Effects of Certain Three-Body Nuclear Interactions in H^3 and He^3†

A. W. SOLBRIG, JR.*
Department of Physics, Vanderbilt University, Nashville, Tennessee

(Received March 17, 1954)

This limited, theoretical study is phenomenological and nonrelativistic. Effects of interactions on wave functions, magnetic dipole moments, and binding energies are obtained by first-order perturbation methods. The interactions satisfy well-known requirements of invariance, contain no power of momentum higher than the first, involve no dependence on charge, and introduce only P state in a first-order calculation. They contain spin-orbit interactions and may give rise to interaction moments. Each interaction contains a scalar radial function of positions f. The unperturbed potential corresponds to pairwise Hooke's law forces between nucleons. The first-order energy perturbation is shown to vanish for all the interactions. It is shown, without further specialization of the f, that no one of fifty-eight of the seventy-five interactions yields observed magnetic moments. No definite conclusion is obtained for eleven of the remaining seventeen interactions, for calculations with plausible f appear difficult. Each of the remaining six interactions yields observed magnetic moments with plausible f.

INTRODUCTION

SEVERAL workers have recently discussed many-body nuclear interactions in connection with the binding energy of He^4, nuclear saturation properties, and the independent-particle model of the nucleus.[1] Although it is not certain that many-body interactions are required in the explanation of nuclear properties, investigation of possible forms of such interactions and of their effects in nuclei is, nevertheless, of interest. The number of possible forms is very great; and the calculation of their effects is likely to be difficult, especially since meson theory in its present state is not well suited to quantitative calculation. The phenomenological and nonrelativistic treatment of the present work is applied only to the comparatively simple nuclei H^3 and He^3.

If the possible existence of three-body forces is ignored and the two-body forces are assumed to be composed of central and tensor parts, the ground states of H^3 and He^3 are mostly mixtures of S and D states.[2] Such ground states combined with the theory of Sachs[3] give the observed value of the sum of the magnetic dipole moments of H^3 and He^3, but they do not give the observed magnetic moments of the individual nuclei. The work of Sachs indicates that a more complicated ground state containing a considerable fraction of P state in addition to the S and D states can yield values for the individual moments which differ only slightly from the observed moments. Avery and Sachs[4] state that relativistic corrections may overcome the difference; but they also conclude, from kinetic energy considerations, that the calculation of the moments by Sachs[3] requires an unreasonably large fraction of P state and that it is more reasonable to invoke interaction moments[5] to obtain agreement. Ross[6] has examined the evidence for nonadditivity of nucleon moments in heavy nuclei. He concludes (a) that the deviations of static moments of heavy nuclei from the Schmidt lines are an unreliable and ambiguous source of information and that they could be ascribed to a nonadditivity effect only if that were a many-body effect and (b) that the observation of certain "forbidden" magnetic dipole transitions in heavy nuclei seems to provide direct evidence for the existence of nonadditivity effects. Although the existence of such effects is fairly well established, the existence of P state attributed to spin-orbit interaction in other very light nuclei is reason to suppose that similar interactions cause P state to appear in the ground state of H^3 and He^3;[7] and the presence of this state can decrease the interaction moment needed to bring agreement with observation. Apparently no calculation of the fraction of P state introduced by plausible nuclear interactions

† Work performed, in part, under U. S. Atomic Energy Commission Predoctoral Fellowship. This report is from a thesis submitted in partial fulfillment of requirements for the Ph.D. degree at Vanderbilt University. Phys. Rev. 92, 855 (1953).

* Now at Los Alamos Scientific Laboratory, Los Alamos, New Mexico.

[1] For example, J. Irving, Proc. Phys. Soc. (London) **A66**, 17 (1953); S. D. Drell and K. Huang, Phys. Rev. **91**, 1527 (1953); L. I. Schiff, Phys. Rev. **84**, 1 (1951); R. E. Peierls, Proc. Phys. Soc. (London) **A66**, 313 (1953).

[2] R. L. Pease and H. Feshbach, Phys. Rev. **88**, 945 (1952), present references and recent work.

[3] R. G. Sachs, Phys. Rev. **72**, 312 (1947). Sachs assumes that the wave functions of the conjugate, or mirror, nuclei H^3 and He^3 are identical in the sense that "the wave function of the one nucleus can be obtained from that of the conjugate nucleus by identifying those variables in the one wave function which refer to the neutrons as the variables referring to the protons in the other wave function, and by treating the proton variables in a similar manner." R. Avery and E. N. Adams II, Phys. Rev. **75**, 1106 (1949), conclude that the difference between the H^3 and He^3 wave functions resulting from Coulomb forces has completely negligible effect on the calculated magnetic dipole moments of these nuclei.

[4] R. Avery and R. G. Sachs, Phys. Rev. **74**, 1320 (1948).

[5] F. Villars, Helv. Phys. Acta **20**, 476 (1947) and Phys. Rev. **86**, 476 (1952); R. G. Sachs, Phys. Rev. **74**, 433 (1948) and Phys. Rev. **75**, 1605 (1949); Blanchard, Avery, and Sachs, Phys. Rev. **78**, 292 (1950); R. K. Osborne and L. L. Foldy, Phys. Rev. **79**, 795 (1950); N. Austern and R. G. Sachs, Phys. Rev. **81**, 710 (1951); A. Russek and L. Spruch, Phys. Rev. **87**, 1111 (1952); N. Austern, Phys. Rev. **92**, 670 (1953).

[6] M. Ross, Phys. Rev. **88**, 935 (1952).

