Two-Body System in Quantum Electrodynamics. Energy Levels of Positronium

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Expressions are obtained for all $\alpha^3 Ry$ contributions to the energy levels of the two-fermion system in electrodynamics. These expressions are evaluated from a relativistic two-body equation which takes binding into account in its interaction operator. They are specifically calculated for the $n=2$ levels of the system. Corrections arise from three sources: (1) improved treatment of pair effects of the Coulomb field and of the exchange of transverse photons, (2) self-energy and vacuum polarization terms, and, in positronium, (3) second-order annihilation processes. The energy shifts resulting from (1) and (2) do not depend on the arbitrary masses through a single parameter like reduced mass. In the limit appropriate to hydrogen, the previously calculated two-body corrections of item (1) are confirmed.

The principal new result is the determination of the $n=2$ levels of positronium. In contrast to hydrogen, where the self-energy effect is dominant, here all three items yield roughly equal corrections. Together, they amount to about 3 percent of the $\alpha^2 Ry$ level splitting.

1. INTRODUCTION

ELECTRODYNAMIC corrections to the energy levels of an atomic system consisting of one particle in an external field have been calculated by many authors[1] and yield results in very close agreement with the observed hydrogen spectrum. It was not until the two-body equation was introduced,[2,3] however, that one could handle recoil effects relativistically, and hence accurately calculate energy levels for a two-body system. Several aspects of this problem have been studied since then. Salpeter[4] has treated to order $\alpha^3 Ry(m/M)$ the splitting in the fine structure of hydrogen due to non-self-energy processes; Karplus and Klein[5] have calculated the singlet-triplet splitting in the ground state of positronium; and Arnowitt[6] has computed this splitting for a system in which the two Dirac particles have arbitrary masses and one has an additional phenomenological magnetic moment.

Recoil corrections[4] to the hydrogen Lamb shift have the same order of magnitude as the discrepancy between theory and experiment. The corrections[6] to hydrogen hyperfine structure are considerably smaller than the experimental error. The two-body effects in positronium are larger and more significant. They give a correction to the hyperfine structure of the ground state of positronium which is confirmed by experiment.[7] Consequently, it seemed desirable to calculate to order $\alpha^3 Ry$ the energy levels of the $2S$ and $2P$ states of positronium. The splitting of these levels is affected by several processes contained in the relativistic two-body equation which do not contribute to the hyperfine splitting of the ground state. Measurement of these shifts may therefore provide a more thorough check of the two-body equation than a study of hyperfine structure does. Also, the present problem requires a more precise cut-off technique for low-energy photons.

Our calculation has been performed for a system of two particles with arbitrary masses. This procedure enabled us also to determine the recoil corrections to self-energy terms in hydrogen, and to compare certain parts of our solution with results previously derived from the two-particle equation. Our findings agree,[8] in the appropriate limits, with each of the three calculations mentioned above.[4–6] The general dependence on the two masses does not indicate any method for reducing the problem to one in which a single parameter plays a role similar to the role of reduced mass in nonrelativistic theory.

The use of a two-body formalism, in which binding is taken into account in intermediate states, is an essential theoretical improvement upon previous treatments. One of us[9] has previously derived, in collaboration with Professor R. Karplus, an improved interaction kernel which accomplishes this purpose by selectively resumming certain classes of terms in the expansion of the kernel in free-particle Green's functions.[10] An alternative derivation, presented in the Appendix, yields this effective interaction operator without expansion and resummation. The introduction of binding removes infrared divergences and yields additions to already convergent results less ambiguously than did previous treatments.

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[1] E. E. Salpeter [Phys. Rev. **89**, 92 (1953)] summarizes and gives references to these calculations.
[2] J. Schwinger, Proc. Natl. Acad. Sci. U.S. **37**, 452, 455 (1951).
[3] E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951).
[4] E. E. Salpeter, Phys. Rev. **87**, 328 (1952). Hereafter referred to as S.
[5] R. Karplus and A. Klein, Phys. Rev. **87**, 848 (1952). Hereafter referred to as KK–III.
[6] R. Arnowitt, Phys. Rev. **92**, 1002 (1953). We understand that a calculation similar to Arnowitt's has been performed by W. A. Newcomb.
[7] M. Deutsch and S. C. Brown, Phys. Rev. **85**, 1047 (1952).
[8] The purely Coulomb shift [our (3.11) and (35) of S] differs from S by a factor of two. This difference amounts to 0.037 Mc/sec in hydrogen and is not numerically significant for it. The reason for this discrepancy appears to be the use in S of $\Lambda_- \to (E-m)/4E$ instead of $\Lambda_- \to (E-m)/2E$.
[9] T. Fulton and R. Karplus, Phys. Rev. **93**, 1109 (1954). Hereafter referred to as FK.
[10] R. J. Eden, Proc. Roy. Soc. (London) **219**, 516 (1953), has independently suggested such a procedure.

2. WAVE EQUATION, INTERACTION, AND PERTURBATION THEORY

In general, we employ the methods and notation of Schwinger,[2] KK-III, and FK in this paper. A brief recapitulation of the relevant formulas seems in order.

The fully renormalized wave equation for two fermions with arbitrary masses is

$$[\bar{F}_1(11')\bar{F}_2(22')-\bar{I}(12,1'2')]\psi(1'2')=0, \quad (2.1)$$

where

$$\bar{F}_{1,2}=p_{1,2}\gamma_{1,2}+m_{1,2}=(G^0_{1,2})^{-1},$$

and m_1 (or m_2) is the experimental mass of particle one (or two). $\bar{I}$ is the renormalized interaction operator.

We introduce center-of-mass and relative coordinates:

$$\eta_1=m_1(m_1+m_2)^{-1}, \quad \eta_2=m_2(m_1+m_2)^{-1},$$
$$\mu=m_1m_2(m_1+m_2)^{-1}, \quad (2.2)$$
$$X=\eta_1x_1+\eta_2x_2, \quad x=x_1-x_2.$$

Every operator, $O(12,1'2')$, which is of interest, depends only on relative coordinates and the difference of center-of-mass coordinates. Hence we define

$$O_K(xx')=\int e^{-iK(X-X')}O(12,1'2')d^4X'. \quad (2.3)$$

The renormalized wave function is

$$\psi(x_1x_2)=(2\pi)^{-3/2}e^{iKX}\varphi_K(x). \quad (2.4)$$

We single out the Coulomb interaction as the part of $\bar{I}$ mainly responsible for the binding and write

$$\bar{I}(12,34)=I^C(12,34)+I'(12,34),$$
$$I^C(xX;x'X')=-i\alpha\delta(X-X')\delta(x-x')\delta(x_0)\gamma_1^0\gamma_2^0/r. \quad (2.5)$$

