\omega)(\omega-p_0-i\epsilon)^{-1}d\omega, \quad (9)$$

where ϵ is an infinitesimal positive constant. We must assume, of course, that the wave function exists for real values of p_0 [i.e., that the integral (9) converges]. From the theory of Stieltjes transforms, we then infer that (9) defines an analytic function of p_0 in the whole complex plane, in the region

$$2\pi>\arg(p_0-\omega_{\min})\geqslant 0. \quad (10)$$

Similarly ϕ_2 is defined in the region

$$-\pi<\arg(p_0-\omega_{\max})<\pi, \quad (11)$$

where

$$-\omega_{\max}=B\mu_b+(m_b^2+\mathbf{p})^{\frac{1}{2}}-m_b>B\mu_b>0.$$

Thus $\phi(p)=\phi(\mathbf{p},p_0)=\phi_1+\phi_2$ is defined in the complex p_0 plane with two cuts from $\omega_{\min}$ to $+\infty$ and from $-\infty$ to $\omega_{\max}$ (Fig. 1). In this case analytic continuation from the lower to the upper half-plane is ensured through the gap between the two cuts. ($B>0$ is essential for the existence of the gap.) Notice also that the sense of rotation implied by (10) and (11) is the opposite of that in the t plane. From the real p_0 axis one goes continuously into the upper half-plane if $p_0>\omega_{\min}>0$, into the lower half-plane if $p_0<\omega_{\max}<0$.

3. TRANSFORMATION OF THE B-S EQUATION

We shall now use the analytic properties of the wave function to transform the B-S equation by a rotation of the axis of integration in the complex p_0 (respectively x_0) plane.

The equation[1] may be written

$$F_aF_b\phi=I_{ab}\phi, \quad (12)$$

where ϕ is the wave function in momentum space, i.e., the Fourier transform of $\chi(x)$; it is a function of the relative momentum p defined by

$$p_a=\mu_aP+p, \quad p_b=\mu_bP-p, \quad (13)$$

P, μ_a and μ_b being the total momentum and the mass ratios previously defined. F_a and F_b are one-particle propagators, which, if one neglects radiative corrections

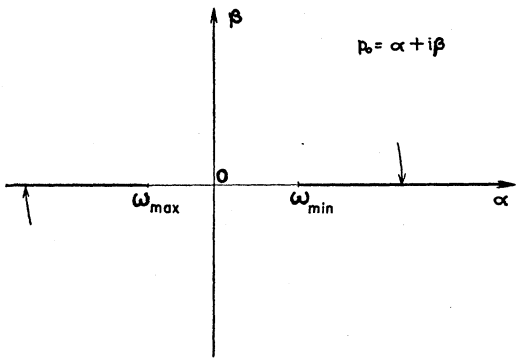

FIG. 1. The complex plane of the variable p_0. The wave function is analytic everywhere, excluding the cuts (heavy lines) on real axis.

reduce to

$$F_a=\gamma_a p_a-im_a, \quad F_b=\gamma_b p_b-im_b, \quad \text{(Dirac particles)} \tag{14}$$

$$F_a=p_a^2+m_a^2, \quad F_b=p_b^2+m_b^2. \quad \text{(Klein-Gordon)}$$

Finally, I_{ab} is the interaction operator, which has different forms, depending on the kind of theory. The following form[4] covers several cases, for the lowest order ("ladder") approximation:

$$I_{ab}\phi(p)=(\lambda/\pi^2)\int \frac{[dk]}{(p-k)^2+\kappa^2}\rho_a\rho_b\phi(k), \tag{15}$$

where $[dk]=idk_0d\mathbf{k}$. The various cases are obtained from the various possible assumptions about the "photon mass" κ, and the factors $\rho_a\rho_b(\rho_a\rho_b=1$, scalar interaction, etc.).

For simplicity we shall carry out the transformation under the assumptions (14), (15), but the proof can be easily generalized to include radiative corrections to any desired order.[7]

Let us consider the right-hand side of Eq. (12), as given by (15). The poles of the interaction kernel are at

$$k_0=p_0\pm[(\mathbf{p}-\mathbf{k})^2+\kappa^2]^{\frac{1}{2}}. \tag{16}$$

Let us carry out the integration over k_0 first. The integration is along the real axis in the plane of the complex k_0 variable, passing just under the cut on the negative axis and above the cut on the positive axis. It is also important to remember that κ in (16) is assumed to have an infinitesimal negative imaginary part, so that the pole with the larger real part lies under the integration path and the pole with the smaller real part above the path. Suppose for instance $p_0>0$, then depending on the relative magnitude of the two terms in (16) the poles will lie as in Fig. 2(a) or 2(b). For simplicity, the cuts of Fig. 1 are not indicated in Fig. 2, but they do not interfere with the following operations. First the integral path may be deformed along the dashed line [there is an *assumption* here, that $\phi(k)$ tends to zero at least like k_0^{-2} when $k_0\to\infty$ in any direction]. Now we move p_0 upwards along a circle so as to end on the positive imaginary axis. In Fig. 2(b) the path need not be changed. In Fig. 2(a) the left pole, around which the path is bent, moves to the left of the imaginary axis, and the path can be straightened. In both cases we end up with p_0 on the positive imaginary axis, and the integral over k_0 along the imaginary axis, from $-i\infty$ to $+i\infty$.

A similar consideration applies when p_0 is on the negative real axis; it then moves to the negative imaginary axis. The net result is a counter-clockwise rotation of the axis on which the wave function is used, on both sides of the equation.

