Quantum Effects in the Interaction between Electrons and High-Frequency Fields. I

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The passage of an electron through a cavity resonator is considered for the purpose of studying the appearance of quantum mechanical effects in an essentially classical experiment as the frequency of the electromagnetic field becomes high. Both the electron and the field must be represented, initially, by wave packets—in space and momentum coordinates for the former, and in electric field and vector potential coordinates for the latter—in order to correspond to classically meaningful systems. The velocity of the electron at the output of the cavity is investigated. It is found that the expectation value of velocity has a quantum-mechanical correction term. A more significant effect, however, is the nonvanishing of the expectation value of the square of the deviation from the expectation value (mean square deviation from the mean) of the velocity. This deviation is a random phenomenon which will produce noise in an electron beam. An expression for the mean square deviation, in which the results of electron and field quantization are separately apparent, is derived, and its significance is discussed. An order-of-magnitude calculation is made for the ratio of minimum possible mean square deviation of velocity to velocity increment due to the field, and the frequency for which this ratio becomes unity is calculated for a particular set of conditions to be of the order of 3×10^{12}/sec.

THE trend, in recent years, toward the generation of higher and higher frequencies by means of microwave generators, has brought about the following questions: How high must the frequency be in order that quantum mechanical effects become apparent, and what will these effects be? Several authors[1–4] have discussed these questions, but, as has been pointed out elsewhere,[5] in a largely unsatisfactory manner. The purpose of the present paper is to analyze a problem the solution of which will help answer the broad questions just raised. Additional pertinent problems will be treated subsequently.

An essential aspect of many microwave generators and amplifiers is the passage of an electron beam through an oscillating cavity, resulting in a velocity modulation of the beam. The beam is then examined as it passes an analyzer, perhaps another cavity. The effect on the analyzer depends on the velocity of the electron after it has passed through the first cavity. We, therefore, consider the problem of an electron passing through an oscillating cavity, and investigate its velocity.

Due to the fact that in a quantum mechanical description of a system all dynamical variables cannot be specified exactly, there is a large variety of physically different ways in which to describe a system in quantum mechanics, the proper choice depending on the preparation of the system. Thus, a moving particle may be described by a plane wave or by one of a variety of wave packets. Similarly, the state of the radiation oscillators in the cavity can be described by means of one of many types of wave functions of the vector potential or electric field coordinates. A problem analogous to the one of selecting the proper wave function to describe a simple system does not ordinarily exist in classical mechanics, where it is assumed that all dynamical variables may be specified exactly, so that no guidance can be found in the classical treatment.

The only type of state which has any meaning in classical mechanics is one for which there is a wave packet of reasonably small width for every dynamical variable, since the preparation of the system in a classical experiment is such that, in principle, it must be possible to measure all dynamical variables. States which accurately describe systems in classical experiments must, consequently, be such wave packets. If, therefore, we are looking for quantum effects in an essentially classical experiment, we must represent the system by wave packets, both for the electron and the field. It becomes apparent that the quantum effects will be due to the necessarily finite widths of the wave packets.

Since our interest lies in those situations in which quantum mechanical effects first begin to appear, and are therefore small, we will retain only first order terms describing these effects. Also, we will regard the interaction between the electron and the field as a small perturbation. The analysis will be nonrelativistic.

Let the electric and magnetic fields of the cavity be given, respectively, by

$$\mathbf{E}=-4\pi c\mathbf{P}, \quad \mathbf{H}=\nabla\times\mathbf{A}. \tag{1}$$

Then, in the usual way,[6] we expand

$$\mathbf{A}=\sum_j q_j\mathbf{u}_j(\mathbf{r}), \quad \mathbf{P}=\sum_j p_j\mathbf{u}_j(\mathbf{r}), \tag{2}$$

where the subscript j refers to the jth normal mode of the cavity, $\mathbf{u}_j(\mathbf{r})$ is a normalized function describing the spatial dependence of the field,[7] and q and p are the

[1] L. P. Smith, Phys. Rev. **69**, 195 (1946).
[2] D. Gabor, Phil. Mag. **41**, 1180 (1950).
[3] C. Shulman, Phys. Rev. **82**, 116 (1951).
[4] J. C. Ward, Phys. Rev. **80**, 119 (1950).
[5] I. R. Senitzky, Phys. Rev. **91**, 1309 (1953); in particular, reference 7.

[6] See, for instance, L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949), Sec. 50.
[7] The function, $\mathbf{u}_j(\mathbf{r})$, is a solution of the equation, $\nabla^2\mathbf{u}+(\omega/c)^2\mathbf{u}=0$, subject to the conditions $\mathbf{n}\times\mathbf{u}=0$ on the boundary of the cavity, $\int|\mathbf{u}|^2d^3\mathbf{r}=1$, and $\mathrm{div}\mathbf{u}=0$. The origin of the coordinate system lies outside of the cavity and is farther from it than the initial width of the electron wave packet.

coordinate and momentum operators of the radiation oscillators, satisfying the commutation relationship $[q_j,p_j]=i\hbar$, all other pairs commuting. We are looking for an initial state of the field for which both the field strength and energy are approximately specified. If the field is left undisturbed, we want these quantities to continue to be specified with the same degree of approximation. If, in addition, we require that the product of the uncertainties in q and in p be a minimum, then the main features of the state of the field are essentially determined,[8] and we proceed to describe it.

In the p representation of the Schrödinger picture, we specify the initial state of the field for the jth mode by the wave function

$$\varphi_j(p_j,0)=\left(\frac{4c^2}{\hbar\omega_j}\right)^{\frac{1}{4}}\exp\left[-\frac{4\pi c^2}{\hbar\omega_j}\left(\tfrac{1}{2}p_j{}^2-\frac{i\delta_{js}E_0}{4\pi c}p_j\right)\right], \quad (3)$$

where E_0 is a nonvanishing constant, and the sth mode is the oscillating one. If the field is not subjected to any external influence, the absolute value of the wave packet described by Eq. (3) remains unchanged for the non-oscillating modes, and oscillates sinusodially about the origin without change in shape with frequency $\omega_s/2\pi$ and amplitude $E_0/4\pi c$ for the oscillating mode; that is, it can be shown[9] that

$$|\varphi_j(p_j,t)|^2=\left(\frac{4c^2}{\hbar\omega_j}\right)^{\frac{1}{2}}\times\exp\left[-\frac{4\pi c^2}{\hbar\omega_j}\left(p_j+\frac{\delta_{js}E_0}{4\pi c}\sin\omega_j t\right)^2\right]. \quad (4)$$