[7] L. Rosenfeld, *Nuclear Forces* (Interscience Publishers, Inc., New York, 1948), Sec. A2.251, and Physica **17**, 461 (1951); D. R. Inglis, Revs. Modern Phys. **25**, 390 (1953).

which yield observed magnetic moments of H^3 and He^3 has been published.[8] Adams[9] has made a study of certain hyperfine-structure effects in tritium according to several theories of the origin of the triton moment anomaly. He concludes that the effects do not serve as a possible means of distinguishing among the several theoretical accounts of the triton moment anomaly.

The purposes of the present work are (a) to construct certain types of three-body interactions which give rise to interaction moments and which introduce only P state in a first-order calculation and (b) to study, in a very limited and circumscribed manner, the effects of these interactions on the wave functions, magnetic dipole moments, and binding energies of H^3 and He^3. This work may be of use in further investigations in which these interactions are combined with other interactions to describe nuclear properties in a more comprehensive fashion. We assume throughout that the wave functions of the two nuclei are identical.[3] Our interactions turn out to involve spin-orbit interactions, and development of the independent-particle model has focused attention on this sort of interaction.[10] Spin-orbit interactions are also useful in describing high-energy nucleon-nucleon scattering.[11]

INTERACTIONS AND OPERATORS

To limit the possible forms of interaction, we use the eight general invariance requirements listed by Eisenbud and Wigner.[12] In the present study we consider only those three-body interactions which (a) contain no power of momentum higher than the first, (b) introduce only P state in a first-order calculation, (c) involve no dependence on charge, and (d) contain real scalar radial factors of the sort $f(r_{12},r_{23},r_{31})$, where $r_{ij}=\sqrt{(\mathbf{r}_{ij}\cdot\mathbf{r}_{ij})}$ and $\mathbf{r}_{ij}$ gives the position of particle i relative to particle j. Restriction (b) implies that the interaction must transform in ordinary space[13] like an axial vector. Since the interaction must be invariant under rotations in combined spin-ordinary space, any interaction of the sort considered here may be written as a sum of terms of the form $H'=f(r_{12},r_{23},r_{31})$(axial vector operator in spin space)·(axial vector operator in ordinary space).

It is not difficult to show that, for a three-nucleon system, there are just twenty linearly independent isotopic operators which satisfy the general conditions of invariance. In spite of the general conditions that the interaction be symmetric under time reversal and under interchange of any pair of particles, the number of products (axial vector operator in spin space)·(axial vector operator in ordinary space)(isotopic operator) is large. In order to reduce the number of such products available, we have ignored the isotopic formalism by taking the isotopic operator to be 1. An interaction with this isotopic factor must be symmetric under interchange of any pair of *identical* particles. Two other important simplifications are obtained by omission of the isotopic formalism: (a) We wish to follow Sachs[3] and assume that the wave functions of the nuclei H^3 and He^3 are identical. Some isotopic factors other than 1 lead to perturbed wave functions for the two nuclei which are essentially not identical. (b) The form of interaction includes no space exchange operator; hence only nonexchange interaction moments appear.

In the wave function of $H^3(He^3)$ we designate the proton (neutron) variables by a subscript 1, and we define B_{ij} as $(1+\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)/2$. In spin space there are axial vector operators $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$, $\boldsymbol{\sigma}_3$, and scalar operators **1**, $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$, $\boldsymbol{\sigma}_2\cdot\boldsymbol{\sigma}_3$, $\boldsymbol{\sigma}_3\cdot\boldsymbol{\sigma}_1$; but there are no polar vector or pseudoscalar operators. The axial vector operator in spin space must then be a linear combination containing one or more of the nine linearly independent operators listed in Table I. Any product among these operators is reducible by the commutation relations to a linear combination of the operators listed, and each of the operators is Hermitian. In ordinary space there are useful polar vector, axial vector, scalar, and pseudoscalar operators. Examples are, respectively, $\mathbf{r}_{12}$, $\mathbf{p}_{12}$; $\mathbf{r}_{31}\times\mathbf{r}_{12}$, $\mathbf{r}_{12}\times\mathbf{p}_{12}$; **1**, $\mathbf{r}_{12}\cdot\mathbf{p}_{31}$; and $\mathbf{r}_{31}\cdot\mathbf{r}_{12}\times\mathbf{p}_{31}\equiv(\mathbf{r}_{31}\mathbf{r}_{12}\mathbf{p}_{31})$. The axial vector operator in ordinary space must then be a linear combination containing one or more of twenty-one linearly independent operators. The condition that the operator be Hermitian reduces from twenty-one to thirteen the number of linearly independent operators available. The axial vector operator in ordinary space must then be a linear combination containing one or more of the thirteen linearly independent Hermitian axial vector operators in ordinary space listed in Table II.

The general interaction available under restrictions imposed up to this point is a sum of terms of the sort $f_{ij}\mathbf{S}_i\cdot\mathbf{K}_j$, where $\mathbf{S}_i$ is one of the nine spin operators of Table I and $\mathbf{K}_j$ is one of the thirteen ordinary operators of Table II. No loss of generality is implied by the restriction that each of the axial vector operators contain only one of the tabulated operators, for linear com-

TABLE I. Axial vector operators in spin space.