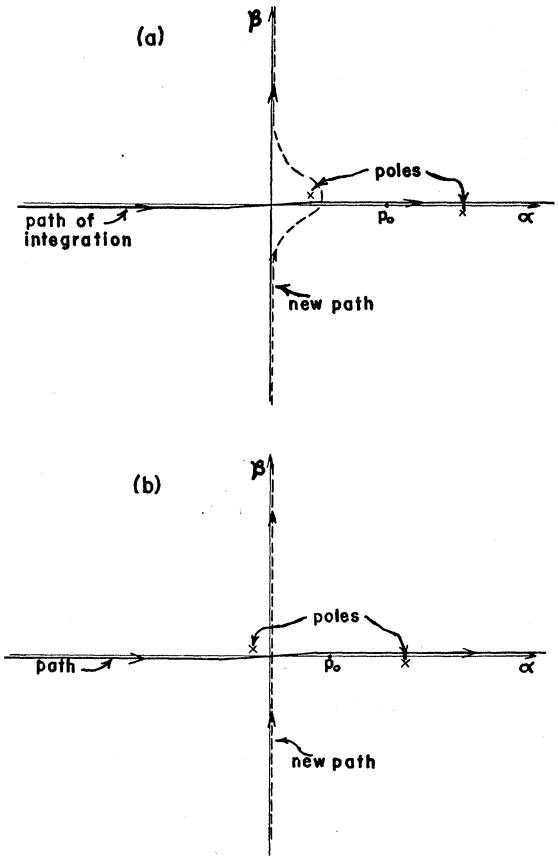

FIG. 2. Integration paths for the variable k_0 in Eq. (15).

Equation (12) is thus reduced to an integral equation in a Euclidean vector space, with the metric

$$p^2=p_1^2+p_2^2+p_3^2+p_4^2. \tag{17}$$

One does not really have to change anything to the equation, except for the understanding that a real vector now has a real component p_4 and that, in Eq. (15)

$$[dk]=dk_1dk_2dk_3dk_4 \tag{18}$$

the integral over k_4 being from $-\infty$ to $+\infty$. The fixed vector $P=(\mathbf{0}, iE)$ is now, of course, regarded as pure imaginary.

One sees at once several advantages of this transformation. The singularities of the interaction kernel [and with them the difficulties mentioned under (b) in the Introduction] are eliminated, and what is equally important, the zeros of the Klein-Gordon factors (14), i.e., the singularities of the inverses F_a^{-1}, F_b^{-1}, have similarly disappeared from the space of real p vectors. Furthermore the symmetry group of the equation is no longer the Lorentz group, but the group

[7] A higher-order term includes, in general, a number of integrations over fourth components k_0, k_0', k_0'', . . . The proof is most easily carried out if all these are regarded as complex variables and their integration paths are rotated simultaneously. An examination of higher-order corrections also requires a closer look at the factors F_a, F_b. The analytical nature of the propagators F_a^{-1}, F_b^{-1} (i.e., of the S_F', Δ_F' functions) is well understood (see reference 14), and it is easy to show that they have no singularities that stand in the way of our transformation.

of real rotations in four dimensions.[8] This is important in the first place, because the group determines the polar variables, which may be used with advantage. In the Lorentz case integrals over a surface $p^2=$const, or $x^2=$const are usually divergent; there are no orthogonality theorems for spherical harmonics, no completeness theorems, etc. Here instead we have the whole familiar machinery at our disposal.

Other advantages appear in the configuration space formulation of the equation, as we shall presently see.

4. DISCUSSION OF THE EIGENVALUE PROBLEM

We shall now examine several cases and show that the transformed equation presents us with an eigenvalue problem, to which many of the ordinary methods and conclusions can be applied.

We shall begin, like Goldstein,[9] with the extreme case $E=0$, where the equation acquires full four-dimensional symmetry in relative momentum space. Unlike Goldstein, however, and for reasons to appear later, we shall choose in Eq. (14) the K.G. (Klein-Gordon) form of the factors F_a and F_b. That is, we assume that a and b have zero spin. The equation for $E=0$ thus has the form

$$(p^2+m_a^2)(p^2+m_b^2)\phi(p)=\lambda\pi^{-2}\int\frac{[dk]}{(p-k)^2+\kappa^2}\phi(k). \quad (19)$$

We shall often use, in the following, the abbreviation $\lambda I_\kappa\phi$ for the right-hand side of (19). In particular I_0 shall designate the interaction operator when the "photon" mass κ is zero.

We can now, of course, separate ϕ, using polar variables, and reduce the problem to a one-dimensional integral equation. If for example ϕ is a function of p^2 only, the integration over angular variables on the right-hand side of (19) is quite elementary. For simplicity we shall write the one-dimensional integral equation for this case only. Let $p^2=s$, $\phi(p)=u(s)$; then

$$(s+m_a^2)(s+m_b^2)u(s)=2\lambda\int_0^\infty tu(t)dt/\{s+t+\kappa^2+[(s+t+\kappa^2)^2-4st]^{\frac{1}{2}}\}. \quad (20)$$

With the further change of variables

$$x=f(s),\quad y=f(t),$$

$$f(s)=\int_0^s ds'/(s'+m_a^2)(s'+m_b^2), \quad (21)$$

$$s^{\frac{1}{2}}(s+m_a^2)(s+m_b^2)u(s)=v(x),$$

Eq. (20) becomes a symmetric integral equation,

$$v(x)=\lambda\int_0^a K(x,y)v(y)dy, \quad (22)$$

with the finite kernel,

$$K(x,y)=2(st)^{\frac{1}{2}}/\{s+t+\kappa^2+[(s+t+\kappa^2)^2-4st]^{\frac{1}{2}}\}, \quad (23)$$

and the finite interval $a=f(\infty)$. Fredholm's theory can then be applied, to conclude that (22) has a discrete eigenvalue spectrum. The case where ϕ is proportional to a four-dimensional spherical harmonic can be similarly handled.

It may be pointed out that if $\kappa=0$, Eq. (20) can be reduced to a second order differential equation either by differentiating twice, or by a parametric representation of the solution. Both methods will be used later, and especially in the subsequent paper by Cutkosky,[6] to obtain more precise information about this case.

Let us now consider briefly Goldstein's Eq. (10), which applies to the case of two Dirac particles. When written in our notation, the equation is quite similar to (19) except that it contains only one quadratic factor in p on the left. Goldstein manages to reduce the equation to the one dimensional form, his Eq. (14), in exact analogy to our Eq. (20); the transformation in the usual frame, however, is far from trivial.[10] Unlike Eq. (20), however, Goldstein's (14) is not reducible to the Fredholm type. The difference in behavior is not an effect of our transformation, but is really due to the different power of p^2 on the left-hand side. The difficulties which Goldstein encounters in defining the eigenvalue spectrum, and which he surmounts by a special cut-off procedure, are thus not a general property of the B-S equation, but rather of the special case considered by him.