Since $|\varphi_j(p_j,t)|^2$ is the probability of finding a given p at time t, we see that the most probable electric field is just the classical one, $\delta_{js}E_0\mathbf{u}_j(\mathbf{r})\sin\omega_j t$, and the uncertainty in magnitude of the field is determined by the Gaussian distribution of Eq. (4). In the Heisenberg representation, the state of the oscillating mode of the undisturbed field is described by the vector of which the nth component is given by

$$A_n{}^{(s)}=\xi^{\frac{1}{2}n}e^{-\frac{1}{2}\xi}(n!)^{-\frac{1}{2}},$$

where $\xi=E_0{}^2(8\pi\hbar\omega_s)^{-1}$. For the non-oscillating modes, we have $A_n{}^{(j)}=\delta_{n0}$, $j\neq s$. The probability of finding in the cavity an energy $n\hbar\omega_s$, not counting the zero point energy, is given by $|A_n{}^{(s)}|^2$. This is a Poisson distribution, with maximum and mean at $n=\xi=E_0{}^2/8\pi\hbar\omega_s$. We see, thus, that the instantaneous field strength and the energy are both approximately specified by their classical values. This is the closest one can come to a classically meaningful field.[10]

For the state of the electron, we want a wave packet for which both position and momentum are approximately specified. Here, however, we cannot require that the degree of approximation remain constant if the electron is left undisturbed, for the width of a free electron wave packet in coordinate space changes with time. The best we can do is assume a wave packet for which the product of the uncertainties in position and in momentum is a minimum at the initial time $t=0$ and for which both uncertainties are reasonably small. We take the initial state of the electron to be described by the wave packet[11]

$$\varphi(\mathbf{r},0)=b^{-\frac{3}{2}}\pi^{-\frac{3}{4}}\exp(-r^2/2b^2+ik_0x), \quad (5)$$

where b and k_0 are constants related to the width of the wave packet and its velocity, respectively. If the electron is left undisturbed, the probability of finding it at the point $\mathbf{r}$ at time t is given by[12]

$$|\varphi(\mathbf{r},t)|^2=b^{-3}\pi^{-\frac{3}{2}}\left[1+\left(\frac{ht}{mb^2}\right)^2\right]^{-\frac{3}{2}}\times\exp\left[\frac{-(\mathbf{r}-\mathbf{v}_0t)^2}{b^2+(ht/mb)^2}\right], \quad (5a)$$

where $\mathbf{v}_0$ is the vector $(\hbar k_0/m,0,0)$; and the probability of finding the electron with a momentum $\mathbf{p}$ is independent of the time and is given by

$$|\varphi(\mathbf{p})|^2=b^3\hbar^{-3}\pi^{-\frac{3}{2}}\exp[-(b/\hbar)^2(\mathbf{p}-m\mathbf{v}_0)^2]. \quad (6)$$

We note that both of these probabilities are Gaussian distributions, each with a maximum at the respective classical value.

The initial state for the entire system will now be described by the wave function

$$\varphi(\mathbf{r},p_j,0)=\varphi(\mathbf{r},0)\prod_j\varphi_j(p_j,0).$$

[8] The only arbitrariness left is that which corresponds to the choice of initial phase in the classical field; if we consider the Schrödinger picture of a free field which meets the above requirements, then the state of the field at any instant of time can serve as the initial state. A particular choice of initial time does not reduce the generality of our problem, however, since the entrance time of the electron wave packet is yet to be specified.

[9] Equation (4) and the subsequent expressions for $A_n{}^{(s)}$ may be derived by methods similar to those used in the analysis of the harmonic oscillator, such as those found in reference 6, Sec. 14, for example.

[10] In classical mechanics, the width of the wave packets is initially—and continues to be—zero. The closest approach, therefore, of a quantum mechanical situation to the classical one is that in which the wave packets referring to all the dynamical variables have the least possible width. This will occur when the product of the uncertainties in q and p is a minimum, which is the condition used. Moreover, such a situation will give the least possible correction to the classical result, and from this point of view, too, it may be considered closest to the classical situation. It is thus of particular interest in indicating the fundamental quantum mechanical limitations to the classical phenomena.

[11] We have used a spherically symmetrical wave packet for simplicity. The analysis can easily be generalized to the case where the initial wave packet is specified by

$$(b_1b_2b_3)^{-\frac{1}{2}}\pi^{-\frac{3}{4}}\exp(-x^2/2b_1{}^2-y^2/2b_2{}^2-z^2/2b_3{}^2+ik_0x).$$

Note that the wave packet is initially outside of the cavity.

[12] A. Sommerfeld, *Wellenmechanik* (Frederick Ungar Publishing Company, New York, 1947), p. 166.

By starting with the Hamiltonian,

$$H=\tfrac{1}{2}m^{-1}[\mathbf{p}+(e/c)\textstyle\sum_j q_j\mathbf{u}_j]^2 +\sum_j[2\pi c^2p_j^2+(\omega_j^2/8\pi c^2)q_j^2], \quad (7)$$

and assuming that the electron velocity is sufficiently small so that we can neglect the effect of the magnetic field on the electron as compared to that of the electric field, the equations of motion in operator form in the Heisenberg picture become

$$d^2\mathbf{r}(t)/dt^2=4\pi c(e/m)\textstyle\sum_j p_j(t)\mathbf{u}_j(t), \quad (8)$$

where, by $\mathbf{u}(t)$ we mean $\mathbf{u}(\mathbf{r}(t))$. These, of course, are formally similar to the classical equations of motion, in view of Eqs. (1) and (2). From Eq. (8) we obtain for the velocity at time t,

$$\mathbf{v}(t)=\mathbf{v}(0)+4\pi c\frac{e}{m}\int_0^t dt_1\sum_j p_j(t_1)\mathbf{u}_j(t_1). \quad (9)$$

We can now use Eq. (9) to calculate the expectation value of the velocity at time t. We can also use the same equation to calculate the expectation value of the square of the deviation from the expectation value (mean square deviation) of the velocity at time t. In a purely classical consideration, this deviation would, of course, vanish. The reason that it does not vanish in a quantum mechanical calculation is that the wave packets cannot have infinitesimal width in both coordinate and momentum spaces. This deviation, when interpreted statistically, is a random phenomenon analogous to a noise current, and we see that it is a purely quantum mechanical effect. We now proceed to calculate both the expectation value and the mean square deviation of the x component of velocity.