Symbol and operator	Symmetry: Time reversal	Symmetry: Interchange of particles 2 and 3
$\mathbf{S}_1=\boldsymbol{\sigma}_1$	−	+
$\mathbf{S}_2=\boldsymbol{\sigma}_2+\boldsymbol{\sigma}_3$	−	+
$\mathbf{S}_3=\boldsymbol{\sigma}_2-\boldsymbol{\sigma}_3$	−	−
$\mathbf{S}_4=\boldsymbol{\sigma}_2\times\boldsymbol{\sigma}_3$	+	−
$\mathbf{S}_5=\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2-\boldsymbol{\sigma}_3\times\boldsymbol{\sigma}_1$	+	+
$\mathbf{S}_6=\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2+\boldsymbol{\sigma}_3\times\boldsymbol{\sigma}_1$	+	−
$\mathbf{S}_7=B_{23}\boldsymbol{\sigma}_1$	−	+
$\mathbf{S}_8=B_{12}\boldsymbol{\sigma}_3+B_{31}\boldsymbol{\sigma}_2$	−	+
$\mathbf{S}_9=B_{12}\boldsymbol{\sigma}_3-B_{31}\boldsymbol{\sigma}_2$	−	−

[8] Blanchard, Avery, and Sachs, reference 5, discuss calculations of interaction moments performed with two-body interactions that can introduce P state; but they neglect effects of this P state on calculated magnetic moments.

[9] E. N. Adams II, Phys. Rev. **81**, 1 (1951).

[10] I. Bloch has suggested in a private communication (1952) that the special spin-orbit interactions of the shell model may be the result of many-body spin-orbit interactions.

[11] For example, K. M. Case and A. Pais, Phys. Rev. **80**, 203 (1950).

[12] L. Eisenbud and E. P. Wigner, Proc. Natl. Acad. Sci. **27**, 281 (1941).

[13] By "ordinary space" we mean "ordinary three-space in which transformations are independent of spin space."

binations may be obtained by suitable choice of the f_{ij} in the sum. Since the interaction must be symmetric under time reversal, only $(3\times1)+(6\times12)=75$ permissible products $\mathbf{S}_i\cdot\mathbf{K}_j$ can be formed. Each of these products has a definite symmetry under interchange of identical particles. It follows that the scalar radial functions f_{ij} associated with each product must have a definite symmetry under interchange of identical particles, for the interaction itself must be symmetric under this interchange. In order to simplify further the following discussion, we consider hereinafter only those three-body interactions which are not a sum of terms, but a *single* term of the sort $H_{ij}'=f_{ij}\mathbf{S}_i\cdot\mathbf{K}_j$.

The interaction moment operators $\mathbf{M}_{ij}$ arising from the interactions considered may be written by use of the results of Sachs and Austern.[14] These operators are listed below in units of the nuclear magneton, $e\hbar/2mc$. The F is $mf_{ij}/\hbar$, and $\mathbf{R}_n$ is the coordinate of particle n with respect to the center of mass of the system. The superscript on H^3 and He^3 is omitted for convenience.

$$\mathbf{M}_{i1}(\mathrm{H})=\mathbf{M}_{i1}(\mathrm{He})=0,$$
$$\mathbf{M}_{i2}(\mathrm{H})=F\mathbf{R}_1\times[(\mathbf{r}_{31}-\mathbf{r}_{12})\times\mathbf{S}_i],$$
$$\mathbf{M}_{i2}(\mathrm{He})=F[\mathbf{R}_2\times(\mathbf{r}_{12}\times\mathbf{S}_i)-\mathbf{R}_3\times(\mathbf{r}_{31}\times\mathbf{S}_i)],$$
$$\mathbf{M}_{i3}(\mathrm{H})=F\mathbf{R}_1\times(\mathbf{r}_{23}\times\mathbf{S}_i),$$
$$\mathbf{M}_{i3}(\mathrm{He})=F[\mathbf{R}_2\times(\mathbf{r}_{12}\times\mathbf{S}_i)+\mathbf{R}_3\times(\mathbf{r}_{31}\times\mathbf{S}_i)],$$
$$\mathbf{M}_{i4}(\mathrm{H})=F\mathbf{R}_1\times[(\mathbf{r}_{12}-\mathbf{r}_{31})\times\mathbf{S}_i],$$
$$\mathbf{M}_{i4}(\mathrm{He})=F[\mathbf{R}_2\times(\mathbf{r}_{31}\times\mathbf{S}_i)-\mathbf{R}_3\times(\mathbf{r}_{12}\times\mathbf{S}_i)],$$
$$\mathbf{M}_{i5}(\mathrm{H})=F\mathbf{R}_1\times(\mathbf{r}_{23}\times\mathbf{S}_i),$$
$$\mathbf{M}_{i5}(\mathrm{He})=F[\mathbf{R}_2\times(\mathbf{r}_{31}\times\mathbf{S}_i)+\mathbf{R}_3\times(\mathbf{r}_{12}\times\mathbf{S}_i)],$$
$$\mathbf{M}_{i6}(\mathrm{H})=F[(\mathbf{r}_{31}\times\mathbf{r}_{12})\times\mathbf{R}_1][(\mathbf{r}_{31}-\mathbf{r}_{12})\cdot\mathbf{S}_i],$$
$$\mathbf{M}_{i6}(\mathrm{He})=F(\mathbf{r}_{31}\times\mathbf{r}_{12})\times[\mathbf{R}_2(\mathbf{r}_{12}\cdot\mathbf{S}_i)-\mathbf{R}_3(\mathbf{r}_{31}\cdot\mathbf{S}_i)],$$
$$\mathbf{M}_{i7}(\mathrm{H})=F[(\mathbf{r}_{31}\times\mathbf{r}_{12})\times\mathbf{R}_1](\mathbf{r}_{23}\cdot\mathbf{S}_i),$$
$$\mathbf{M}_{i7}(\mathrm{He})=F(\mathbf{r}_{31}\times\mathbf{r}_{12})\times[\mathbf{R}_2(\mathbf{r}_{12}\cdot\mathbf{S}_i)+\mathbf{R}_3(\mathbf{r}_{31}\cdot\mathbf{S}_i)],$$
$$\mathbf{M}_{i8}(\mathrm{H})=F[(\mathbf{r}_{31}\times\mathbf{r}_{12})\times\mathbf{R}_1][(\mathbf{r}_{12}-\mathbf{r}_{31})\cdot\mathbf{S}_i],$$
$$\mathbf{M}_{i8}(\mathrm{He})=F(\mathbf{r}_{31}\times\mathbf{r}_{12})\times[\mathbf{R}_2(\mathbf{r}_{31}\cdot\mathbf{S}_i)-\mathbf{R}_3(\mathbf{r}_{12}\cdot\mathbf{S}_i)],$$
$$\mathbf{M}_{i9}(\mathrm{H})=F[(\mathbf{r}_{31}\times\mathbf{r}_{12})\times\mathbf{R}_1](\mathbf{r}_{23}\cdot\mathbf{S}_i),$$
$$\mathbf{M}_{i9}(\mathrm{He})=F(\mathbf{r}_{31}\times\mathbf{r}_{12})\times[\mathbf{R}_2(\mathbf{r}_{31}\cdot\mathbf{S}_i)+\mathbf{R}_3(\mathbf{r}_{12}\cdot\mathbf{S}_i)],$$
$$\mathbf{M}_{i10}(\mathrm{H})=\mathbf{M}_{i10}(\mathrm{He})=0,$$
$$\mathbf{M}_{i11}(\mathrm{H})=-\mathbf{M}_{i11}(\mathrm{He})=F(\mathbf{r}_{31}\mathbf{r}_{12}\mathbf{S}_i)(\mathbf{r}_{23}\times\mathbf{R}_1),$$
$$\mathbf{M}_{i12}(\mathrm{H})=\mathbf{M}_{i12}(\mathrm{He})=0,$$
$$\mathbf{M}_{i13}(\mathrm{H})=\tfrac{1}{2}\mathbf{M}_{i13}(\mathrm{He})=\mathbf{M}_{i11}(\mathrm{H}).$$