This separation serves two purposes. In the first place, we can use perturbation theory[11] with the "unperturbed" equation,

$$[F_{K^C}(xx')-I^C{}_{K^C}(xx')]\varphi_{K^C}(x')=0, \quad (2.6)$$

as a starting point. F_{K^C} is the center-of-mass transform of $\bar{F}_1\bar{F}_2$. To the accuracy of our calculation, the wave function φ_K may be approximated by

$$\varphi_K(x)=-i\int\Lambda_K(xx')\delta(x_0')\varphi_{K^C}(x')d^4x', \quad (2.7)$$

where

$$\Lambda(12,1'2')=G_1^0(11')\gamma_1^0\delta(22')+\delta(11')G_2^0(22')\gamma_2^0. \quad (2.8)$$

Energy corrections to order α^3Ry to K^C are given by[4,6,9]

$$\Delta E=K_0-K_0{}^C=i\int\bar{\varphi}_K(x)J_K(xx')\varphi_K(x')d^4xd^4x'$$
$$=-i\int\varphi_{K^C}(\mathbf{r},0)\delta(x_0)\mathcal{K}_K(xx')\delta(x_0')$$
$$\times\varphi_{K^C}(\mathbf{r}',0)d^4xd^4x', \quad (2.9)$$

[11] Alternative approaches to perturbation theory are presented in S and reference 6.

where

$$K^C=(\mathbf{0},K_0{}^C),$$

$$\mathcal{K}(12,1'2')=\Lambda(12,34)\gamma_1^0\gamma_2^0J(34,3'4')\Lambda(3'4',1'2'), \quad (2.10)$$

$$J(12,1'2')=I'(12,1'2')+I'(12,34)G^C(34,3'4')\times I'(3'4',1'2').$$

G^C is the Green's function of Eq. (2.6).

Secondly, the separation (2.5) enables us to express I' in a way that takes the Coulomb binding into account. Such a form for the interaction kernel is exhibited to lowest order in the Appendix.[12]

The complete kernel to order α^3Ry including binding, is prohibitively complicated. However, binding is not important everywhere. It need be taken into account only when an interaction term contains significant corrections from intermediate states with low-frequency quanta. Such interaction terms may be easily recognized; when treated by Born approximation, they lead to infrared divergent energy shifts for S states. When this happens, we find that the energy shifts of other states may not be calculated from these Born-approximated kernels either. Though neglect of binding does not produce divergences, it is not sufficiently accurate. On the other hand, kernels which do not yield infrared divergences for S states when calculated by Born approximation describe processes of high-momentum transfer. Low-energy quanta, for which binding is important, give a negligible correction to these kernels. As an initial step of simplification, then, we use Born approximation on those parts of J which, when approximated, yield convergent S-state shifts. Two terms cannot be treated this crudely: one describes the exchange of a single quantum; the other arises from vertex parts. Binding is significant for these two, and the correct kernel must be retained.

We have introduced binding by using, as a first approximation to the two-body system, one in which the instantaneous Coulomb potential is the total interaction. Naturally, this potential is not covariant. As a result, we find that, except in those terms of the renormalized equation which are separately ultraviolet divergent, the noncovariant radiation gauge is more convenient than the Lorentz gauge. The divergent terms, of course, can only be recognized and eliminated unambiguously when expressed in covariant form. We therefore use Lorentz gauge in self-energy, vacuum polarization, and virtual annihilation terms.

With the above remarks in mind, we write down the interaction kernel J correct to order α^3Ry:[13]

$$J=J_C+J_B+J_L+J_V+J_A. \quad (2.11)$$

[12] The form is essentially equivalent to those proposed in references 9 and 10.

[13] $\gamma(\xi)=\gamma(\xi,ab)=\gamma_{ab}\delta(x_\xi-x_a)\delta(x_a-x_b)$. The notation, $\gamma(\xi,ab)$ of KK-III is forced upon us by the annihilation interaction. Where the annihilation interaction is not considered, the shorter notation, $\gamma(\xi)$, of reference 2 will be used. The latter employs the standard matrix summation convention for the particle coordinates and spins.

J_C is given by

$$J_C=(4\pi i\alpha)^2\gamma_1^0(\xi)\gamma_2^0(\bar{\xi}')G_1^0G_2^0\gamma_1^0(\xi')\times\gamma_2^0(\bar{\xi})D_C(\xi\xi')D_C(\bar{\xi}\bar{\xi}'),\quad(2.12)$$

where

$$D_C(\xi\xi')=\frac{1}{(2\pi)^4}\int\frac{e^{ik(\xi-\xi')}}{k^2}d^4k.$$

J_C involves corrections to the energy levels due to pair production by the Coulomb field.[14] It is not included in Eq. (2.6). J_B is defined by

$$J_B=J_{B1}+J_{B2},$$

where

$$J_{B1}=4\pi i\alpha[\gamma_1^T(\xi)F_2G^CF_1\gamma_2^T(\xi')+F_1\gamma_2^T(\xi)G^C\gamma_1^T(\xi')F_2-\gamma_1^T(\xi)\gamma_2^T(\xi')]D_T(\xi\xi'),\quad(2.13)$$

$$J_{B2}=(4\pi i\alpha)^2[\gamma_1^T(\xi)\gamma_2^T(\bar{\xi}')G_1^0G_2^0\gamma_1^T(\bar{\xi})\gamma_2^T(\xi')+\gamma_1^T(\xi)\gamma_2^T(\xi')G_1^0G_2^0\gamma_1^T(\bar{\xi})\gamma_2^T(\bar{\xi}')]\times D_T(\xi\xi')D_T(\bar{\xi}\bar{\xi}'),\quad(2.14)$$

$$\gamma_1^T(\xi)\gamma_2^T(\xi')D_T(\xi\xi')=\frac{1}{(2\pi)^4}\int\frac{e^{ik(\xi-\xi')}}{k_\mu^2}\gamma_1^i(\xi)\gamma_2^j(\xi')\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)d^4k.$$