The Hamiltonian of Eq. (7) is a constant, so that $p_j(t)$ and $\mathbf{u}_j(t)$ can be obtained from the initial operators by the relationships

$$p_j[t]=\exp[(i/\hbar)Ht]p_j(0)\exp[-(1/\hbar)Ht], \quad (10)$$

$$\mathbf{u}_j(t)=\exp[(i/\hbar)Ht]\mathbf{u}_j[\mathbf{r}(0)]\exp[-(i/\hbar)Ht]. \quad (11)$$

Since we are going to substitute from Eqs. (10) and (11) into the perturbation term in Eq. (9), we can approximate by neglecting the interaction term in the exponents of Eqs. (10) and (11). The right sides of these equations can then be evaluated without much difficulty. Equation (10) becomes that for a freely oscillating field, and $p_j(t)$ can be obtained directly from the equations of motion for a free field to be

$$p_j(t)=p_j(0)\cos\omega_j t-(\omega_j/4\pi c^2)q_j(0)\sin\omega_j t. \quad (12)$$

The right side of Eq. (11) may be evaluated by expressing $\mathbf{u}_j(\mathbf{r})$ in terms of its Fourier transform $\mathbf{U}_j(\mathbf{k})$:

$$\mathbf{u}_j(\mathbf{r})=\int d^3\mathbf{k}\mathbf{U}_j(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{r}).$$

The exponential operators can be taken under the integral sign to operate on $\exp(i\mathbf{k}\cdot\mathbf{r})$. Using the relationship,

$$e^Ae^B=e^Be^Ae^{[A,B]}e^{-\frac{1}{2}([A,[A,B]]+[B,[A,B]])}\cdots,$$

where the dots indicate exponentials of higher commutators, we obtain

$$\mathbf{u}_j(t)=\int d^3\mathbf{k}\mathbf{U}(\mathbf{k})\exp\left(\frac{itk^2\hbar}{2m}+i\mathbf{k}\cdot\mathbf{r}\right)\exp\left(\frac{it\mathbf{k}\cdot\mathbf{p}}{m}\right). \quad (13)$$

We can now find the expectation value of the velocity $\langle\mathbf{v}\rangle$. Bringing the expectation value operation under the time integral in Eq. (9), we have

$$\langle\mathbf{v}(t)\rangle=\langle\mathbf{v}(0)\rangle+4\pi c\frac{e}{m}\int_0^t dt_1\sum_j\langle p_j(t_1)\mathbf{u}_j(t_1)\rangle. \quad (14)$$

Since we have neglected the interaction term in the Hamiltonian occurring in Eqs. (10) and (11), we can write

$$\langle p_j(t)\mathbf{u}_j(t)\rangle=\langle p_j(t)\rangle\langle\mathbf{u}_j(t)\rangle. \quad (15)$$

The expectation value of $p_j(t)$ on the right side of Eq. (15) is that for a free field. It can be obtained immediately in the p representation of the Schrödinger picture by making use of Eq. (4). We obtain

$$\langle p_j(t)\rangle=-\delta_{js}(E_0/4\pi c)\sin\omega_j t. \quad (16)$$

This, as was to be expected, is just the classical time-dependence of the field for the oscillating mode.[13] The expectation value for the non-oscillating modes vanishes, of course.

The expectation value of $\mathbf{u}_j(t)$ can be evaluated in the $\mathbf{r}$ representation.[14] Using Eqs. (13) and (5), we have

$$\langle\mathbf{u}_j(t)\rangle=b^{-3}\pi^{-\frac{3}{2}}\int d^3\mathbf{r}\int d^3\mathbf{k} \times\exp\left(\frac{-r^2}{2b^2}-ik_0x\right)\mathbf{U}(\mathbf{k})\exp\left(\frac{itk^2\hbar}{2m}+i\mathbf{k}\cdot\mathbf{r}\right) \times\exp\frac{it\mathbf{k}\cdot\mathbf{p}}{m}\exp\left(\frac{-r^2}{2b^2}+ik_0x\right). \quad (17)$$

If we express the function $\exp(-r^2/2b^2)$, on which $\exp(it\mathbf{k}\cdot\mathbf{p}/m)$ operates, by means of its Fourier transform,

$$\exp(-r^2/2b^2)=b^3(2\pi)^{-\frac{3}{2}}\int d^3\mathbf{k}_1 \times\exp(-k_1^2b^2/2)\exp(i\mathbf{k}_1\cdot\mathbf{r}), \quad (18)$$

[13] The expectation value of the field at a well-defined point, for which we have $\langle\mathbf{u}(\mathbf{r})\rangle=\mathbf{u}(\mathbf{r})$, is just the classical field $E_0\mathbf{u}(\mathbf{r})\sin\omega t$.

[14] The calculation is in the Heisenberg picture. Although it is, perhaps, simpler to calculate $\langle u(t)\rangle$ in the Schrödinger picture, later calculations involving u will require use of the Heisenberg picture, so that for uniformity we use it in the present case, too.

the result of the $\mathbf{p}$ operation may be written down immediately, since $\exp(it\mathbf{k}\cdot\mathbf{p}/m)$ now operates on eigenvalues of $\mathbf{p}$. Integrating then with respect to $\mathbf{r}$ and $\mathbf{k}_1$, we obtain

$$\langle\mathbf{u}_j(t)\rangle=\int d^3\mathbf{k}\mathbf{U}_j(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{v}_0t)\exp[-k^2\delta^2(t)], \quad (19)$$

where

$$\delta^2(t)=(b/2)^2+(ht/2mb)^2.$$

It is to be noted that if the second exponential factor under the integral sign were missing, $\langle\mathbf{u}_j(t)\rangle$ would be equal to $\mathbf{u}_j(\mathbf{v}_0t)$, the classical value. It is this factor which accounts for the quantum mechanical effect. It is interesting to note that $2\delta(t)$ is just the half-width at time t of the free electron probability density, as can be seen from Eq. (5a). The first term in $\delta^2(t)$ is due to the original width of the wave packet, and the second term is due to the spreading of the wave packet. The effect of this exponential on the integral is small if $\mathbf{U}_j(\mathbf{k})$ becomes small for those value of k^2 for which the absolute value of the exponent is no longer small compared to unity. We assume that the absolute value of $k^2\delta^2(t)$ is small compared to unity for the range of k^2 from which significant contributions to the integral arise. Then