THE UNPERTURBED SYSTEM

We consider the nucleus H^3 (or He^3) as a system of three particles each of mass m[15] with position vectors

TABLE II. Axial vector operators in ordinary space.

Symbol and operator	Symmetry: Time reversal	Symmetry: Interchange of particles 2 and 3
$\mathbf{K}_1=\mathbf{r}_{31}\times\mathbf{r}_{12}$	+	−
$\mathbf{K}_2=\mathbf{r}_{12}\times\mathbf{p}_{12}+\mathbf{r}_{31}\times\mathbf{p}_{31}$	−	+
$\mathbf{K}_3=\mathbf{r}_{12}\times\mathbf{p}_{12}-\mathbf{r}_{31}\times\mathbf{p}_{31}$	−	−
$\mathbf{K}_4=\mathbf{r}_{31}\times\mathbf{p}_{12}+\mathbf{r}_{12}\times\mathbf{p}_{31}$	−	+
$\mathbf{K}_5=\mathbf{r}_{31}\times\mathbf{p}_{12}-\mathbf{r}_{12}\times\mathbf{p}_{31}$	−	−
$\mathbf{K}_6=-2\hbar i(\mathbf{r}_{31}\times\mathbf{r}_{12})+\mathbf{r}_{12}(\mathbf{r}_{31}\mathbf{r}_{12}\mathbf{p}_{12})+\mathbf{r}_{31}(\mathbf{r}_{31}\mathbf{r}_{12}\mathbf{p}_{31})$	−	−
$\mathbf{K}_7=\mathbf{r}_{12}(\mathbf{r}_{31}\mathbf{r}_{12}\mathbf{p}_{12})-\mathbf{r}_{31}(\mathbf{r}_{31}\mathbf{r}_{12}\mathbf{p}_{31})$	−	+
$\mathbf{K}_8=\hbar i(\mathbf{r}_{31}\times\mathbf{r}_{12})+\mathbf{r}_{31}(\mathbf{r}_{31}\mathbf{r}_{12}\mathbf{p}_{12})+\mathbf{r}_{12}(\mathbf{r}_{31}\mathbf{r}_{12}\mathbf{p}_{31})$	−	−
$\mathbf{K}_9=\mathbf{r}_{31}(\mathbf{r}_{31}\mathbf{r}_{12}\mathbf{p}_{12})-\mathbf{r}_{12}(\mathbf{r}_{31}\mathbf{r}_{12}\mathbf{p}_{31})$	−	+
$\mathbf{K}_{10}=(\mathbf{r}_{31}\times\mathbf{r}_{12})(-8\hbar i+\mathbf{r}_{12}\cdot\mathbf{p}_{12}+\mathbf{r}_{31}\cdot\mathbf{p}_{31})$	−	−
$\mathbf{K}_{11}=(\mathbf{r}_{31}\times\mathbf{r}_{12})(\mathbf{r}_{12}\cdot\mathbf{p}_{12}-\mathbf{r}_{31}\cdot\mathbf{p}_{31})$	−	+
$\mathbf{K}_{12}=(\mathbf{r}_{31}\times\mathbf{r}_{12})(4\hbar i+\mathbf{r}_{31}\cdot\mathbf{p}_{12}+\mathbf{r}_{12}\cdot\mathbf{p}_{31})$	−	−
$\mathbf{K}_{13}=(\mathbf{r}_{31}\times\mathbf{r}_{12})(\mathbf{r}_{31}\cdot\mathbf{p}_{12}-\mathbf{r}_{12}\cdot\mathbf{p}_{31})$	−	+