For the purpose of obtaining a more general viewpoint, let us now examine the problem in configuration space, i.e., in terms of the function $\chi(x)$. Consider first again the case $E=0$. The Fourier transform of Eq. (19) is

$$[(-\Box+m_a^2)(-\Box+m_b^2)-\lambda V(R)]\chi(x)=0, \quad (24)$$

where the "potential" $V(R)$ is

$$V(R)=4\kappa R^{-1}K_1(\kappa R),\quad R=(x_\mu x_\mu)^{\frac{1}{2}}, \quad (25)$$

K_1 being a modified Hankel function. The expression for $V(R)$ in the case $\kappa=0$,

$$V(R)=4R^{-2}, \quad (25a)$$

also gives the singularity of V at the origin in the general case.

Goldstein's Eq. (10) becomes similarly

$$[\Box-m^2+\lambda V(R)]\chi(x)=0, \quad (26)$$

[8] Four-dimensional rotations must be applied simultaneously, of course, to the relative momentum p and to the total P. If one uses the c.m. system to begin with, so that P is pure imaginary, it will stay pure imaginary after a real rotation. For Dirac particles, a linear transformation of the χ or ϕ function must accompany the rotation; this can be established in the usual way. Contrary to the Lorentz case, however, the transformation here is always *unitary*.

[9] See reference 4. Like Goldstein, we find it convenient, in general, to regard E as given, λ as the eigenvalue to be found.

[10] The author is indebted to Dr. Goldstein for various interesting conversations, and in particular for pointing out to him the peculiar "Euclidean" nature of his Eq. (14). This remark was one of the early motivations for the present study.

which presents a striking analogy to the ordinary three-dimensional Schrödinger equation. With x_4 real, (26) is, of course, an elliptic differential equation. This, together with the boundary condition $\chi(x)\to 0$ at infinity, allows a discussion of the eigenvalue problem along familiar lines.

A special difficulty, also encountered by Goldstein, is presented by the boundary condition at the origin $R=0$, about which we have unfortunately no definite indication from general field-theoretic considerations. The difficulty arises because of the Fuchsian singularity (25a); if the potential were regular everywhere, there would be little doubt that $\chi(x)$ must be regular too.

One can see at once, however, that the singularity of the potential affects (24) and (26) in a very different manner. Consider, for example, spherically symmetric solutions. The radial equation corresponding to (26), or

$$[d^2/dR^2+(3/R)(d/dR)-m^2+\lambda V(R)]\chi=0, \quad (26a)$$

has two solutions near the origin, of the type $\chi=R^\alpha \times(1+c_1R+\cdots)$ with

$$\alpha=-1\pm(1-4\lambda)^{\frac{1}{2}}. \quad (27)$$

Thus, if $\lambda<\frac{1}{4}$, it is possible to make a distinction between the "regular" (less singular) and the "irregular" solution. If $\lambda>\frac{1}{4}$, it seems highly unlikely that a plausible condition to determine the right solution can be found. In the case $\kappa=0$, moreover, the equation can be solved explicitly,[4] the "regular" solution being $R^{-1}J_n(iR)$, where $n=+(1-4\lambda)^{\frac{1}{2}}$. This solution, however, never satisfies the condition at infinity. We thus reach the conclusion that no value $\lambda<\frac{1}{4}$ is an eigenvalue. In our opinion, for $\lambda>\frac{1}{4}$ the eigenvalue problem becomes ill-defined. We shall not try to discuss further here[11] whether the limiting case $\lambda=\frac{1}{4}$ can actually be regarded as an eigenvalue.[4]

In Eq. (24), on the other hand, the singularity (25a) does not affect the indicial equation. The radial equation for a spherically symmetric solution, for example, has four independent solutions near origin, say $\chi_1, \chi_2, \chi_3, \chi_4$, behaving respectively like R^2, R^0, $\ln R$, and R^{-2}. If there were no potential, we would clearly say that the acceptable solution is a linear combination $c_1\chi_1+c_2\chi_2$ of the two "regular" solutions. We shall make the same assumption when there is a potential.[12] Likewise we can define, for large R values, four solutions behaving respectively like $R^{-\frac{3}{2}}\exp(\pm\mu_aR)$ and $R^{-3}\exp(\pm\mu_bR)$. The solution $c_1\chi_1+c_2\chi_2$ will be a linear combination of these four. In order to satisfy the condition $\chi\to 0$ at infinity, two coefficients must be zero; that is, we have two conditions. One of these may be satisfied by a suitable choice of c_1/c_2; the remaining one gives a condition on λ. This will, in general, determine a discrete spectrum of eigenvalues.

We shall see later that for $\kappa=0$ the analysis can be carried much further. Let us now turn to the more interesting general case $E\neq 0$. Let us write (in the c.m. system)

$$P=(\mathbf{0},iE)=i(m_a+m_b)\eta, \quad (28)$$

where η is the four vector

$$\eta=(\mathbf{0},\epsilon), \quad \epsilon=E/(m_a+m_b).$$

Notice that

$$\eta^2=\epsilon^2<1. \quad (29)$$

The factor on the left of Eq. (19) now becomes, remembering (13):

$$(m_a^2+p_a^2)(m_b^2+p_b^2)=p^4+(m_a^2+m_b^2)(1-\eta^2)p^2 +4m_am_b(p\eta)^2+m_a^2m_b^2(1-\eta^2)^2 +2i(m_a-m_b)(p^2-m_am_b)(p\eta). \quad (30)$$

It is at first sight rather puzzling that the equation now contains an imaginary term whose presence depends on m_a being $\neq m_b$. In configuration space this means that the operator corresponding to (30) is self-adjoint only when $m_a=m_b$. One can show that this feature is connected with the time-reversal properties of the equation.