$$\begin{aligned}\langle\mathbf{u}_j(t)\rangle&\cong\int d^3\mathbf{k}\mathbf{U}_j(\mathbf{k})[1-k^2\delta^2(t)]\exp(i\mathbf{k}\cdot\mathbf{r})\,|_{\mathbf{r}=\mathbf{v}_0t}\\&=\int d^3\mathbf{k}\mathbf{U}_j(\mathbf{k})[1+\delta^2(t)\nabla^2]\exp(i\mathbf{k}\cdot\mathbf{r})\,|_{\mathbf{r}=\mathbf{v}_0t}\\&=[1+\delta^2(t)\nabla^2]\mathbf{u}_j(\mathbf{r})\,|_{\mathbf{r}=\mathbf{v}_0t}\\&=[1-\delta^2(t)(\omega^2/c^2)]\mathbf{u}_j(\mathbf{v}_0t), \quad (20)\end{aligned}$$

where, in the last step, we have made use of the properties of $\mathbf{u}(\mathbf{r})$ described in footnote 7.

For $\mathbf{v}(0)$, we have

$$\langle\mathbf{v}(0)\rangle=(1/m)\langle\mathbf{p}(0)\rangle+(e/mc)\langle\mathbf{A}(0)\rangle.$$

Since the electron wave packet is initially outside the cavity, $\langle\mathbf{A}(0)\rangle$ vanishes, and $\langle\mathbf{p}(0)\rangle$ is seen immediately from Eq. (6) to be $m\mathbf{v}_0$. From Eqs. (14), (15), (16), and (20) we therefore obtain

$$\begin{aligned}\langle\mathbf{v}(t)\rangle=\mathbf{v}_0&+E_0\frac{e}{m}\int_0^t dt_1\mathbf{u}_s(\mathbf{v}_0t_1)\sin\omega t_1\\&-E_0\frac{e}{m}\frac{\omega^2}{c^2}\int_0^t dt_1\delta^2(t_1)\mathbf{u}_s(\mathbf{v}_0t_1)\sin\omega t_1. \quad (21)\end{aligned}$$

The first two terms on the right side of Eq. (21) are the classical expression for the velocity of an electron, with initial velocity $\mathbf{v}_0$, traversing a cavity with electric field $E_0\mathbf{u}_s(\mathbf{r})\sin\omega t$. The last term gives a first order quantum mechanical correction. We could study this effect in detail. However, the dispersion in velocity, as measured by the mean square deviation in velocity, is a much more significant effect. As far as the production of velocity modulation of an electron beam is concerned, the former effect merely changes the signal amplitude slightly, while the latter effect produces noise.

We therefore consider the expectation value of the square of the deviation from $\langle\mathbf{v}(t)\rangle$. Writing

$$v_x(t)=v_x(0)+\Delta v_x, \quad (22)$$

where, from Eq. (9),

$$\Delta v_x=4\pi c\frac{e}{m}\int_0^t\sum_i u_{jx}(t)p_j(t),$$

we have

$$\begin{aligned}\langle[v_x(t)-\langle v_x(t)\rangle]^2\rangle&=[\langle v_x^2(0)\rangle-\langle v_x(0)\rangle^2]\\&\quad+[\langle(\Delta v_x)^2\rangle-\langle\Delta v_x\rangle^2]\\&\quad+[\langle v_x(0)\Delta v_x+\Delta v_xv_x(0)\rangle-2\langle v_x(0)\rangle\langle\Delta v_x\rangle], \quad (23)\end{aligned}$$

since $v_x(0)$ and Δv_x do not commute. The first square bracket on the right side of Eq. (23) can be evaluated easily. From Eq. (6), we obtain

$$\langle v_x^2(0)\rangle-\langle v_x(0)\rangle^2=\hbar^2/2m^2b^2. \quad (24)$$

We proceed now to evaluate the second square bracket. For simplicity of notation, the x subscript will be dropped from the u's henceforth, with the understanding that u stands for the x component of $\mathbf{u}$, and likewise for the $\mathbf{U}$'s. We have, using the expression for Δv_x from Eq. (22),

$$\begin{aligned}\langle(\Delta v_x)^2\rangle=16\pi^2c^2\frac{e^2}{m^2}&\int_0^t dt_1\int_0^t dt_2\\&\times\sum_i\sum_j\langle u_i(t_1)u_j(t_2)p_i(t_1)p_j(t_2)\rangle.\end{aligned}$$

Neglecting the interaction term in the Hamiltonian of Eq. (11), just as in the case of Eq. (15), the integrand becomes

$$\sum_i\sum_j\langle u_i(t_1)u_j(t_2)\rangle\langle p_i(t_1)p_j(t_2)\rangle.$$

If $i\neq j$,

$$\langle p_i(t_1)p_j(t_2)\rangle=\langle p_i(t_1)\rangle\langle p_j(t_2)\rangle=0,$$

since either the ith or the jth mode, or both, are in their lowest energy states (non-oscillating). We thus have

$$\begin{aligned}\langle(\Delta v_x)^2\rangle=16\pi^2c^2\frac{e^2}{m^2}&\int_0^t dt_1\int_0^t dt_2\\&\times\sum_i\langle u_j(t_1)u_j(t_2)\rangle\langle p_j(t_1)p_j(t_2)\rangle. \quad (25)\end{aligned}$$