J_{B1} and J_{B2} may be said to represent the exchange by the particles of one and two transverse photons respectively. (The replacement of G^C by $G_1^0G_2^0$ reduces J_{B1} to the more familiar form $4\pi i\alpha\gamma_1^T(\xi)\gamma_2^T(\xi')D_T(\xi\xi')$.) The two parts of J_{B2} are the so-called "crossed" and "uncrossed" terms of the second-order interaction. The kernel J_L, where

$$J_L=-4\pi i\alpha\{\gamma_2(\xi)[F_1(G^C-G_1^0G_2^0)F_1+4\pi i\alpha G_2^0\gamma_1^T(\bar{\xi})\gamma_2^T(\bar{\xi}')G_2^0D_T(\bar{\xi}\bar{\xi}')]\gamma_2(\xi')+\gamma_1(\xi)[F_2(G^C-G_1^0G_2^0)F_2+4\pi i\alpha G_1^0\gamma_1^T(\bar{\xi})\gamma_2^T(\bar{\xi}')G_1^0D_T(\bar{\xi}\bar{\xi}')]\gamma_1(\xi')-(\alpha/2\pi)(B_1+B_2)\gamma_1(\xi)\gamma_2(\xi')\}D(\xi\xi'),\quad(2.15)$$

includes the contribution of all vertex parts. (The first square bracket in Eq. (2.15) represents self-energy contributions of the second particle. It would become the more usual $4\pi i\alpha G_2^0\gamma_1(\bar{\xi})\gamma_2(\bar{\xi}')G_2^0D(\bar{\xi}\bar{\xi}')$, if we made the approximation $G^C=G_1^0G_2^0+G_1^0G_2^0I^CG_1^0G_2^0$, and noted the equivalence

$$\gamma_1^T(\xi)\gamma_2^T(\xi')D_T(\xi\xi')-\gamma_1^0(\xi)\gamma_2^0(\xi')D_C(\xi\xi')=\gamma_1(\xi)\gamma_2(\xi')D(\xi\xi').$$

The second square bracket would reduce in similar fashion.) The last group of terms in Eq. (2.15) results from wave-function renormalization corrections to single quantum exchange. They originate in the replacement of $\psi(x_1x_2)$ by $[1-\alpha(B_1+B_2)/4\pi]\psi(x_1x_2)$ and are included in J_L rather than elsewhere to facilitate computation.

[14] J_C corresponds to $G_{CC}^{(2)}$ of S. It is the "crossed Coulomb" term. Note also that $\gamma_1^0(\xi)\gamma_2^0(\xi')D_C(\xi\xi')=I^C$.

The vacuum polarization correction to the exchange of a single quantum is

$$J_{V\neq}=(4\pi i\alpha)\gamma_1^\nu(\xi)\gamma_2^\mu(\xi')\{(4\pi i\alpha)D(\xi\bar{\xi})\times[\mathrm{tr}(\gamma_1^\nu(\bar{\xi})G_1^0\gamma_1^\mu(\bar{\xi}')G_1^0)+\mathrm{tr}(\gamma_2^\nu(\bar{\xi})G_2^0\gamma_2^\mu(\bar{\xi}')G_2^0)]D(\bar{\xi}'\xi')-(\alpha/2\pi)(A_1+A_2)\delta_{\mu\nu}D(\xi\xi')\}.\quad(2.16)$$

The term containing A_1 and A_2 arises from the already performed charge renormalization and is handled in a fashion analogous to the renormalization term in J_L.

Up to this point, all kernels have the same form, whether m_1 is equal to m_2 or not. However, two modifications must be made in treating positronium. The first concerns J_V. Since there is only a single fermion field in this case, the vacuum is polarized by the emission and absorption of virtual pairs of only one kind of particle. Thus J_V becomes

$$J_{V\mathrm{pos}}=(4\pi i\alpha)\gamma_1^\nu(\xi)\gamma_2^\mu(\xi')\{4\pi i\alpha D(\xi\bar{\xi})\times\mathrm{tr}[\gamma^\nu(\bar{\xi})G^0\gamma^\mu(\bar{\xi}')G^0]D(\bar{\xi}'\xi')-(\alpha/2\pi)A\delta_{\mu\nu}D(\xi\xi')\}.\quad(2.17)$$

In addition, a new interaction, representing virtual annihilation, arises from the particle-antiparticle relationship of the constituents of the system. The lowest-order contribution of this interaction has been calculated by several authors[15–17] and higher-order corrections have been obtained in KK-III. The complete annihilation kernel is

$$J_A=I_{A1}+J_{A2},$$

$$I_{A1}(12,34)=(4\pi i\alpha)\gamma(\xi,11')C(1'2)D(\xi\xi')\times C^{-1}(43')\gamma(\xi',3'3),\quad(2.18)$$

$$J_{A2}=I_{A2}+I_{A1}G_1^0G_2^0I_{A1}+I_{A1}G_1^0G_2^0I_{B1}+I_{B1}G_1^0G_2^0I_{A1}-(\alpha/\pi)(B+\tfrac{1}{2}A)I_{A1},$$

$$I_{A2}(12,34)=(4\pi i\alpha)^2\gamma(\xi,11')G(1'1'')\gamma(\bar{\xi},1''2')\times C(2'2)\,D(\xi\xi')D(\bar{\xi}\bar{\xi}')[C^{-1}(33')\gamma(\bar{\xi}'3'4')\times G(4'4'')\gamma(\xi',4''4)-C^{-1}(44')\gamma(\bar{\xi}',4'3')\times G(3'3'')\gamma(\xi',3''3),\quad(2.19)$$

$$I_{B1}(12,34)=(4\pi i\alpha)\gamma(\xi,13)\gamma(\xi',24)D(\xi\xi').$$

3. COULOMB CORRECTIONS

In order to calculate energy levels of Eq. (2.1) correct to order α^3Ry, we must first determine the "unperturbed" energy K_0^C to that order. For this purpose, we write (2.6) in the form

$$[K_0^C-H_1(\mathbf{p})-H_2(-\mathbf{p})]\varphi_{K^C}(\mathbf{r}0)+(2\pi)^{-3}\int e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}[\Lambda_1^+(\mathbf{k})\Lambda_2^+(-\mathbf{k})-\Lambda_1^-(\mathbf{k})\Lambda_2^-(-\mathbf{k})](\alpha/r')\varphi_{K^C}(\mathbf{r}'0)=0,\quad(3.1)$$

[15] V. B. Berestetski and L. D. Landau, J. Exptl. Theoret. Phys. (U.S.S.R.) **19**, 673 (1949).
[16] J. Pirenne, Arch. sci. phys. et nat. **29**, 207 (1947).
[17] R. H. Ferrel, Ph.D. thesis, Princeton, 1951 (unpublished).

where

$$H_{1,2}(\mathbf{p})=\boldsymbol{\alpha}_{1,2}\mathbf{p}+m_{1,2}\gamma_{1,2}{}^{0},$$

$$\Lambda_{1,2}{}^{+}(\mathbf{k})=[E_{1,2}(\mathbf{k})\pm H_{1,2}(\mathbf{k})]/2E_{1,2}(\mathbf{k}),$$

$$\mathbf{p}=-i\boldsymbol{\nabla}, \quad \text{and} \quad E_{1,2}(\mathbf{k})=(m_{1,2}{}^{2}+k^{2})^{\frac{1}{2}}.$$