We shall point out, when the occasion arises, the differences produced by the term in m_a-m_b. For the moment, we shall consider only the case $m_a=m_b$ ($=m$, say). The analog of Eq. (24) then is

$$\{[-\Box+m^2(1-\eta^2)]^2-4m^2\eta^2\partial^2/\partial x_4^2\}\chi(x) =\lambda V(R)\chi(x). \quad (31)$$

Since complete separation of variables is impossible, a solution must now be a superposition $\chi=\sum_n f_n(R)Y_n$ of four-dimensional spherical harmonics Y_n of different orders. The radial functions f_n satisfy a system of coupled fourth-order differential equations, and it is no longer possible to discuss the eigenvalue problem in terms of a single radial function. This is a considerable complication, but one may notice, nevertheless, that the term in (31) which produces the coupling is of second order only, so that the indicial equation for each radial function f_n is the same as in Eq. (24). If one writes $f_n(R)=R^\alpha(1+c_1R+\ldots)$ the possible values for α are $\pm n$, $\pm(n+2)$; we may assume that only the positive values are allowed in a "regular" solution, just as in Eq. (24). Thus there is no qualitative difference between the two equations, with regard to the behavior of solutions near $R=0$.

The asymptotic behavior of $\chi(x)$ at infinity, on the other hand, is more interesting. It will be shown below that when x tends to infinity, χ behaves asymptotically like $\exp[-R\varphi(\theta_4)]$, i.e., it tends to zero exponentially but with a coefficient depending on the direction, more specifically on the angle θ_4 with the "4" axis. For our present purpose, however, it is only interest-

[11] It may be remarked that in reference 6 Goldstein's eigenvalue is also obtained from Eq. (19) in the limit $m_b/m_a\to 0$ (and $\kappa=0$).

[12] One can argue that $\chi\sim\ln R$ is not really a solution of (24) since it gives an additional term $\sim\delta_4(x)$. $\chi\sim R^{-2}$ gives a term $\Box\delta_4(x)$.

ing to notice that $\varphi(\theta_4)$ has a positive lower limit $\varphi \geqq 1-\epsilon$ so that, in a certain sense, there is again no fundamental difference in behavior between the solutions of (31) and those of (24), and we may expect that in both cases the boundary conditions at $R=0$ and $R=\infty$ will determine a discrete λ spectrum.

The elementary considerations developed previously seemed of interest, because of the analogy with considerations often made with regard to the ordinary Schrödinger equation. In this sense we may say that (31) presents an analogy to the Schrödinger equation for a particle in an asymmetric field, where again the reduction of the eigenvalue problem to a simple one-dimensional Sturm-Liouville problem is not feasible.

In either case, a rigorous discussion of the eigenvalue problem can only be achieved by less elementary means, such as the reduction of the problem to an integral equation. We do not wish to carry out such a study here, but we may point out along what lines it could be carried out.

We already have, of course, in Eq. (19) and its generalization for $E\neq 0$, an integral formulation of the problem. In the case $m_a=m_b$ corresponding to Eq. (31), the equation can be reduced to the real symmetric form

$$\Phi(p)=\lambda\int H(p,k)\Phi(k)[dk], \tag{32}$$

where

$$\begin{aligned}\Phi(p)&=f^{\frac{1}{2}}(p)\phi(p)\\ f(p)&=[p^2+m^2(1-\eta)]^2+4m^2\eta^2p_4^2\\ H(p,k)&=\pi^{-2}[f(p)]^{-\frac{1}{4}}[(p-k)^2+\kappa^2]^{-1}[f(k)]^{-\frac{1}{4}}.\end{aligned} \tag{33}$$

Now by counting powers of p and k it is easy to see that

$$\int H^2(p,k)[dp][dk]<\infty, \tag{34}$$

which together with other similar inequalities, which the mathematically inclined reader can readily discover, may be used to show that (32) is "nonsingular" and thus possesses a discrete λ spectrum. Furthermore all eigenvalues are real. Finally, one can see that the kernel is positive-definite[13] so that $\lambda>0$.

An alternative integral formulation can be obtained as usual in configuration space. In fact, Eq. (31) together with the regularity condition at the origin and the boundary condition $\chi(x)\to 0$ at infinity, can be replaced by an integral equation,

$$\chi(x)=\lambda\int G(x-x')V(R')\chi(x')[dx'], \tag{35}$$

or $\chi=\lambda GV\chi$, where G is the inverse of the differential operator on the left-hand side of (31). When $m_a=m_b$, the function $G(x)$ is even: $G(x)=G(-x)$, so that (35) can be easily symmetrized. The function $G(x)$ is constructed in the Appendix, and it may be seen from Eqs. (A7) and (A8) there that $G(x)$ has a very weak singularity at the origin (it is in fact finite at $x=0$) and tends to zero at infinity like

$$G(x)\sim ge^{-R\varphi(\theta_4)}, \tag{36}$$

where g is a factor which varies slowly compared to the exponential and

$$\begin{aligned}\varphi(\theta_4)&=m(1-\epsilon\cos\theta_4) \quad |\cos\theta_4|>\epsilon\\ &=m(1-\epsilon^2)^{\frac{1}{2}}\sin\theta_4 \quad |\cos\theta_4|<\epsilon.\end{aligned} \tag{37}$$

If $V(R)\to 0$ sufficiently rapidly when $R\to\infty$, the asymptotic behavior of $\chi(x)$ as given by the integral in Eq. (35) will reflect that of $G(x)$, from which the conclusions previously mentioned may be obtained. Incidentally it may be noticed that in the nonrelativistic limit, $\epsilon\approx 1$, the lower form in Eq. (37) covers almost the whole solid angle, and furthermore $\varphi\approx(mB)^{\frac{1}{2}}\sin\theta_4$, $R\varphi(\theta_4)=(mB)^{\frac{1}{2}}r$, where $r^2=x_1^2+x_2^2+x_3^2$. We thus find the typical exponential of the three-dimensional Schrödinger function. It is indeed rather remarkable that in this region, i.e., with the exception of a narrow cone around the x_4 axis, the asymptotic form of $\chi(x)$ is not time-dependent.