We evaluate the factors $\langle u_j(t_1)u_j(t_2)\rangle$ and $\langle p_j(t_1)p_j(t_2)\rangle$ separately. From Eqs. (3) and (12), we have [omitting the argument in $p(0)$ and $q(0)$ for simplicity of nota-

tion]:

$$\langle p_j(t_1)p_j(t_2)\rangle$$
$$=\left(\frac{\beta}{\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}dp_j\exp\left[-\beta\left(\frac{p_j^2}{2}+\frac{i\delta_{js}E_0p_j}{4\pi c}\right)\right]$$
$$\times\{p_j^2\cos\omega_jt_1\cos\omega_jt_2-\beta^{-1}\hbar^{-1}[q_jp_j\sin\omega_jt_1\cos\omega_jt_2$$
$$+p_jq_j\cos\omega_jt_1\sin\omega_jt_2]+\beta^{-2}\hbar^{-2}q_j^2\sin\omega_jt_1\sin\omega_jt_2\}$$
$$\times\exp\left[-\beta\left(\frac{p_j^2}{2}-\frac{i\delta_{js}E_0p_j}{4\pi c}\right)\right],$$

where $\beta=4\pi c^2/\hbar\omega_j$. Making use of the fact that in the p representation q is represented by $i\hbar\partial/\partial p$, we obtain

$$\langle p_j(t_1)p_j(t_2)\rangle=\delta_{js}(4\pi c)^{-2}E_0^2\sin\omega_jt_1\sin\omega_jt_2$$
$$+(8\pi c^2)^{-1}\hbar\omega_j\exp[i\omega_j(t_2-t_1)]. \quad (26)$$

We note that the first term on the right, which is equal to $\langle p_j(t_1)\rangle\langle p_j(t_2)\rangle$, is the classical value, and the second term is of purely quantum mechanical origin. We also note that although the first term vanishes for non-oscillating modes (as it should, being classical), the second term does not.

In order to evaluate $\langle u_j(t_1)u_j(t_2)\rangle$, we use Eqs. (13) and (5), obtaining

$$\langle u_j(t_1)u_j(t_2)\rangle$$
$$=b^{-3}\pi^{-\frac{3}{2}}\int d^3\mathbf{r}\int d^3\mathbf{k}_1\int d^3\mathbf{k}_2U_j(\mathbf{k}_1)U_j(\mathbf{k}_2)$$
$$\times\exp\left(\frac{-r^2}{2b^2}-ik_0x\right)\exp\left(\frac{it_1k_1^2\hbar}{2m}+i\mathbf{k}_1\cdot\mathbf{r}\right)$$
$$\times\exp\left(\frac{it_1\mathbf{k}_1\cdot\mathbf{p}}{m}\right)\exp\left(\frac{it_2k_2^2\hbar}{2m}+i\mathbf{k}_2\cdot\mathbf{r}\right)$$
$$\times\exp\left(\frac{it_2\mathbf{k}_2\cdot\mathbf{p}}{m}\right)\exp\left(\frac{-r^2}{2b^2}+ik_0x\right).$$

As in the case of evaluation involved in Eq. (17), we replace $\exp(-r^2/2b^2)$ on the right of the $\mathbf{p}$ operator by its Fourier integral representation given by Eq. (19), with the variable of integration changed to $\mathbf{k}_3$, say. The $\mathbf{p}$ operation can then be carried through. An integration now with respect to $\mathbf{r}$ and $\mathbf{k}_3$, successively, yields

$$\langle u_j(t_1)u_j(t_2)\rangle$$
$$=\int d^3\mathbf{k}_1\int d^3\mathbf{k}_2U_j(\mathbf{k}_1)\exp(i\mathbf{k}_1\cdot\mathbf{v}_0t_1)U_j(\mathbf{k}_2)$$
$$\times\exp(i\mathbf{k}_2\cdot\mathbf{v}_0t_2)\exp[-k_1^2\delta^2(t_1)]\exp[-k_2^2\delta^2(t_2)]$$
$$\times\exp\{-2\mathbf{k}_1\cdot\mathbf{k}_2\delta^2[(t_1t_2)^{\frac{1}{2}}]\}$$
$$\times\exp[i(\hbar/2m)\mathbf{k}_1\cdot\mathbf{k}_2(t_1-t_2)]. \quad (26a)$$

It is interesting to note that if we omit the last four factors of the integrand, the integral represents the classical value, namely $u_j(\mathbf{v}_0t_1)u_j(\mathbf{v}_0t_2)$. If we omit the last two factors only, we obtain $\langle u_j(t_1)\rangle\langle u_j(t_2)\rangle$. The last two factors of the integrand produce an interference effect between $u_j(t_1)$ and $u_j(t_2)$. The last four factors in the integrand all produce quantum mechanical effects to the extent that they are different from unity in the ranges of $\mathbf{k}_1$ and $\mathbf{k}_2$ which contribute significantly to the integral. As in the case of Eq. (19) for $\langle u_j(t)\rangle$, we assume that these effects are small; that is, that the exponents are small compared to unity in the significant ranges of $\mathbf{k}_1$ and $\mathbf{k}_2$. Applying the same reasoning as used in Eq. (20), we obtain[15]

$$\langle u_j(t_1)u_j(t_2)\rangle\cong\left(1+\left\{\frac{i\hbar(t_2-t_1)}{2m}\right.\right.$$
$$\left.\left.+2\left[\left(\frac{b}{2}\right)^2+\left(\frac{\hbar}{2mb}\right)^2t_1t_2\right]\right\}\nabla_1\cdot\nabla_2\right)$$
$$\times\langle u_j(\mathbf{r}_1)\rangle\langle u_j(\mathbf{r}_2)\rangle\,|_{\mathbf{r}_1=\mathbf{v}_0t_1,\mathbf{r}_2=\mathbf{v}_0t_2},$$

where the subscript on ∇ indicates the coordinate on which it operates. Using Eq. (20) and keeping lowest order terms only, we have

$$\langle u_j(t_1)u_j(t_2)\rangle\cong\left[1+\left(\frac{b}{2}\right)^2(\nabla_1+\nabla_2)^2\right.$$
$$\left.+\left(\frac{\hbar}{2mb}\right)^2(t_1\nabla_1+t_2\nabla_2)^2+\frac{i\hbar(t_2-t_1)}{2m}\nabla_1\cdot\nabla_2\right]$$
$$\times u_j(\mathbf{r}_1)u_j(\mathbf{r}_2)\,|_{\mathbf{r}_1=\mathbf{v}_0t_1,\mathbf{r}_2=\mathbf{v}_0t_2}.$$

We can now evaluate $\langle(\Delta v_x)^2\rangle-\langle\Delta v_x\rangle^2$. From Eqs. (25), (22), and (16), we have

$$\langle(\Delta v_x)^2\rangle-\langle\Delta v_x\rangle^2$$
$$\cong16\pi^2c^2\frac{e^2}{m^2}\int_0^t dt_1\int_0^t dt_2[\sum_j\langle u_j(t_1)u_j(t_2)\rangle$$
$$\times\langle p_j(t_1)p_j(t_2)\rangle-\langle u_s(t_1)\rangle\langle u_s(t_2)\rangle$$
$$\times\langle p_s(t_1)\rangle\langle p_s(t_2)\rangle]. \quad (27)$$