Equation (3.1) is very similar to the Breit equation without magnetic interaction,

$$[K_0{}'^{C}-H_1(\mathbf{p})-H_2(-\mathbf{p})+\alpha/r]\varphi_B(\mathbf{r})=0. \quad (3.2)$$

The energy levels of (3.2) may be expressed as power series in α^2 with their leading terms of order Ry;[18] the energy levels of (3.1) may be expanded in powers of α. The corresponding energy levels are equal to order $\alpha^2 Ry$; to the order Ry, they are the reduced mass energy levels of the Schroedinger equation. We compute by perturbation theory the difference, to order $\alpha^3 Ry$, between $K_0{}'^{C}$ and $K_0{}^{C}$, the former having no $\alpha^3 Ry$ term in its expansion. We rewrite (3.1) as

$$[K_0{}'^{C}-H_1(\mathbf{p})-H_2(-\mathbf{p})+(\alpha/r)]\varphi_K{}^{C}(\mathbf{r}0)$$

$$-\int\Omega(\mathbf{r}\mathbf{r}')(\alpha/r')\varphi_K{}^{C}(\mathbf{r}'0)=0, \quad (3.1a)$$

where

$$\Omega(\mathbf{r}\mathbf{r}')=(2\pi)^{-3}\int e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}[\Lambda_1{}^{+}(\mathbf{k})\Lambda_2{}^{-}(-\mathbf{k})$$

$$+\Lambda_1{}^{-}(\mathbf{k})\Lambda_2{}^{+}(-\mathbf{k})+2\Lambda_1{}^{-}(\mathbf{k})\Lambda_2{}^{-}(-\mathbf{k})].$$

$\Omega(\alpha/r')$ is a small correction to the Hamiltonian of (3.2). By first-order perturbation theory we get

$$\Delta E_{Ca}=K_0{}^{C}-K_0{}'^{C}=\int\varphi_B{}^{*}(\mathbf{r})\Omega(\mathbf{r}\mathbf{r}')(\alpha/r')\varphi_B(\mathbf{r}'). \quad (3.3)$$

Let

$$\varphi_{++}(\mathbf{r})=(2\pi)^{-3}\int e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\Lambda_1{}^{+}(\mathbf{k})\Lambda_2{}^{+}(-\mathbf{k})\varphi_B(\mathbf{r}').$$

The "large component" $[\varphi_{++}{}^{*}(\mathbf{r})]$ of $\varphi_B{}^{*}(\mathbf{r})$ is orthogonal to $\Omega(\mathbf{r}\mathbf{r}')$. The small components, correct to the required order, are obtained by iteration. We write (3.2) as an integral equation,

$$\varphi_B(\mathbf{r})=(2\pi)^{-3}\int e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}[H_1(\mathbf{k})+H_2(-\mathbf{k})$$

$$-K_0{}'^{C}]^{-1}(\alpha/r')\varphi_B(\mathbf{r}'), \quad (3.4)$$

replace $\varphi_B(\mathbf{r}')$ by the Pauli wave function $\varphi_P(\mathbf{r}')$ on the right-hand side of (3.4), and apply the proper projection operators. Using these wave functions for $\varphi_B{}^{*}(\mathbf{r})$ in (3.3) and also substituting $\varphi_P(\mathbf{r}')$ for $\varphi_B(\mathbf{r}')$ in this equation, we get[4]

$$\Delta E_{Ca}=\frac{\alpha^2}{(2\pi)^3}\int\varphi_P{}^{*}(\mathbf{r})\frac{1}{r}e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\frac{[\Lambda_1{}^{+}(\mathbf{k})\Lambda_2{}^{-}(-\mathbf{k})+\Lambda_1{}^{-}(\mathbf{k})\Lambda_2{}^{+}(-\mathbf{k})+2\Lambda_1{}^{-}(\mathbf{k})\Lambda_2{}^{-}(-\mathbf{k})]}{H_1(\mathbf{k})+H_2(-\mathbf{k})-K_0{}'^{C}}\frac{1}{r'}\varphi_P(\mathbf{r}'). \quad (3.5)$$

Since contributions to the integral come only from large momenta $k\cong m$, φ_P may be replaced by the amplitude of the Pauli wave function at the origin. With these approximations, the energy shift ΔE_{Ca} becomes

$$\Delta E_{Ca}=\frac{\alpha^2|\varphi(0)|^2}{2\pi}\int\frac{d^3k}{k^4}\frac{1}{E_1E_2}\left[\frac{(E_1+m_1)(E_2-m_2)}{(E_1-E_2)-(m_1+m_2)}\right.$$

$$\left.+\frac{(E_1-m_1)(E_2+m_2)}{(E_2-E_1)-(m_1+m_2)}-\frac{2(E_1-m_1)(E_2-m_2)}{E_1+E_2+m_1+m_2}\right];$$

$$E_{1,2}=E_{1,2}(k). \quad (3.6)$$

This integral can be evaluated exactly and yields

$$\Delta E_{Ca}=-\frac{2}{3}\alpha^2|\varphi(0)|^2\left(\frac{2}{m_1{}^{2}}+\frac{1}{m_1m_2}+\frac{2}{m_2{}^{2}}\right). \quad (3.7)$$

The kernel J_C gives rise to an integral which is very similar to Eq. (3.6). Because only large photon momenta contribute, we may set

$$\Delta E=-i\int\varphi_K{}^{C}{}^{*}(\mathbf{r}0)\delta(x_0)\mathcal{K}_K(xx')\delta(x_0')$$

$$\times\varphi_K{}^{C}(\mathbf{r}'0)d^4xd^4x'\cong-i|\varphi(0)|^2\int J_K(xx')d^4xd^4x'. \quad (3.8)$$

Some simplification yields the integral

$$\Delta E_{Cb}=\frac{i\alpha^2}{\pi^2}|\varphi(0)|^2\int\frac{d^4k}{k^4}\left\langle\frac{\Lambda_1{}^{+}(\mathbf{k})\Lambda_2{}^{-}(\mathbf{k})+\Lambda_1{}^{-}(\mathbf{k})\Lambda_2{}^{+}(\mathbf{k})}{(k_0-\eta_1K_0{}^{C}+H_1(\mathbf{k}))(k_0-\eta_2K_0{}^{C}+H_2(\mathbf{k}))}\right\rangle,$$

$$\cong\frac{\alpha^2}{2\pi}|\varphi(0)|^2\int\frac{d^3k}{k^4}\frac{1}{E_1E_2}\left[\frac{(E_1+m_1)(E_2-m_2)}{(E_1-m_1)+(E_2+m_2)}+\frac{(E_1-m_1)(E_2+m_2)}{(E_1+m_1)+(E_2-m_2)}\right],$$

$$=\frac{2}{3}\alpha^2|\varphi(0)|^2\left(\frac{2}{m_1{}^{2}}-\frac{1}{m_1m_2}+\frac{2}{m_2{}^{2}}\right). \quad (3.9)$$

[18] T. Ishidzu, Progr. Theoret. Phys. **6**, 154 (1951).