In the foregoing discussion we have, perhaps, laid too much stress on the special case of two spinless particles with the special interaction I_κ of Eq. (19). It is clear that none of the conclusions we have reached as to discreteness of the λ-spectrum, etc., must necessarily remain true if we change the propagators F_a, F_b or the interaction kernel.

If, for example, we write the analog of (32) with Dirac propagators, the conclusion that the equation is nonsingular no longer holds true. As pointed out above, Goldstein[4] already met this situation for the special case $E=0$. It is, of course, also possible to formulate the problem in a form similar to (35), namely,

$$\chi(x)=\lambda\int G_D(x-x')V(R')\chi(x')[dx'], \tag{38}$$

where

$$\begin{aligned}G_D(x)=&[\gamma_a(\partial/\partial x)-m_a(1+\gamma_a\eta)]\\ &\times[\gamma_b(\partial/\partial x)+m_b(1+\gamma\eta)]G(x).\end{aligned} \tag{39}$$

In this case the singular character of the equation comes about because $G_D(x)$ has a much stronger singularity than $G(x)$, near $x=0$. When this is combined with the $1/R^2$ singularity of $V(R)$ [Eqs. (25) and (25a)], Eq. (38) becomes singular. This does not mean that discrete eigenvalues of λ will not exist, but only that a much more detailed study of the equation will be necessary. One could, of course, also consider the possibility of less singular potentials $V(R)$, in which case the general theory of integral equations might again be applicable.

It seems pointless at present to investigate in detail such possibilities. One will bear in mind, however, that within the framework of our transformed system of

[13] G. C. Wick, Nuovo cimento (to be published).

coordinates, such questions can be attacked by ordinary mathematical methods.

5. APPROXIMATION METHODS

It is also possible to show that our transformed equation has several advantages if one wants to employ approximate methods of solution. We have in mind, in particular: (a) a perturbation expansion in the neighborhood of $E=0$ (see also reference 4), (b) variational principles, (c) nonrelativistic approximations, without special restrictions as to the form of $V(R)$. These questions will be discussed in a paper which the author hopes to present shortly in another periodical.[13]

6. EXACT SOLUTIONS FOR $\kappa=0$

A comparison of Eqs. (25) and (25a) suggests that the problem of solving the B-S equation exactly may be far more elementary in the latter ($\kappa=0$) case. This is borne out by Goldstein's solution[4] for Eq. (26), and we shall see in a moment that also Eq. (24) has quite simple solutions if $\kappa=0$ and $m_a=m_b$. And, of course, one will remember that the ordinary nonrelativistic Schrödinger problem is far more elementary with a Coulomb than with a Yukawa potential.

At first, however, one would regard this analogy as encouraging only for the special case $E=0$, when the B-S equation is separable. We were, therefore, quite surprised when we first realized that for $\kappa=0$ even the nonseparable Eq. (31) can be reduced to a one-dimentional integral equation, or alternatively to a one-dimensional eigenvalue problem of the Sturm-Liouville type. We shall explain the basic idea for the simplest type of solution and for $m_a=m_b$ only. The extension to other cases was carried out by Cutkosky and is described in the accompanying paper.

Choosing $m_a=m_b$ ($=m$, say), let us first examine the separable case, Eq. (24). In momentum space, the equation has the form

$$(p^2+m^2)^2\phi(p)=\lambda I_0\phi(p), \tag{40}$$

which is very similar to the nonrelativistic hydrogen equation in momentum space. The latter, of course, is a three-dimensional equation and does not have the square power on the left, but it will appear that the analogy is closest when the two changes are made simultaneously.

In particular, the ground-state wave function of hydrogen: $\phi(p)=(p^2+p_0^2)^{-2}$, is duplicated here by the solution

$$\phi(p)=(p^2+m^2)^{-3} \tag{41}$$

corresponding to the eigenvalue $\lambda=2m^2$. That (41) satisfies Eq. (40) can be verified most easily if one first writes, à la Feynman:

$$(k^2-2k\cdot p+p^2)^{-1}(k^2+m^2)^{-3}=\int_0^1 3(1-x)^2dx \times[(k-xp)^2+(1-x)(m^2+xp^2)]^{-4}. \tag{42}$$

One then finds easily that

$$I_0\phi=(1/2m^2)(p^2+m^2)^{-1}, \tag{43}$$

showing that Eq. (40) is satisfied.

More generally, one can see that I_0 applied to $(p^2+2p\cdot q+M^2)^{-3}$, where M^2 and the vector q are constants, gives $(p^2+2p\cdot q+M^2)^{-1}$, apart from a proportionality factor. This peculiar self-reproducing property of a quadratic form in p, under the operation I, is characteristic of the case $\kappa=0$.

Consider now the equation for $E\neq 0$. For simplicity let $m=1$ from now on. The equation is

$$[p^2+2ip\cdot\eta+1-\eta^2][p^2-2ip\cdot\eta+1-\eta^2]\phi=I_0\phi. \tag{44}$$

Clearly ϕ cannot be a function of p^2 alone; it must be at least a function of p^2 and $p\cdot\eta$ (for an S state). The above considerations suggest that we may be able to generalize solution (41) by writing ϕ as a superposition of terms[14] of the type $(p^2+2p\cdot q+M^2)^{-3}$ where q is parallel to η, say, $q=iz\eta$. That is

$$\phi(p)=\int dzdM^2g(z,M^2)[p^2+2izp\cdot\eta+M^2]^{-3}. \tag{45}$$

One then sees immediately that

$$I_0\phi=\tfrac{1}{2}\int dzdM^2g_1(z,M^2)[p^2+2izp\cdot\eta+M^2]^{-1}, \qquad g_1(z,M^2)=g(z,M^2)/(M^2+z^2\eta^2). \tag{46}$$