[15] In the previous expansion of the exponential functions [Eq. (20)], we retained terms up to the first power of the exponent, and have considered each term of the exponent to be of the same order; that is $(k\hbar t/2mb)^2$ and $(kb/2)^2$ are of the same order of smallness compared to unity for the significant range of $\mathbf{k}$. In the present case the two terms in the exponent of the real exponential function (the next to the last factor in the integrand of Eq. 26a) which is being expanded are obviously of the same order as the above two terms. The imaginary exponent (of the last factor) must be examined separately, however. We have

$$k_1k_2\hbar(t_2-t_1)/2m\sim k^2\hbar t/2m\sim(k\hbar t/2mb)(kb/2).$$

Since we have considered $(kb/2)\sim(k\hbar t/2mb)$, we see that the absolute value of the imaginary exponent is of the same order as the other exponents.

If we set $\langle p_j(t_1)\rangle\langle p_j(t_2)\rangle \equiv C_j$, $(\hbar\omega_j/8\pi c^2)\exp[i\omega_j(t_2-t_1)] \equiv Q_j$, then, from Eq. (26),

$$\langle p_j(t_1)p_j(t_2)\rangle = C_j + Q_j.$$

C_j is the classical term and Q_j is the quantum mechanical correction. Substituting into Eq. (32), and bearing in mind that $C_j=0$ for $j\neq s$, we obtain

$$\langle(\Delta v_x)^2\rangle - \langle\Delta v_x\rangle^2 = 16\pi^2c^2\frac{e^2}{m^2}\int_0^t dt_1\int_0^t dt_2\{C_s[\langle u_s(t_1)u_s(t_2)\rangle - \langle u_s(t_1)\rangle\langle u_s(t_2)\rangle] + \textstyle\sum_j Q_j\langle u_j(t_1)u_j(t_2)\rangle\}.$$

We retain only first-order quantum mechanical correction terms, so that in the summation term under the integral sign we replace $\langle u_j(t_1)u_j(t_2)\rangle$ by $u_j(\mathbf{v}_0t_1)u_j(\mathbf{v}_0t_2)$. We thus obtain

$$\langle(\Delta v_x)^2\rangle - \langle\Delta v_x\rangle^2 = 16\pi^2c^2\frac{e^2}{m^2}\int_0^t dt_1\int_0^t dt_2\left\{\frac{E_0^2}{8\pi^2c^2}\sin\omega_s t_1\sin\omega_s t_2 \times\left[t_1t_2\left(\frac{\hbar}{2mb}\right)^2+\left(\frac{b}{2}\right)^2+\frac{i\hbar(t_2-t_1)}{4m}\right] \times\nabla_1\cdot\nabla_2 u_s(\mathbf{r}_1)u_s(\mathbf{r}_2)\,|_{\mathbf{r}_1=\mathbf{v}_0t_1,\mathbf{r}_2=\mathbf{v}_0t_2} + \sum_j\frac{\hbar\omega_j}{8\pi c^2}\exp[i\omega_j(t_2-t_1)]u_j(t_1)u_j(t_2)\right\}.$$

The imaginary term drops out in the integration, and we have, after simplification,

$$\langle(\Delta v_x)^2\rangle - \langle\Delta v_x\rangle^2 = \frac{2\pi e^2}{m^2v_0^2}\sum_j\hbar\omega_j\left|\int_{x_1}^{x_2}dx u_j(x,0,0)\exp[i(\omega_j/v_0)x]\right|^2 + 2E_0^2\frac{e^2}{m^2}\left\{\left[\frac{b}{2v_0}\int_{x_1}^{x_2}dx\left(\sin\frac{\omega_s}{v_0}x\right)\nabla u_s(\mathbf{r})\right]^2 + \left[\frac{\hbar}{2mbv_0^2}\int_{x_1}^{x_2}dx\left(x\sin\frac{\omega_s}{v_0}x\right)\nabla u_s(\mathbf{r})\right]^2\right\}_{y=z=0}, \quad (28)$$

where x_1 and x_2 mark the limits of the part of the x axis lying inside the cavity ($u=0$ outside these limits). We now evaluate the last bracket on the right side of Eq. (23). By means of reasoning similar to that used previously, we obtain[16]

[16] The $\partial/\partial x$ in the expression $\langle(\partial/\partial x)u_s(t)\rangle$ operates on everything in front of it, not merely on $u_s(t)$.

$$\langle v_x(0)\Delta v_x\rangle = 4\pi ic\frac{e\hbar}{m^2}\int_0^t dt_1\langle p_s(t_1)\rangle\left\langle\left(\frac{\partial}{\partial x}\right)u_s(t_1)\right\rangle = -4\pi c\frac{e\hbar}{m^2}\int_0^t dt_1\langle p_s(t_1)\rangle\int d^3\mathbf{k}U(\mathbf{k}) \times\exp[ik_xv_0t_1-k^2\delta^2(t_1)]\left(k_0+\frac{i\hbar k_xt_1}{2mb^2}+\tfrac{1}{2}k_x\right).$$

Similarly,

$$\langle\Delta v_xv_x(0)\rangle = 4\pi ic\frac{e\hbar}{m^2}\int_0^t dt_1\langle p_s(t_1)\rangle\left\langle u_s(t_1)\frac{\partial}{\partial x}\right\rangle = -4\pi c\frac{e\hbar}{m^2}\int_0^t dt_1\langle p_s(t_1)\rangle\int d^3\mathbf{k}U(\mathbf{k})\exp(ik_xv_0t_1) \times\exp[-k^2\delta^2(t_1)]\left(k_0+\frac{i\hbar k_xt_1}{2mb^2}-\tfrac{1}{2}k_x\right).$$

Making use of Eqs. (21) and (16), and retaining only the lowest order term, we obtain

$$\langle\Delta v_xv_x(0)\rangle+\langle v_x(0)\Delta v_x\rangle-2\langle v_x(0)\rangle\langle\Delta v_x\rangle = \frac{-E_0e\hbar^2}{m^3b^2v_0^2}\int_{x_1}^{x_2}dx\left(x\sin\frac{\omega_s}{v_0}x\right)\frac{\partial}{\partial x}u_s(x,0,0). \quad (29)$$