The brackets, $\langle\ \rangle$, denote expectation values between Pauli wave functions. The energy shift of order $\alpha^3 Ry$ arising purely from the Coulomb interaction[8] is

$$(\Delta E_C)_{2S}=-\frac{1}{6\pi}\frac{\alpha^5\mu^2}{m_1+m_2} \quad \text{for } 2S \text{ states}, \tag{3.10a}$$

$$(\Delta E_C)_{2P}=0 \quad \text{for } 2P \text{ states} \tag{3.10b}$$

4. EFFECTS OF TRANSVERSE QUANTA

The kernel $\mathcal{K}_{B1}$, derived from Eqs. (2.8), (2.10), and (2.13) is given in matrix notation by

$$\begin{aligned}\mathcal{K}_{B1}=\{&\alpha_2{}^T(\xi)G^C\gamma_1{}^0\gamma_2{}^0\alpha_1{}^T(\xi')\\&+\gamma_2{}^0\gamma_2{}^T(\xi)[G_1{}^0+G^CI^CG_1{}^0]\gamma_1{}^T(\xi')G_1{}^0\gamma_1{}^0\\&+G_2{}^0\gamma_1{}^0\gamma_2{}^T(\xi)G_2{}^0I^CG^C\gamma_1{}^T(\xi')\gamma_2{}^0\\&+G_2{}^0\gamma_1{}^0\gamma_2{}^T(\xi)G_2{}^0[I^C+I^CG^CI^C]\\&\times G_1{}^0\gamma_1{}^T(\xi')G_1{}^0\gamma_1{}^0+(1\leftrightarrow 2)\}D_T(\xi\xi'),\end{aligned} \tag{4.1}$$

where we have set $\alpha^T=\gamma^0\gamma^T$ and used the relations

$$\begin{aligned}G^C&=G_1{}^0G_2{}^0+G_1{}^0G_2{}^0I^CG^C=G_1{}^0G_2{}^0+G^CI^CG_1{}^0G_2{}^0\\&=G_1{}^0G_2{}^0+G_1{}^0G_2{}^0[I^C+I^CG^CI^C]G_1{}^0G_2{}^0.\end{aligned}$$

Our first step in calculating the energy change due to $\mathcal{K}_{B1}$ is to use Born approximation on the terms of this kernel which involve pair processes, and to neglect processes which require three quanta. Thus we let

$$\begin{aligned}\mathcal{K}_{B1}\rightarrow[&\alpha_2{}^T(\xi)G^C\gamma_1{}^0\gamma_2{}^0\alpha_1{}^T(\xi')\\&+G_1{}^0\gamma_2{}^0\gamma_2{}^T(\xi)\gamma_1{}^T(\xi')G_1{}^0\gamma_1{}^0\\&+G_1{}^0\gamma_2{}^0\gamma_2{}^T(\xi)G_2{}^0I^CG_1{}^C\gamma_1{}^T(\xi')G_1{}^0\gamma_1{}^0\\&+G_2{}^0\gamma_1{}^0\gamma_2{}^T(\xi)G_2{}^0I^CG_1{}^0\gamma_1{}^T(\xi')G_2{}^0\gamma_2{}^0\\&+G_2{}^0\gamma_1{}^0\gamma_2{}^T(\xi)G_2{}^0I^CG_1{}^0\gamma_1{}^T(\xi')G_1{}^0\gamma_1{}^0\\&+(1\leftrightarrow 2)]D_T(\xi\xi').\end{aligned} \tag{4.1a}$$

By substituting $G^C=G_1{}^0G_2{}^0+G_1{}^0G_2{}^0I^CG_1{}^0G_2{}^0$ in (4.1a) and rearranging terms, one may obtain the kernels which give rise to the infrared divergent terms ΔE_B and ΔE_{CT} of S. To avoid these divergences, we treat the first term of (4.1a) more carefully, writing

$$\begin{aligned}\Delta E=4\pi\alpha\int&\varphi_K{}^{C*}\delta(x_0)\{[\alpha_2{}^T(\xi)G^C\gamma_1{}^0\gamma_2{}^0\alpha_1{}^T(\xi')\\&+(1\leftrightarrow 2)]D_T(\xi\xi')\}_K\delta(x_0')\varphi_K{}^C\\=i\alpha(2\pi)^{-3}\int&\varphi_K{}^{C*}(\mathbf{r},0)[\alpha_2{}^iG^C{}_{-\mathbf{k},K_0'}(\mathbf{r}0;\mathbf{r}'0)\\&\times\alpha_1{}^ie^{-i\mathbf{k}\cdot(\eta_1\mathbf{r}+\eta_2\mathbf{r}')}+\alpha_1{}^iG^C{}_{-\mathbf{k},K_0'}(\mathbf{r}0;\mathbf{r}'0)\\&\times\alpha_2{}^ie^{i\mathbf{k}\cdot(\eta_2\mathbf{r}+\eta_1\mathbf{r}')}]\exp[i(K_0{}^C-K_0')(X_0-X_0')]\\&\times e^{-ik|X_0-X_0'|}k^{-1}(\delta_{ij}-k_ik_j/k^2)\varphi_K{}^C(\mathbf{r}'0)\\&\times d^3\mathbf{r}d^3\mathbf{r}'d^3\mathbf{k}dX_0'dK_0'.\end{aligned} \tag{4.2}$$

For wave functions, we use the approximate form

$$\varphi_K{}^C(\mathbf{r}0)\cong(1+\boldsymbol{\alpha}_1\cdot\mathbf{p}/2m_1)(1-\boldsymbol{\alpha}_2\cdot\mathbf{p}/2m_2)\varphi_P(\mathbf{r}). \tag{4.3}$$