$\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, where particles 2 and 3 are identical. We take the unperturbed Hamiltonian of the system to be[16] $H=[(p_1^2+p_2^2+p_3^2)/2m]+[k(r_{12}^2+r_{23}^2+r_{31}^2)/2]+D$, where k is a real positive constant and D is a real constant. In order to separate and solve the unperturbed Schrödinger equation, we transform to normal coordinates in which $\boldsymbol{\varrho}_i$ has components x_i, y_i, and z_i by the relations

$$x_1=-X_2/\sqrt{2}+X_3/\sqrt{2},$$
$$x_2=2X_1/\sqrt{6}-X_2/\sqrt{6}-X_3/\sqrt{6},$$
$$x_3=X_1/\sqrt{3}+X_2/\sqrt{3}+X_3/\sqrt{3},$$

and by similar relations for y and z components. The space-dependent factor of the wave function is

$$u_{t1,\,t2}=u_{n1}(x_1)u_{p1}(y_1)u_{q1}(z_1)u_{n2}(x_2)u_{p2}(y_2)u_{q2}(z_2)\times u(x_3,y_3,z_3)=u_{n1,\,p1,\,q1,\,n2,\,p2,\,q2}u(x_3,y_3,z_3),$$

where

$$(t1)=(n1)+(p1)+(q1),\quad (t2)=(n2)+(p2)+(q2),$$
$$u(x_3,y_3,z_3)=C\exp(i\mathbf{P}_3\cdot\boldsymbol{\varrho}_3/\hbar),$$
$$u_{n1}(x_1)=N_{n1}H_{n1}(x_1/a)\exp(-x_1^2/2a^2),$$
$$(n1)=0,1,2,\cdots,$$
$$N_{n1}=[(n1)!2^{n1}a\pi^{\frac{1}{2}}]^{-\frac{1}{2}},\quad a=(\hbar^2/3km)^{\frac{1}{4}},$$

H_{n1} is an Hermite polynomial, and the remaining u's are similar to $u_{n1}(x_1)$. If we assume that the momentum of the center of mass of the nucleus $\mathbf{P}_3$ is zero and that $C=1$, then the space-dependent factor is normalized and the energy eigenvalues are

$$E_{t1,\,t2}=[(t1)+(t2)+3](\hbar^2/ma^2)+D.$$

The spin-dependent factor of the wave function must be a linear combination containing one or more of the eight linearly independent spin functions for three

[14] R. G. Sachs and N. Austern, Phys. Rev. **81**, 705 (1951), Sec. II. It is not difficult to demonstrate that our three-body interactions satisfy the general consequences of gauge invariance given in Sec. III of this reference.

[15] We assume $m=m_p$, for $|m_n-m_p|/(m_n+m_p)<0.1$ percent.

[16] Although this Hamiltonian is unrealistic, it leads to a fairly reasonable ground-state wave function. Hamiltonians of this type have the advantage of allowing calculation of a set of energy eigenfunctions for any nucleus.

TABLE III. Spin functions for three particles.

Symbol and spin function	Eigenvalues of $\mathbf{S}^2/\hbar^2$	Eigenvalues of $S_z/\hbar$	Symmetry under interchange of particles 2 and 3
$s_1=\alpha\alpha\alpha$	15/4	3/2	+
$s_2=\beta\beta\beta$	15/4	$-3/2$	+
$s_3=(\alpha\alpha\beta+\alpha\beta\alpha+\beta\alpha\alpha)/\sqrt{3}$	15/4	1/2	+
$s_4=(\beta\beta\alpha+\beta\alpha\beta+\alpha\beta\beta)/\sqrt{3}$	15/4	$-1/2$	+
$s_5=(\alpha\alpha\beta+\alpha\beta\alpha-2\beta\alpha\alpha)/\sqrt{6}$	3/4	1/2	+
$s_6=(\beta\beta\alpha+\beta\alpha\beta-2\alpha\beta\beta)/\sqrt{6}$	3/4	$-1/2$	+
$s_7=(\alpha\alpha\beta-\alpha\beta\alpha)/\sqrt{2}$	3/4	1/2	−
$s_8=(\beta\beta\alpha-\beta\alpha\beta)/\sqrt{2}$	3/4	$-1/2$	−

particles: $\alpha\alpha\alpha$, $\alpha\alpha\beta$, $\alpha\beta\alpha$, $\beta\alpha\alpha$, $\beta\beta\alpha$, $\beta\alpha\beta$, $\alpha\beta\beta$, $\beta\beta\beta$.[17] It is convenient to group these functions into combinations that are simultaneous eigenfunctions of $\frac{1}{4}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2+\boldsymbol{\sigma}_3)^2 \equiv \mathbf{S}^2/\hbar^2$ and $\frac{1}{2}(\sigma_{1z}+\sigma_{2z}+\sigma_{3z})\equiv S_z/\hbar$. These combinations may be generated with the aid of a spin raising operator $(\sigma_{1x}+\sigma_{2x}+\sigma_{3x})+i(\sigma_{1y}+\sigma_{2y}+\sigma_{3y})$. In Table III are shown a set of such linearly independent orthonormal spin functions. Now the magnetic moment is conventionally described as the expectation value of the z component of a magnetic moment operator over a state in which the z component of total angular momentum has its maximum value $J_z=J$. Since the total angular momentum of each of the nuclei H^3 and He^3 is $\hbar/2$ and since the common wave function of the nuclei must be antisymmetric under interchange of particles 2 and 3, we write the normalized unperturbed ground-state wave function of the nuclei as $\psi_0=s_7u_{000,\,000}$.