Substituting Eqs. (24), (28), and (29) into Eq. (23), we finally get

$$\langle v_x^2(t)\rangle - \langle v_x(t)\rangle^2 = \frac{\hbar^2}{2m^2b^2}\left\{1-\frac{2E_0e}{mv_0^2}\int_{x_1}^{x_2}dx\left(x\sin\frac{\omega_s}{v_0}x\right)\frac{\partial}{\partial x}u_s(x,0,0) + \frac{E_0^2e^2}{m^2v_0^4}\left[\int_{x_1}^{x_2}dx\left(x\sin\frac{\omega_s}{v_0}x\right)\nabla u_s(\mathbf{r})\right]^2_{y=z=0}\right\} + \frac{E_0^2e^2b^2}{2m^2v_0^2}\left[\int_{x_1}^{x_2}dx\left(\sin\frac{\omega_s}{v_0}x\right)\nabla u_s(\mathbf{r})\right]^2_{y=z=0} + \frac{2\pi e^2}{m^2v_0^2}\sum_j\hbar\omega_j\left|\int_{x_1}^{x_2}dx u_j(x,0,0)\times\exp[i(\omega_j/v_0)x]\right|^2. \quad (30)$$

We can see that the first and second terms on the right side of this equation are due to the quantization of the electron, and the third term, the derivation of which has been reported previously,[17] is due to the quantization of the field. The first term may be attributed to both the initial uncertainty in velocity and the spreading of the wave packet subsequent to the time $t=0$; the second term is due to the initial width of the wave packet (or the initial uncertainty in position). The third term contains the effect not only of the oscillating mode but of all the non-oscillating modes. The reason

[17] I. R. Senitzky, Phys. Rev. **90**, 386 (1953).

is that even for the lowest energy modes there is a finite probability of finding a nonzero field. However, the physical meaning of the third term is obscured by the fact that in the absence of a cavity the electron interacts with all the lowest energy modes in free space, and nevertheless the motion of the electron is considered uniform. What we are interested in is the difference between the electron motion in the presence of the cavity and the motion in free space. For the very high frequency modes the apertures of the cavity become large compared to the wavelength; the field (as represented by the function $\mathbf{u}(\mathbf{r})$) leaks out, and does not differ much from the free space field. Therefore our summation should cut off at a certain frequency. Below that frequency, the effect of field quantization in Eq. (30) is small compared to $\hbar^2/2m^2b^2$, and will be neglected, unless there is a resonance effect for a particular mode, that is, unless $u_j(x,0,0)$ has the same periodicity as $\exp[i(\omega_j/v_0)x]$, and the contribution of a particular integral in the summation becomes large. Since there is no resonance effect in free space, this is certainly a real effect due to the cavity and should be taken into consideration.

If we separate the $\partial/\partial x$ terms in the gradients from the other terms in Eq. (30), we can perform an integration by parts on these terms. If, in addition, we assume that the cavity is of such structure that the derivatives of $u(\mathbf{r})$ with respect to y and z vanish at $y=z=0$,[18] we obtain

$$\langle v_x^2(t)\rangle-\langle v_x(t)\rangle^2=\frac{\hbar^2}{2m^2b^2}\left(1-\frac{E_0eu_0}{mv_0\omega_s}A\right)^2+\frac{E_0^2e^2b^2u_0^2}{2m^2v_0^2}B^2+2\pi\hbar\frac{e^2}{m^2}\sum_i\frac{u_0^2}{\omega_j^2}|D_j|^2, \quad (31)$$

where A, B, and D are dimensionless quantities defined by

$$A=\int_{\theta s1}^{\theta s2}d\theta f_s(\theta)[\sin\theta+\theta\cos\theta],$$

$$B=\int_{\theta s1}^{\theta s2}d\theta f_s(\theta)\cos\theta, \quad D_j=\int_{\theta j1}^{\theta j2}d\theta f_j(\theta)e^{i\theta}.$$

We have used the following notation: $\theta_j=\omega_j t$, $u_j(v_0t,0,0)=u_0f_j(\theta_j)$, where $u_0=V^{-\frac{1}{2}}$ and V is the volume of the cavity. The function $f_j(\theta)$ is of the order of unity, since $\mathbf{u}_j(\mathbf{r})$ is normalized over the volume of the cavity.

It is obvious that the mean square deviation in velocity of the electron at the output of the cavity depends on a detailed consideration of the spatial dependence of the field, strength of the field, velocity of the electron, and width of the initial wave packet. In order to obtain an idea of the order of magnitude of the mean square deviation, without going into the many details of a particular experiment, we will introduce some simplifications and assumptions. We consider a case in which there is no resonance interaction between the electron and field. We therefore neglect the quantized field term and assume that A and B are of the order of unity. We also assume that the derivatives of u with respect to y and z vanish, so that we can use Eq. (31). Because of the assumption, made at the beginning of our analysis, that the interaction between electron and field produces a small perturbation, we have $E_0eu_0/mv_0\omega\ll1$.[19] Thus, we obtain (the symbol $\sim$ indicates order of magnitude)

$$\langle v_x^2(t)\rangle-\langle v_x(t)\rangle^2\sim\hbar^2/2m^2b^2+\tfrac{1}{2}E_0^2e^2u_0^2b^2/m^2v_0^2.$$

This, of course, is a function of the initial electron wave-packet half-width b. We can minimize this quantity with respect to b. The result is

$$[\langle v_x^2(t)\rangle-\langle v_x(t)\rangle^2]_{\min}\sim\hbar E_0eu_0m^{-2}v_0^{-1}.$$

The ratio of this expression to the square of the expectation value of velocity increment which the electron undergoes in passing through the cavity is significant, since this ratio is essentially a comparison of noise to signal. From Eq. (21) we have

$$\langle\Delta v_x\rangle\cong E_0\frac{e}{m}\int_0^t dt_1u_s(\mathbf{v}_0t)\sin\omega t=\frac{E_0eu_0}{m\omega_s}\int_{\theta s1}^{\theta s2}d\theta f_s(\theta)\sin\theta\sim E_0eu_0/m\omega_s.$$

We obtain, therefore,

$$\eta\equiv\frac{[\langle v_x^2(t)\rangle-\langle v_x(t)\rangle^2]_{\min}}{\langle\Delta v_x\rangle^2}\sim\frac{\hbar\omega_s^2}{E_0u_0ev_0}.$$

We see that this ratio increases with increase in frequency and decreases with increase in field strength. Rewriting it as $\hbar\omega_s/E_0u_0e(v/\omega_s)$, one notes that it is of the same order of magnitude as the ratio of the energy of a photon to the energy which the electron may receive from the field in a half-cycle. We can also write

$$\eta^2\sim137\pi^2\frac{V}{\lambda^3}\left(\frac{c}{v_0}\right)^2\frac{\hbar\omega_s}{E_0^2/8\pi}. \quad (32)$$

The last factor in this expression is the ratio of the energy of a photon to the most probable energy in the cavity.