We divide the integration over k into two parts: a region where k is less than and another where k is greater than some given constant. This separation of regions of high and low photon frequencies simultaneously divides G^C into regions of high and low recoil momentum. In the high-frequency, high-recoil-momentum region, G^C can be expanded in terms of free particle Green's functions. In the region of low-frequency quanta and low recoil momenta, large values of $K_0{}^C$ (high-energy intermediate states) do not contribute, and a nonrelativistic approximation to the Green's function may be made. This nonrelativistic approximation involves setting

$$\begin{aligned}G_C(\mathbf{r}0,X;\mathbf{r}'0,X')\cong&-(2\pi)^{-3}\int d^3K''\Sigma_n\varphi\overline{\mathbf{K}}''{}_{,n}(\mathbf{r}0)\\&\times\varphi^*\overline{\mathbf{K}}''{}_{,n}(\mathbf{r}'0)e^{iK''(X-X')}, \quad X_0>X_0',\\\cong&0, \quad X_0<X_0',\end{aligned} \tag{4.4}$$

where $K_0''=K_n$ is the energy of the nth state. The $\varphi\mathbf{K}''{}_{,n}(\mathbf{r}0)$ may be taken to satisfy

$$\begin{aligned}[H_1(\eta_1\mathbf{K}''+\mathbf{p})+H_2(\eta_2\mathbf{K}''-\mathbf{p})-\alpha/r]&\varphi_{\mathbf{K}'',n}(\mathbf{r}0)\\&=K_n\varphi_{\mathbf{K}'',n}(\mathbf{r}0).\end{aligned} \tag{4.5}$$

In fact, the approximate solutions

$$\begin{aligned}\varphi_{\mathbf{K}'',n}(\mathbf{r}0)\cong&[1+\boldsymbol{\alpha}_1\cdot(\eta_1\mathbf{K}''+\mathbf{p})/2m_1]\\&\times[1+\boldsymbol{\alpha}_2\cdot(\eta_2\mathbf{K}''-\mathbf{p})/2m_2]\varphi_P(\mathbf{r}),\end{aligned}$$

where

$$K_n\cong E_n+K''{}^2/2(m_1+m_2), \quad E_n=m_1+m_2-\alpha^2\mu/2n^2,$$

are sufficient.

The boundary between the relativistic and nonrelativistic regions is not critical. Thus the constant separating the two regions can be as large as $B\cong\alpha^{\frac{1}{2}}\mu$, or as small as $A=\alpha^{\frac{3}{2}}\mu$. As indicated in S, the region A to B, in either approximation, yields the same shift.

We are now in a position to outline the details of the calculation. We combine the lowest order, high energy part of the first term in expression (4.1a), $\alpha_1{}^iG_1{}^0G_2{}^0\alpha_2{}^j$, with the second term, and the corresponding terms with the particles interchanged. Employing techniques of KK-III, we then derive to the required order of accuracy a "high-energy, single transverse quantum" contribution,

$$\begin{aligned}\Delta E_{B1}{}^{TH}=\alpha(2\pi^2)^{-1}\int&\phi^*(\mathbf{p}')\delta(\mathbf{p}'-\mathbf{p}''+\mathbf{k})(2k)^{-1}\\&\times R(p'p''k)S(\mathbf{p}'\mathbf{p}''\mathbf{k})\phi(\mathbf{p}'').\end{aligned} \tag{4.6}$$

where

$$R(p'p''k)=\frac{1}{4E_1'E_2''}\frac{(E_1'+m_1)(E_2''+m_2)}{k+E_1'+E_2''-K_0{}^C}+\frac{1}{2E_1'E_1''}\frac{m_1{}^2-E_1'E_1''}{k+E_1'+E_1''}+\frac{1}{4E_1'E_2''}\times\frac{(m_1-E_1')(m_2-E_2'')}{k+E_1'+E_2''+K_0{}^C}+(1\leftrightarrow 2),$$

$$S(\mathbf{p}'\mathbf{p}''\mathbf{k})\delta(\mathbf{p}'-\mathbf{p}''+\mathbf{k})=\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)\left\langle\left(1+\frac{\boldsymbol{\alpha}_1\cdot\mathbf{p}'}{2m_1}\right)\left(1-\frac{\boldsymbol{\alpha}_2\cdot\mathbf{p}'}{2m_2}\right)\times\alpha_1{}^i\alpha_2{}^j\left(1+\frac{\boldsymbol{\alpha}_1\cdot\mathbf{p}''}{2m_1}\right)\left(1-\frac{\boldsymbol{\alpha}_2\cdot\mathbf{p}''}{2m_2}\right)\right\rangle\delta(\mathbf{p}'-\mathbf{p}''+\mathbf{k}),$$

and

$$S(\mathbf{p}'\mathbf{p}''\mathbf{k})=-\frac{1}{4m_1m_2}\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)\times\langle[2\mathbf{p}'-i(\boldsymbol{\sigma}_1\times\mathbf{k})]_i[2\mathbf{p}''-i(\boldsymbol{\sigma}_2\times\mathbf{k})]_j\rangle.$$

The wave function $\phi(\mathbf{p})$ is the Fourier transform of $\varphi_P(\mathbf{r})$. The first approximation to (4.6), obtained by letting $R\rightarrow 2k^{-1}$, is the part of the Breit interaction energy which arises from the momentum region $k>B$:

$$\mathcal{E}_{B\infty}=\frac{\alpha}{2\pi^2}\int_{k>B}d^3k\int\varphi_B{}^*(\mathbf{r})\frac{\alpha_1{}^i\alpha_2{}^j}{k^2}\times\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)e^{i\mathbf{k}\cdot\mathbf{r}}\varphi_B(\mathbf{r}). \quad (4.7)$$

In this notation, the total Breit interaction energy is $\mathcal{E}_{0\infty}$.

At energies greater than B, the approximation procedures outlined in KK-III (in the paragraph below Eq. (4.8)) are applicable. Corrections to $\mathcal{E}_{B\infty}$ in this relativistic region ($k>B$) arise only for $p'\gg p''$, or $p''\gg p'$. In these regions we let

$$\delta(\mathbf{p}'-\mathbf{p}''+\mathbf{k})\rightarrow\delta(\mathbf{p}'+\mathbf{k}),$$

and

$$\delta(\mathbf{p}'-\mathbf{p}''+\mathbf{k})\rightarrow\delta(-\mathbf{p}''+\mathbf{k}),$$

respectively. Consequently, in terms correcting the Breit energy, we can allow

$$S\delta\rightarrow(4m_1m_2)^{-1}\langle(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)k^2-(\boldsymbol{\sigma}_1\cdot\mathbf{k})(\boldsymbol{\sigma}_2\cdot\mathbf{k})\rangle\delta. \quad (4.8)$$