For the sake of simplicity the unperturbed ground-state wave function written just above is used in all calculations in this report, although it is possible to carry out the calculations using a slightly more general unperturbed Hamiltonian.[18] Let us take k_{nn1}, k_{pp1}, k_{np1}, and k_{np3} to be force constants between two neutrons in a relative singlet spin state, two protons in a singlet state, a neutron and a proton in a singlet state, and a neutron and a proton in a triplet state, respectively. If $k_{nn1}=k_{pp1}=k_1$ and $k_{np1}=k_{np3}=k_2\neq k_1$, then the normal coordinates are just those defined above; but the eigenfunctions of the altered Hamiltonian contain two distinct a's: $a_1=\{\hbar^2/[(2k_1+k_2)m]\}^{\frac{1}{4}}$ and $a_2=(\hbar^2/3k_2m)^{\frac{1}{4}}$. This choice of force constants seems unacceptable, for it is believed[19] that, although the singlet forces are nearly equal in strength, the triplet force is about 60 percent stronger than the singlet forces. If we attempt to simulate spin-dependent forces by using $k_{nn1}=k_{pp1}=k_{np1}=k_{np3}/1.6$, then the Hamiltonian is not symmetric under interchange of identical particles and the eigenfunctions of the Hamiltonian are not antisymmetric under interchange of identical particles. From these eigenfunctions we may, however, construct antisymmetric functions which are in a sense approximate eigenfunctions of the Hamiltonian. The spin dependence of the forces is properly taken into account by use of a Hamiltonian which contains spin operators and which is symmetric under interchange of identical particles.

[17] Notation for spin functions and operators is that of L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949), Sec. 33, with (+) here replaced by α and (−) by β.

[18] For example, W. V. Houston, Phys. Rev. **47**, 942 (1935), treats the case of three different force constants and three different masses.

[19] See L. Rosenfeld, *Nuclear Forces*, reference 7, Sec. 3.3 and Tables 6.432, 7.13, and 8.1.

THE PERTURBED SYSTEM

The Ground-State Wave Function and Binding Energies

The perturbed ground-state wave function may be written $\psi=\psi_0+\psi_{ij}'$, where the perturbing wave function ψ_{ij}' is obtained by the standard first-order method. It is easy to show that $\mathbf{p}_{bc}u_0=\hbar i a^{-2}\mathbf{r}_{bc}u_0$,[20] a result which is helpful in writing the products $\mathbf{K}_i u_0$ of u_0 by the ordinary operators of Table II. These products reveal that only thirty-two of our seventy-five permissible interactions may yield a nonvanishing perturbing wave function. These interactions contain either a product of any one of the spin operators $\mathbf{S}_1$, $\mathbf{S}_3$, $\mathbf{S}_7$, $\mathbf{S}_8$, $\mathbf{S}_9$ by any one of the ordinary operators $\mathbf{K}_5$, $\mathbf{K}_6$, $\mathbf{K}_8$, $\mathbf{K}_{10}$, $\mathbf{K}_{11}$, $\mathbf{K}_{12}$ or one of the products $\mathbf{S}_4\cdot\mathbf{K}_1$ or $\mathbf{S}_6\cdot\mathbf{K}_1$. For brevity we list these products as $\mathbf{S}_{1,3,7,8,9}\cdot\mathbf{K}_{5,6,8,10,11,12}$, $\mathbf{S}_{4,6}\cdot\mathbf{K}_1$.

The unperturbed potential leads to an infinite binding energy. The first order-energy perturbation vanishes because ψ_0 is spherically symmetric while H_{ij}' transforms like a P state. The perturbed potential is unsuited to energy calculations because it yields an infinite binding energy in a first-order calculation.[21] In view of this fact we shall not consider the second-order energy perturbation.

Magnetic Dipole Moments

In the following calculations we take the observed magnetic dipole moments in nuclear magnetons to be $\mu_p=2.7928$, $\mu_n=-1.9129$, $\mu(H^3)\equiv\mu_H=2.9789$, and $\mu(He^3)\equiv\mu_{He}=-2.1276$.[22] It is not likely that any of these values is in error by much more than ± 0.0002.

Our perturbed wave function contains only ${}^2\tilde{S}$, ${}^2\bar{P}$, ${}^2\tilde{P}$, and ${}^4\bar{P}$ states, where $\sim$ or — indicates[23] that the spin-dependent factor is antisymmetric or symmetric, respectively, under interchange of particles 2 and 3. The spin and orbital contribution to the magnetic dipole moment may be written by use of the results of Sachs.[3] If the interaction is one which yields a vanishing perturbing wave function, then from spin and orbital contribution alone $\mu_H(\text{calc})=\mu_p<\mu_H$, $\mu_{He}(\text{calc})=\mu_n>\mu_{He}$, and $\mu_H(\text{calc})+\mu_{He}(\text{calc})=\mu_p+\mu_n>\mu_H+\mu_{He}$. If the interaction is one of the thirty-two which may yield a nonvanishing perturbing wave function, we can reach

[20] This result implies at once that all two-body interactions which satisfy the general requirements of invariance and are of first degree in momenta yield no perturbing wave function on application to our ground state. The same conclusion holds for any ground state with space dependence a function of $r_{12}^2+r_{23}^2+r_{31}^2=3(\rho_1^2+\rho_2^2)$; but it does not hold (a) if the function contains $(x_1x_2+y_1y_2+z_1z_2)/(\rho_1\rho_2)$, which is the q of Avery and Sachs, reference 4, or (b) if, as discussed above, the unperturbed Hamiltonian contains two or more distinct force constants.