It is interesting to consider a numerical example, and find the conditions for which η becomes unity. We assume that the volume of the cavity is of the order of λ^3, and that the initial velocity of the electron is one-tenth that of light. We can then see from Eq. (32) that η becomes unity when the most probable number of photons in the cavity is of the order of 10^5. If, in addition, we make an assumption about the Q of the cavity[20]

[18] This assumption corresponds to the usual experimental arrangement, in which the path of the electron lies where the field is strongest.

[19] This ratio is of the order of the ratio of velocity increment which the electron may obtain in the cavity to the initial velocity.

[20] The unloaded Q of a cavity is defined as $\omega W/\bar{P}$ where W is the energy stored in the cavity, and $\bar{P}$ is the average power loss in the walls. Now (energy stored) $\propto$ (volume) $\propto\lambda^3$. Also, (power loss) $\propto$ (wall area)$\times$(skin depth)$^{-1}\propto\lambda^2\times\lambda^{-\frac{1}{2}}$. Thus $Q\propto\lambda^{\frac{1}{2}}$. We assume

and consider a specific amount of power being fed into the cavity, we can calculate the frequency for which η becomes unity. If we take $Q \sim 10^3\lambda^{\frac{1}{2}}$, where λ is given in centimeters, and consider the case in which 10^{-4} watts is fed into the cavity, then η becomes unity for $\lambda \sim 10^{-2}$ cm. The electron wave packet half-width b which minimizes the mean square deviation in velocity for this wavelength and the above assumptions is of the order of 10^{-4} cm. Thus, the minimum mean square deviation of the velocity becomes comparable to the modulation in velocity, for the particular conditions assumed, when the wavelength is of the order of a tenth of a millimeter.

the same frequency dependence for the loaded Q, which is the one pertinent to the present discussion.

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Penetration of 6-Mev Gamma Rays in Water

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The penetration of 6-Mev gamma rays has been studied out to 190 cm in water. The dose rate has been measured with an anthracene scintillation detector as a function of the distance from the N^{16} source. The results agree closely out to 160 cm with the distribution calculated according to the theory of gamma-ray penetration as developed by Spencer and Fano.

A MEASUREMENT has been made of the broad beam penetration of high-energy (~6 Mev) gamma rays from N^{16} in water. Very little experimental data have been heretofore available on broad beam penetration with simple geometries and none at all in this energy region. The results are also of interest as a check on the theoretical method of calculation of gamma-ray penetration as developed by Spencer and Fano.[1]

The N^{16} source is obtained by circulating demineralized water through the high-flux region of the Materials Testing Reactor and then piping it into a flat cylindrical disk, 30 cm in diameter, which serves as the source. The disk is made of a tightly wound coil of $\frac{1}{4}$-in. i.d. Saran tubing. It is located in a large body of water with a minimum of 4 ft of water in all directions from the coil. A detector is positioned so that it can be moved along the axis of the cylindrical disk. Variations in source intensity due to changes in water flow or reactor flux are compensated for by means of a monitor.

The detector and monitor are anthracene scintillation counters. The anthracene cylinders are $1\frac{1}{2}$ in. in diameter and 1 in. high and are optically attached to RCA 5819 photomultiplier tubes. The output current of the photomultiplier is read on a low drift ac electrometer. This current is a measure of the dose rate.

The following important reactions are expected to occur upon neutron irradiation in the reactor: $O^{16}(n,p)N^{16}$, $O^{17}(n,p)N^{17}$, and $O^{18}(n,\gamma)O^{19}$. The N^{16} production should be predominant. In order to verify this and to check on the existence of any spurious activity from impurities, decay curves have been

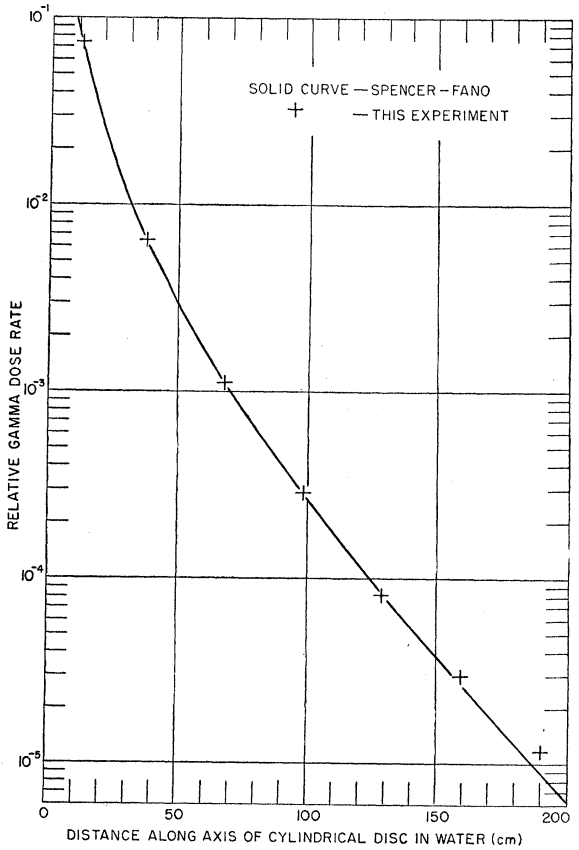

FIG. 1. Relative gamma dose rate as a function of distance from source.

[1] L. V. Spencer and U. Fano, J. Research Natl. Bur. Standards **46**, 446 (1951); Phys. Rev. **81**, 464 (1951).