Inserting on the right of (44) and dividing by the two quadratic factors on the left, one then tries to reduce the result again to the form (45) by reassembling the three quadratic denominators into a cube [in a similar way as in Eqs. (42) and (A3) in the Appendix]. One sees at once that if $M^2=1-\eta^2$ the "mass term" reproduces itself. Thus we set

$$g(z,M^2)=g(z)\delta(M^2-1+\eta^2). \tag{47}$$

Carrying out the transformations indicated above and writing

$$Q(z)=1-\eta^2+z^2\eta^2, \tag{48}$$

we find

$$\phi(p)=\tfrac{1}{2}\lambda\int Q^{-1}(z)g(z)dz\int_{-1}^{+1}dy\int_0^1 \times xdx[p^2+2i\zeta p\cdot\eta+1-\eta^2]^{-3}, \qquad \zeta=xy+(1-x)z. \tag{49}$$

[14] An expression of this type has a certain resemblance to the parametric representations for S_F' and Δ_F' developed by M. Gell-Mann and F. E. Low, Phys. Rev. **95**, 1300 (1954). G. Källén [Helv. Phys. Acta **25**, 417 (1952)] has previously used similar representations for other quantities that are a little less closely related to the B-S wave function, Eq. (1). In the case of these quantities, and of the functions $S_F'\Delta_F'$, it is possible as the above-mentioned authors have shown, to derive the general form of the parametric representation from the definition of the quantities, and from considerations of relativistic invariance. The author has not been able to do the same for Eq. (1). Nevertheless the analogy with S_F' and Δ_F' was used to "guess" the form of Eq. (45).

Eliminating y in favor of ζ, and carrying out the integrations over x and z first, (49) acquires indeed the general form required by (45) and (47). Writing that the two expressions are identical gives an integral equation for $g(z)$.

To this end notice that if z in (45) is allowed to vary between -1 and $+1$, ζ will also vary between the same limits. Writing $d\zeta=xdy$ and noting that for given z and ζ,

$$\int dx=R(\zeta,z)=\begin{cases}(1+\zeta)/(1+z) & \text{if } z>\zeta\\ (1-\zeta)/(1-z) & \text{if } z<\zeta,\end{cases} \tag{50}$$

one finds

$$\phi(p)=\int_{-1}^{+1}\gamma(\zeta)[p^2+2i\zeta p\cdot\eta+1-\eta^2]^{-3}d\zeta, \tag{51}$$

where $\gamma(\zeta)$ is given by the right-hand side of Eq. (52) below. The condition $g=\gamma$ thus gives the integral equation

$$g(\zeta)=\tfrac{1}{2}\lambda\int_{-1}^{+1}R(\zeta,z)Q^{-1}(z)g(z)dz. \tag{52}$$

This is, of course, an integral equation of Fredholm's type, and has a discrete λ spectrum. We thus have achieved the surprising result that the B-S equation (44), although nonseparable (as far as we can tell), can be reduced to a one-dimensional problem.

Further Reduction of the Problem

Equations (45), (47), and (52), of course, do not give all the solutions; they do not even give all the S states. The necessary generalizations, however, are natural and will be described in the accompanying paper. Let us instead study (52) a little further. From (50) and (52) one can see that

$$g(+1)=g(-1)=0. \tag{53}$$

Furthermore, differentiating (52) twice, we get

$$g''(z)+\lambda(1-z^2)^{-1}Q^{-1}(z)g(z)=0. \tag{54}$$

These equations formulate the problem as a Sturm-Liouville eigenvalue problem. Thus it is easy to predict qualitatively the dependence of λ on η^2.

Thus consider first $\eta^2=0$; then $g(z)=(1-z^2)$ is a solution, and clearly it corresponds to the lowest eigenvalue since it has no nodes. The higher solutions are also polynomials.[15] The lowest eigenvalue is $\lambda=2$, as we know already. The "potential" $Q^{-1}(z)$ is an increasing function of the parameter η^2. Hence every eigenvalue λ must decrease as η^2 increases.

When $\eta^2\to 1$, $Q^{-1}(z)$ develops a singularity at $z=0$, in fact,

$$Q^{-1}(z)\approx(1-\eta^2+z^2)^{-1}\approx\pi(1-\eta^2)^{-\frac{1}{2}}\delta(z). \tag{55}$$

[15] See the general discussion in reference 6.

The lowest eigenfunction simply develops a kink at $z=0$, while the behavior of the higher states is more complicated; if one inserts the approximation (55) into (52), one finds

$$g(\zeta)=\tfrac{1}{2}\pi(1-\eta^2)^{-\frac{1}{2}}g(0)\lambda(1-|\zeta|), \tag{56}$$

which requires

$$\lambda=(2/\pi)(1-\eta^2)^{\frac{1}{2}}. \tag{57}$$

This is, of course, just what one expects from the nonrelativistic Balmer formula for the lowest eigenvalue.

Clearly the limit $\eta^2\to 1$ requires a more careful treatment for the higher eigenvalues. The reason is that all the nodes of the eigenfunction tend to concentrate near $z=0$ so that the approximation (55) is not adequate.

It is easy to see that λ does not tend to zero for the higher eigenvalues. Thus, none of the higher eigenvalues of Eq. (54) has anything to do with the states known from the nonrelativistic case. It will be shown by Cutkosky[6] that the other known states are contained in other families of solutions of the B-S Equation; each of these families, however, contains in addition "abnormal" solutions that have no nonrelativistic limit.