[21] H. Margenau and D. T. Warren, Phys. Rev. **52**, 790 (1937) and D. T. Warren and H. Margenau, Phys. Rev. **52**, 1027 (1937), consider a two-body central perturbing potential designed to remove this difficulty.

[22] N. F. Ramsey, *Experimental Nuclear Physics*, E. Segrè, Editor (John Wiley and Sons, Inc., New York, 1953), Vol. 1, Part III. The sign of μ_{He} is from Fred, Tomkins, Brody, and Hammermesh, Phys. Rev. **82**, 406 (1951).

[23] Rosenfeld, *Nuclear Forces*, reference 7.

useful conclusions about spin and orbital contributions by considering the general properties of the space dependence and the details of only the spin dependence of the perturbed wave function. It may be shown[24] that any one of the interactions yields by spin and orbital contribution alone $\mu_H(\text{calc}) \leqslant \mu_p < \mu_H$ and $\mu_{He}(\text{calc}) \geqslant \mu_n > \mu_{He}$, although some interactions may possibly give $\mu_H(\text{calc}) + \mu_{He}(\text{calc}) = \mu_H + \mu_{He}$.

Since any one of our interactions yields by spin and orbital contribution alone $\mu_H(\text{calc}) < \mu_H$ and $\mu_{He}(\text{calc}) > \mu_{He}$, it is necessary for agreement with experiment that the interaction contributions $\langle M_z(\text{H})\rangle$ and $\langle M_z(\text{He})\rangle$ be different from zero and of opposite sign. In terms of a normalized perturbed wave function ψ, the interaction contribution is $\langle M_z\rangle = (\psi, M_z\psi) = (\psi_0, M_z\psi_0) + (\psi', M_z\psi') + (\psi', M_z\psi_0) + (\psi_0, M_z\psi')$, where the subscripts ij have been omitted from M_{ijz} and ψ_{ij}' for brevity. The $(\psi', M_z\psi')$ is hereinafter ignored, for it is essentially a third-order term comparable with terms like $(\psi'', M_z\psi_0)$ arising from a second-order perturbed wave function. Fairly straightforward manipulations reveal[24] (a) that an interaction not containing one of the twenty-four products $\mathbf{S}_{1,7,8}\cdot\mathbf{K}_{2,3,4,5,7,9,11,13}$ gives $(\psi_0, M_z\psi_0) = 0$ and (b) that an interaction not containing one of the twelve products $\mathbf{S}_{1,7,8,9}\cdot\mathbf{K}_{5,6,8}$ gives $(\psi', M_z\psi_0) + (\psi_0, M_z\psi') = 0$. Clearly, then, only thirty-three interactions may yield a nonvanishing interaction contribution. Closer examination of these thirty-three interactions, with extensive use of symmetry and transformation properties but without specialization of the radial factor f in the interactions, shows[24] that sixteen of them do not yield the observed moments. The seventeen interactions remaining for detailed calculations are those containing one of the products $\mathbf{S}_{1,7,8}\cdot\mathbf{K}_{2,4,6,8,11}$ or $\mathbf{S}_9\cdot\mathbf{K}_{6,8}$.

In order further to test the remaining interactions, it appears necessary to make some assumption about the radial function. If this function contains factors resembling conventional square, Gaussian, exponential, or Yukawa wells, and if the interaction contains one of the products $\mathbf{S}_{1,7,8}\cdot\mathbf{K}_{6,8,11}$ or $\mathbf{S}_9\cdot\mathbf{K}_{6,8}$, then the perturbing wave function contains, in general, an infinite number of excited functions, $u_{n1,p1,q1,n2,p2,q2}$. If the plausible value of $a^2 = 2.8\times 10^{-26}$ cm^2 is selected,[21] then $E_{t1,t2} - E_0 = [(t1)+(t2)]$ (14.8 Mev); so there may be appreciable fractions of excited states with $(t1)+(t2) > 6$ associated with a not unreasonable kinetic energy of the system. The calculations involving a perturbing wave function expressed as an infinite series in the excited functions appear difficult; and it is not evident how to estimate the error committed in performing a calculation with a truncated series. If the radial function f is a polynomial in the coordinates of the particles, then the perturbed wave function contains only a finite number of excited functions. We carried out detailed calculations[24] with the radial function

$$f = \frac{V}{\hbar a^2}\left(1 + w\frac{\rho_1^2}{a^2} + v\frac{\rho_2^2}{a^2}\right) = \frac{V}{\hbar a^2}\left[1 + \frac{w}{3}\left(\frac{r_{12}^2 + r_{31}^2}{a^2}\right) + \left(\frac{w}{2} - \frac{v}{6}\right)\frac{r_{23}^2}{a^2}\right]$$