Equation (4.6) can then be integrated to give for $\Delta E_{B1}{}^{TH}$ the result of Arnowitt.[6]

$$\Delta E_{B1}{}^{TH}=\mathcal{E}_{B\infty}-\frac{4}{3}\frac{\alpha^2}{m_1m_2}|\varphi(0)|^2\langle\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\rangle\times\left(2+\eta_2\ln\frac{m_1}{2B}+\eta_1\ln\frac{m_2}{2B}\right). \quad (4.9)$$

Equation (4.6) may also be evaluated for photon momenta lying between A and B. In this region, the methods of KK-III are no longer sufficient; p', p'', and k are now all of comparable magnitude and $k\ll\mu$. Here, we set

$$R(p'p''k)\approx\frac{1}{4E_1'E_2''}\frac{(m_1+E_1')(m_2+E_2'')}{k+E_1'+E_2''-K_0{}^C}+(1\leftrightarrow 2)\approx\frac{1}{k}\left(2+\frac{K_0{}^C-E_1'-E_2'}{k}+\frac{K_0{}^C-E_1''-E_2''}{k}\right), \quad (4.10)$$

and obtain

$$\Delta E_{B1}{}^{TH}=\mathcal{E}_{AB}+\frac{\alpha}{(2\pi)^2}\int_{k=A}^{k=B}d^3k\int\phi^*(\mathbf{p}')\times\delta(\mathbf{p}'-\mathbf{p}''+\mathbf{k})\frac{1}{k^3}S(\mathbf{p}'\mathbf{p}''\mathbf{k})[(K_0{}^C-E_1'-E_2')+(K_0{}^C-E_1''-E_2'')]\phi(\mathbf{p}''). \quad (4.11)$$

The remaining interaction terms of Eq. (4.1a) are relevant only at high energies. Combined with the second part of the expansion of the first term, $\alpha_1{}^iG_1{}^0G_2{}^0I^CG_1{}^0G_2{}^0\alpha_2{}^j$, these comprise the so-called Coulomb-transverse interaction. This kernel gives rise to an energy shift

$$\Delta E_{B1}{}^{CTH}=-i\int\varphi_K{}^{c*}\delta(x_0)\{\Lambda\gamma_1{}^0\gamma_2{}^0\times[\gamma_2{}^T(\xi)G_2{}^0I^CG_1{}^0\gamma_1{}^T(\xi')+(1\leftrightarrow 2)]D_T(\xi\xi')\Lambda\}_K\delta(x_0')\varphi_K{}^c$$

$$=-\frac{4\alpha^2i}{(2\pi)^2}|\varphi(0)|^2\left\langle\int_{k>B}d^4k\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)\times\frac{1}{k^2k_\mu{}^2}\left[\alpha_2{}^i\frac{1}{H_1(\mathbf{k})-m_1+k_0}\times\frac{1}{H_2(\mathbf{k})-m_2+k_0}\alpha_1{}^j+(1\leftrightarrow 2)\right]\right\rangle. \quad (4.12)$$

This shift contains only a spin-spin correction,

$$\Delta E_{B1}{}^{CTH}=\frac{4\alpha^2}{3m_1m_2}\langle\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\rangle|\varphi(0)|^2\times\left\{2-\ln\frac{2B}{\mu}-\frac{m_1\ln\eta_1-m_2\ln\eta_2}{m_1-m_2}\right\}. \quad (4.13)$$

In the region A to B, the energy shift arises entirely from

$$\alpha_1{}^iG_1{}^0G_2{}^0I^CG_1{}^0G_2{}^0\alpha_2{}^i+(1\leftrightarrow 2),$$

that is, the part of J_{B1} treated in Eq. (4.2). This part yields approximately

$$\Delta E_{B1}{}^{CTH}=\frac{\alpha^2}{(2\pi)^4}\int_{k=A}^{k=B}d^3k\int\phi^*(\mathbf{p}')\times\frac{S(\mathbf{p}'\mathbf{p}''\mathbf{k})+S(\mathbf{p}''\mathbf{p}'\mathbf{k})}{k^3(\mathbf{p}'-\mathbf{p}''+\mathbf{k})^2}\phi(\mathbf{p}''). \quad (4.14)$$

Thus the high-energy result, the sum of (4.9) and (4.13), is

$$\Delta E_{B1}{}^{H}=\mathcal{E}_{B\infty}-\frac{8\alpha^2|\varphi(0)|^2}{3(m_1{}^2-m_2{}^2)}\langle\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\rangle\ln\frac{m_1}{m_2}. \quad (4.15)$$

The Fourier transforms of Eqs. (4.11) and (4.14) combine to give, after some manipulation,

$$\Delta E_{B1}{}^{M}=\mathcal{E}_{AB}+\frac{2\alpha}{m_1m_2}\frac{1}{(2\pi)^2}\int_{k=A}^{k=B}\int\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)\frac{1}{k^3}\times\varphi_P{}^*(r)e^{-i\mathbf{k}\cdot\mathbf{r}}p_i[\mathfrak{H}\mathcal{C},p_j]\varphi_P(\mathbf{r}), \quad (4.16)$$

for the intermediate region. $\mathfrak{H}\mathcal{C}$ is the Schroedinger Hamiltonian. Equation (4.16) contains an orbit-orbit interaction only; the spin-orbit and spin-spin parts vanish. The equation agrees with (48) of S.

The nonrelativistic contribution of (4.1) may be rewritten, in view of (4.2) and (4.4), in a form which also arises naturally out of three-dimensional perturbation theory:

$$\Delta E=\frac{\alpha}{(2\pi)^2}\int\varphi_K{}^{c*}(\mathbf{r}0)[\alpha_2{}^ie^{-i\mathbf{k}\cdot\eta_1\mathbf{r}}\varphi_{-\mathbf{k},n}(\mathbf{r}0)\times\varphi_{-\mathbf{k},n}{}^*(\mathbf{r}'0)\alpha_1{}^ie^{-i\mathbf{k}\cdot\eta_2\mathbf{r}'}+\alpha_1{}^ie^{i\mathbf{k}\cdot\eta_2\mathbf{r}}\times\varphi_{-\mathbf{k},n}(\mathbf{r}0)\varphi_{-\mathbf{k}n}{}^*(\mathbf{r}'0)\alpha_2{}^ie^{i\mathbf{k}\cdot\eta_1\mathbf{r}'}]\times\frac{1}{k}\sum_n\frac{1}{k+K_n-K_0{}^C}\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)\varphi_K{}^c(\mathbf{r}'0). \quad (4.17)$$