We shall now examine the behavior of the "abnormal" eigenvalues of Eq. (54) when $\eta\to 1$ and show that all these eigenvalues converge to a common limit $\lambda\to\frac{1}{4}$. First we can see that $\lambda<\frac{1}{4}$ cannot be an eigenvalue other than (57). Consider in fact the second eigenvalue; the corresponding eigenfunction must be odd and have a node at $z=0$. Hence we need only examine a solution of (54) with the boundary conditions $g(0)=g(1)=0$. Assuming

$$1-\eta^2\ll 1, \tag{58}$$

we divide the interval $0-1$ into two parts,

$$0<z<z_0 \quad\text{and}\quad z_0<z<1, \tag{59}$$

choosing z_0 to satisfy

$$(1-\eta^2)^{\frac{1}{2}}\ll z_0\ll 1. \tag{60}$$

In the first interval we write the equation with a slight change of variables,

$$\begin{aligned} x&=\eta(1-\eta^2)^{-\frac{1}{2}}z,\\ d^2g/dx^2+\lambda(1+x^2)^{-1}g&=0,\end{aligned} \tag{61}$$

neglecting terms of order $\leq(I-\eta^2)^{\frac{1}{2}}$. (One can see *a posteriori* that this approximation is justified for our purposes.) Equation (61) is of Riemann's type, and the solution we want is

$$\begin{aligned} g&=g^+-g^-,\\ g^\pm&=(1+x^2)_2F_1(\tfrac{3}{2}+\rho,\tfrac{3}{2}-\rho;2;\tfrac{1}{2}(1\pm ix)),\\ \rho&=(\tfrac{1}{4}-\lambda)^{\frac{1}{2}}.\end{aligned} \tag{62}$$

In the second interval we write $Q(z)=z^2$, again neglecting terms of order $(I-\eta^2)^{\frac{1}{2}}$ at most, and write

$$s=z^2, \quad d^2g/ds^2+\tfrac{1}{2}s^{-1}dg/ds+\tfrac{1}{4}\lambda g/(1-s)s^2=0, \quad (63)$$

which again is of Riemann's type. The solution satisfying $g=0$ at $z=1$ is

$$g=(1-z^2)z^{\frac{1}{2}+\rho}{}_2F_1(1\tfrac{1}{4}+\tfrac{1}{2}\rho, \tfrac{3}{4}+\tfrac{1}{2}\rho; 2; 1-z^2). \quad (64)$$

We will first show that if $\lambda<\frac{1}{4}$, the "internal" and "external" solutions (62) and (64) cannot join smoothly at $z=z_0$, i.e., $x=x_0=\eta z_0(1-\eta^2)^{-\frac{1}{2}}$. In fact, since $x_0\gg 1$, we may evaluate (62) by means of the asymptotic formula for the hypergeometric function. One finds, omitting a proportionality factor,

$$g_{\text{int}}\sim x^{\rho+\frac{1}{2}}(1+\cdots)+A(\rho)x^{-\rho+\frac{1}{2}}(1+\cdots). \quad (62')$$

The dots indicate expansions in powers of x^{-1}, and since $\rho<\frac{1}{2}$ it is consistent to keep the first term of the second expansion, while neglecting the higher terms of the first expansion. Furthermore,

$$A(\rho)=2^{2\rho}\tan(\tfrac{1}{4}\pi-\tfrac{1}{2}\pi\rho)\Gamma(2\rho)/\Gamma(-2\rho) \quad (65)$$

is a negative quantity which varies from 0 to -1 as λ varies from 0 to $\frac{1}{4}$.

Similarly, (64) may be evaluated for small values of z by means of the known transformation of $F(a, b, c, 1-s)$ to hypergeometric functions of the variable s. One finds

$$g_{\text{ext}}\sim z^{\rho+\frac{1}{2}}(1+\cdots)+B(\rho)z^{\rho+\frac{1}{2}}(1+\cdots), \quad (64')$$

where the dots now indicate expansions in powers of z^2, and

$$B(\rho)=2^{2\rho}\Gamma(\rho)\Gamma(\tfrac{3}{2}-\rho)/\Gamma(-\rho)\Gamma(\tfrac{3}{2}+\rho) \quad (66)$$

is a quantity which on the whole interval $0<\lambda<\frac{1}{4}$ $(0<\rho<\frac{1}{2})$ stays quite close to -1 (and is in fact <-1).

Rewriting (62′) in terms of the variable z and omitting again a proportionality factor, we find

$$g_{\text{int}}\sim z^{\rho+\frac{1}{2}}+A(\rho)(1-\eta^2)^{\rho}z^{-\rho+\frac{1}{2}}, \quad (67)$$

which is of the same form as (64′), but with a coefficient for the second term which is smaller than $B(\rho)$ in absolute value, for all values of ρ in the stated interval. Hence (64′) and (67) can never join smoothly. In addition it is easy to verify that the slope g'/g is larger for (67) than for (64′), as one expects if λ is *too low* to be an eigenvalue.

Let us now turn to the case $\lambda>\frac{1}{4}$. One can see that essentially the same formulae will hold, except that ρ will be a pure imaginary, say $\rho=i\sigma$, $\sigma=(\lambda-\frac{1}{4})^{\frac{1}{2}}$. One sees, then, that (64′) and (67) take the respective forms

$$g_{\text{ext}}\sim z^{\frac{1}{2}}\sin(\sigma\ln z+\beta) \quad (64'')$$

and

$$g_{\text{int}}\sim z^{\frac{1}{2}}\sin(\sigma\ln z-\tfrac{1}{2}\sigma\ln(1-\eta^2)+\alpha), \quad (67')$$

where α and β are phases depending on σ, which for small values of σ are of the form

$$\alpha=a\sigma, \quad \beta=b\sigma, \quad (68)$$

a and b being constants, whose precise value we shall not determine.

Obviously (64″) and (67′) can be joined smoothly if $\alpha-\beta-\frac{1}{2}\sigma\ln(1-\eta^2)=n\pi$, where n is an integer. If $1-\eta^2$ is so small that $-\ln(1-\eta^2)\gg 1$, the above equation will have small roots σ so that by using (68),

$$\sigma\equiv(\lambda-\tfrac{1}{4})^{\frac{1}{2}}=n\pi/[a-b-\tfrac{1}{2}\ln(1-\eta^2)]. \quad (69)$$

To an even cruder approximation, one has

$$\sigma\sim-2\pi n/\ln(1-\eta^2); \quad \lambda=\tfrac{1}{4}+[2\pi n/\ln(1-\eta^2)]^2. \quad (70)$$

Equation (70), for $n=0, 1, 2, \cdots$ gives an infinity of eigenvalues all tending to $\lambda=\frac{1}{4}$ when $\eta^2\to 1$. It should be pointed out that these correspond to odd eigenfunctions. In a similar way one can show, however, that the same formula, with $2n$ replaced by $2n+1$, gives the eigenvalues for the even eigenfunctions.