for interactions containing one of the products $\mathbf{S}_{1,7,8}\cdot\mathbf{K}_{11}$ or $\mathbf{S}_9\cdot\mathbf{K}_{6,8}$ and with the radial function

$$f = \frac{V}{\hbar a^4}\left(1 + w\frac{\rho_1^2}{a^2} + v\frac{\rho_2^2}{a^2}\right)(\boldsymbol{\varrho}_1\cdot\boldsymbol{\varrho}_2)$$
$$= \frac{V}{\hbar a^4}\left[1 + \frac{w}{3}\left(\frac{r_{12}^2 + r_{31}^2}{a^2}\right) + \left(\frac{w}{2} - \frac{v}{6}\right)\frac{r_{23}^2}{a^2}\right]\left(\frac{r_{12}^2 - r_{31}^2}{2\sqrt{3}}\right)$$

for interactions containing one of the products $\mathbf{S}_{1,7,8}\cdot\mathbf{K}_{6,8}$. For all real values of the parameters V, a^2, w, and v the calculated moments differ from the observed moments by many times the experimental error in the observed moments. Omission of the term unity in the second factor of the f does not alter this result. Since the calculations were carried out with a very specialized form of radial function, and since this form is not very reasonable physically, we have no definite conclusion about agreement between observed moments and moments calculated from interactions containing one of the eleven products $\mathbf{S}_{1,7,8}\cdot\mathbf{K}_{6,8,11}$ or $\mathbf{S}_9\cdot\mathbf{K}_{6,8}$.

Interactions containing one of the products $\mathbf{S}_{1,7,8}\cdot\mathbf{K}_{2,4}$ lead to no perturbing wave function. A radial function which is fairly reasonable physically is

$$f = \frac{V}{\hbar}\left[1 + w\left(\frac{\rho_1^2 - \rho_2^2}{a^2}\right)\right]\exp\left(-\frac{\rho_1^2 + \rho_2^2}{g^2}\right)$$
$$= \frac{V}{\hbar}\left[1 + \frac{w}{3}\left(\frac{2r_{23}^2 - r_{12}^2 - r_{31}^2}{a^2}\right)\right]\times\exp\left(-\frac{r_{12}^2 + r_{23}^2 + r_{31}^2}{3g^2}\right).$$

If we require that calculated and observed moments be equal, easy calculations show that for an interaction containing one of the products $\mathbf{S}_{1,7,8}\cdot\mathbf{K}_2$,

$$\mu_{He} - \mu_n = 2(mVa^2/\hbar^2)[g^2/(g^2+a^2)]^4,$$
$$\frac{(\mu_{He} - \mu_n) - (\mu_H - \mu_p)}{\mu_{He} - \mu_n} = wg^2/(g^2+a^2);$$

and for an interaction containing one of the products $\mathbf{S}_{1,7,8}\cdot\mathbf{K}_4$,

$$(\mu_{He} - \mu_n) - (\mu_H - \mu_p) = 2(mVa^2/\hbar^2)[g^2/(g^2+a^2)]^4,$$
$$\frac{\mu_{He} - \mu_n}{(\mu_{He} - \mu_n) - (\mu_H - \mu_p)} = wg^2/(g^2+a^2).$$

[24] A. W. Solbrig, Jr., thesis, Vanderbilt University, 1953 (unpublished). Available on microfilm from University Microfilms, Inc., Ann Arbor, Michigan.

TABLE IV. Some values of parameters in interactions which yield observed moments of H^3 and He^3. ($a^2=2.8\times10^{-26}$ cm^2.)

	$f\mathbf{S}_{1,7,8}\cdot\mathbf{K}_2$		$f\mathbf{S}_{1,7,8}\cdot\mathbf{K}_4$	
$3g^2$	V (Mev)	w	V (Mev)	w
∞	-1.59	1.87	-2.97	0.54
$30a^2$	-2.33	2.05	-4.35	0.59
$9a^2$	-5.03	2.49	-9.39	0.71
$6a^2$	-8.05	2.80	-15.0	0.80
a^2	-25.5	3.73	-47.5	1.07

Values of V and w required by a few different values of the range parameter $3g^2$ are presented in Table IV for $a^2=2.8\times10^{-26}$ cm^2.[21]

REMARKS

Since no interaction which alters the ground-state wave function accounts for the observed magnetic moments in the above calculations, we are not able to calculate the nonvanishing fraction of P state introduced by an interaction which yields observed magnetic moments.

Several suggestions for further work arise from the present study: (a) The effects of linear combinations of the three-body interactions on the magnetic moments are worth consideration, for these effects are not necessarily additive. Preliminary calculations indicate that the nonadditive effects may lead to better agreement between calculated and observed moments. (b) Study of the properties of interactions containing one or more of the other nineteen permissible three-body isotopic operators remains to be undertaken. (c) Combination of the three-body interactions with well-known two-body interactions may account for nuclear properties in a more comprehensive way. In particular, combination of the three-body interactions with the conventional tensor force can lead to an orbital $^4P^4D$ cross term which may reduce the difference between calculated and observed moments. (d) Use of an unperturbed ground-state wave function which depends on $\boldsymbol{\rho}_1\cdot\boldsymbol{\rho}_2$ or which is not symmetric under interchange of normal coordinates 1 and 2 would cause significant changes in the calculation.[20] It therefore appears desirable to seek other acceptable forms of the unperturbed Hamiltonian and to examine the altered effects of the three-body interactions.

ACKNOWLEDGMENT

The author wishes to thank Professor Ingram Bloch for encouragement, direction, and assistance without which this work would have been impossible.