About the possible significance of these "abnormal" solutions we shall not try to speculate here. Since they occur only for finite values of λ $(\lambda\gtrsim\frac{1}{4})$, it would be unwise to assume that they are a property of the complete B-S Equation. Certainly the ladder approximation cannot be trusted to that extent. If the theory is used only for small values of the coupling constant, the abnormal solutions do not exist, in the case we have studied, and no contradiction with known facts can be established. Nevertheless it would seem that these solutions deserve further study.

ACKNOWLEDGMENTS

The present work was begun while the author was a guest of the Institute for Advanced Study, Princeton, New Jersey. The author is happy to acknowledge his indebtedness to the Director of the Institute, Professor J. R. Oppenheimer, for the stimulation and encouragement he derived from a year's stay at the Institute. Various members of the Institute, in particular, Professor F. Dyson, Professor G. Källén, Professor A. Pais, Professor W. Pauli, and Professor R. Jost gave kind encouragement and invaluable criticism. Special thanks are due to Dr. Murray Gell-Mann for suggesting that the analogy discussed in reference 14 might be of help.

APPENDIX

We shall construct here the Green's function $G(x)$, which is a solution of $(p_a^2+m_a^2)(p_b^2+m_b^2)G(x)=\delta(x)$, p_a and p_b being defined by Eqs. (13) and (28), with $p=-i$ Grad. We shall calculate G for the general case $m_a\neq m_b$, since this involves no additional difficulty. Using Fourier transforms, one sees at once that

$$G(x)=(2\pi)^{-4}\int[(p_a^2+m_a^2)(p_b^2+m_b^2)]^{-1}e^{ipx}[dp]. \quad (A1)$$

In the following we use $\frac{1}{2}(m_a+m_b)$ as the unit of mass, setting

$$m_a=1+\Delta, \quad m_b=1-\Delta. \tag{A2}$$

Furthermore we transform, *à la* Feynman,

$$[(p_a^2+m_a^2)(p_b^2+m_b^2)]^{-1}=\tfrac{1}{2}\int_{-1}^{+1}[p,y,\Delta]^{-2}dy, \tag{A3}$$

where

$$[p,y,\Delta]\equiv p^2+2i(y+\Delta)(p\cdot\eta) +(1-\eta^2)(1+2y\Delta+\Delta^2). \tag{A4}$$

Furthermore, applying to $Q\equiv[p,y,\Delta]$ the formula

$$Q^{-2}=\int_0^\infty e^{-\alpha Q}\alpha d\alpha$$

and inserting into (A1), the integration over p may be performed, with the result

$$G(x)=(32\pi^2)^{-1}\int_{-1}^{+1}dy\int_0^\infty \alpha^{-1}d\alpha \times\exp[-\alpha U-\tfrac{1}{4}R^2\alpha^{-1}+(y+\Delta)(x\eta)], \tag{A5}$$

with

$$U=(1+2y\Delta+\Delta^2)(1-\eta^2)+\eta^2(y+\Delta)^2. \tag{A6}$$

Owing to (29), U is positive for $|y|\leq 1$; hence the integral over α in (A5) is always meaningful.

We then find that

$$G(x)=(4\pi)^{-2}e^{\epsilon\Delta x_4}\int_{-1}^{+1}dy \times e^{y\epsilon x_4}K_0(R(1-\eta^2+\Delta^2+2y\Delta+y^2\eta^2)^{\frac{1}{2}}), \tag{A7}$$

where $K_0(z)$ is the modified Hankel function, $i(\pi/2)\times H_0^1(iz)$. The asymptotic behavior of (A7) when $R\to\infty$ in a specified direction (i.e., keeping x_4/R constant) is found noting that $K_0(z)\sim(\pi/2z)^{\frac{1}{2}}e^{-z}$. The exponential part of (A7) is then

$$G(x)\sim\cdots\int dy\exp[-Rf(y)],$$

where

$$f(y)=[(1-\eta^2)(1-y^2)+(y+\Delta)^2]^{\frac{1}{2}}-\epsilon(y+\Delta)x_4R^{-1}.$$

It is easy to see that $f(y)>0$ in the whole interval $-1\leq y\leq+1$. Hence $G(x)$ satisfies the boundary condition $G\to 0$ as $R\to\infty$ in any direction. If y_m is the point in the interval where $f(y)$ is a minimum, then the strongest factor in the asymptotic dependence of $G(x)$ is

$$G(x)\sim\exp[-Rf(y_m)]. \tag{A8}$$

Notice that y_m depends on the direction. Consider, for example, the simplest case $\Delta=0$. Then if $|x_4|<\epsilon R$, y_m is defined by the minimum condition

$$y_m\epsilon R=x_4(1-\epsilon^2+y^2\epsilon^2)^{\frac{1}{2}}; \tag{A9}$$

that is, writing $x_4/R=\cos\theta_4$, $y_m\epsilon=(1-\epsilon^2)^{\frac{1}{2}}\cot\theta_4$. If $|\cos\theta_4|>\epsilon$ the root (A9) is not inside the interval, so the minimum of $f(y)$ occurs at $y=\pm 1$, according as $\cos\theta_4\gtrless 0$; summarizing, one has

$$\begin{aligned} &|\cos\theta_4|>\epsilon \quad f(y_m)=1-\epsilon|\cos\theta_4| \\ &|\cos\theta_4|<\epsilon \quad f(y_m)=(1-\epsilon^2)^{\frac{1}{2}}\sin\theta_4. \end{aligned} \tag{A10}$$

Notice that in the latter case,

$$G(x)\sim\exp[-(1-\epsilon^2)^{\frac{1}{2}}r\sin\theta_4]=\exp[-(1-\epsilon^2)^{\frac{1}{2}}r],$$

if r is the length of the space component of x. In the former case, instead, $G(x)\sim e^{-R+\epsilon|x_4|}$; in particular, in the time direction $G(x)$ tends to zero like $\exp[-(1-\epsilon)\times|x_4|]$.