We first isolate the part of the Breit energy, $\mathcal{E}$, in (4.17), arising from nonrelativistic momenta. The remainder is treated by two approximations. In the region $0\leqslant k\leqslant A$, we may set $k=0$ in the exponents and $\varphi_{-\mathbf{k},n}$, so that

$$\Delta E_{B1}{}^{L}=\mathcal{E}_{0A}-\frac{2\alpha}{(2\pi)^2}\int_{=0}^{k=A}\frac{d^3k}{k^2}\sum_n\frac{(0|\alpha_1{}^i|n)(n|\alpha_2{}^i|0)}{k+K_n-K_0{}^C}\times(K_n-K_0{}^L)\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)$$

$$=\mathcal{E}_{0A}+\frac{4\alpha}{3\pi m_1m_2}\left\langle(\mathbf{p}\varphi_P(\mathbf{r}))^*\cdot\ln\frac{A}{\mathfrak{H}\mathcal{C}-K_0{}^C}\times(\mathfrak{H}\mathcal{C}-K_0{}^C)(\mathbf{P}\varphi_P(\mathbf{r}))\right\rangle. \quad (4.18)$$

Again, Eq. (4.18) has no spin-orbit or spin-spin contributions. Because $K_n=E_n$ when k is small, the α^3Ry correction [see S, Eq. (46)] is[19]

$$(\Delta E_{B1}{}^{L})_{2S}=\frac{\alpha^5\mu^2}{3\pi(m_1+m_2)}\ln\left[\frac{A}{k_0(2,0)}\frac{Ry_\infty}{Ry_\mu}\right], \quad (4.19a)$$

and

$$(\Delta E_{B1}{}^{L})_{2P}=(\alpha^5\mu^2/3\pi(m_1+m_2))\ln(Ry_\infty/k_0(2,1)). \quad (4.19b)$$

In $2P$ states, the contribution is independent of A, provided A is large compared to the binding energy. The A we have chosen fulfills this requirement.

Only certain states in the summation over n [in Eq. (4.18)] contribute significantly in the intermediate region to the correction to the Breit energy. In these states, and for this region, $K_n-K_0{}^C$ is much smaller than k. We therefore let

$$\frac{1}{k^2}\frac{K_n-K_0{}^C}{k+K_n-K_0{}^C}\rightarrow\frac{K_n-K_0{}^C}{k^3}. \quad (4.20)$$

Use of the approximation (4.20) yields the expression

$$\Delta E_{B1}{}^{M}=\mathcal{E}_{AB}+\frac{\alpha^2}{(2\pi)^2}\int_{k=A}^{k=B}\frac{d^3k}{k^3}\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)\times\{(0|\alpha_1{}^ie^{ik\cdot\eta_1\mathbf{r}}|n,-k)(n,-k|\alpha_2{}^ie^{ik\cdot\eta_2\mathbf{r}}|0)+(0|\alpha_2{}^ie^{-i\mathbf{k}\cdot\eta_2\mathbf{r}}|n,-k)\times(n,-k|\alpha_1{}^ie^{-i\mathbf{k}\cdot\eta_1\mathbf{r}}|0)\}(K_n-K^0). \quad (4.21)$$

Evaluation of Eq. (4.21) shows that spin-spin and spin-orbit terms vanish, while the orbit-orbit part agrees with that in Eq. (4.16). This again confirms[4] the equivalence of the two methods for the intermediate region. The α^3Ry contribution of Eq. (4.16) is like that given in S in Eqs. (48a) and (49a):

$$(\Delta E_{B1}{}^{M})_{2S}=\frac{\alpha^5\mu^2}{3\pi(m_1+m_2)}\left[\frac{25}{12}+\ln\frac{\alpha\mu}{A}\right], \quad (4.22a)$$

and

$$(\Delta E_{B1}{}^{M})_{2P}=-\frac{2\alpha^2}{3\pi m_1m_2}\left\langle\frac{1}{r^3}\right\rangle_{2P}=-\frac{\alpha^5\mu^2}{36\pi(m_1+m_2)}. \quad (4.22b)$$

The energy shift due to J_{B2} is simpler to calculate than that due to J_{B1}. The extremely low energy region does not contribute, and the effects of both the intermediate and relativistic regions can be gotten from the free particle expansions of these interaction terms in-

[19] $k_0(2,0)$ and $k_0(2,1)$ are given by Bethe, Brown, and Stehn, Phys. Rev. **77**, 370 (1950). They are calculated using the electron mass. The factor Ry_∞/Ry_μ serves as the reduced mass correction.

volving two transverse photons. In the relativistic region $k>B$ it is sufficient to set

$$\Delta E_{B2}{}^{H}=-i|\varphi(0)|^2\int(J_{B2})_K(xx')d^4xd^4x'. \quad (4.23)$$

This approximation neglects the relative time dependence of the wave function. It also disregards p' and p'', the momenta of the initial and final states, relative to k and k', the photon momenta. Substituting the explicit form of J_{B2} in Eq. (4.23), we obtain

$$\Delta E_{B2}{}^{H}=\frac{4\alpha^2 i}{(2\pi)^2}|\varphi(0)|^2\int_{k>B}\frac{d^4k}{(k_\mu{}^2)^2}\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)\left(\delta_{lm}-\frac{k_lk_m}{k^2}\right)\Bigg\langle\frac{\gamma_1{}^i[m_1+\gamma_1{}^0(m_1-k_0)]\gamma_1{}^l}{[E_1{}^2-(m_1-k_0)^2]}$$
$$\times\frac{\gamma_2{}^m[m_2+\gamma_2{}^0(m_2-k_0)]\gamma_2{}^i}{[E_2{}^2-(m_2-k_0)^2]}+\frac{\gamma_1{}^i[m_1+\gamma_1{}^0(m_1-k_0)]\gamma_1{}^l\gamma_2{}^i[m_2+\gamma_2{}^0(m_2+k_0)]\gamma_2{}^m}{[E_1{}^2-(m_1-k_0{}^2][E_2{}^2-(m_2+k_0)^2]}\Bigg\rangle. \quad (4.24)$$

The integral may be evaluated exactly; it has the value

$$\Delta E_{B2}{}^{H}=\frac{2\alpha^2|\varphi(0)|^2}{m_1m_2}\left[\ln\frac{2B}{\mu}+\frac{m_1{}^2\ln\eta_1-m_2{}^2\ln\eta_2}{m_1{}^2-m_2{}^2}\right.$$
$$\left.+\frac{1}{3}\frac{m_1m_2}{m_1{}^2-m_2{}^2}\langle\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\rangle\ln\frac{m_1}{m_2}\right]. \quad (4.25)$$

The expression $\ln(2B/\mu)$ arises from two pair terms only. These same terms are the only ones which are significant in the region $A<k<B$. In this region, we still neglect retardation corrections, but no longer omit p' and p'' relative to k and k'. This means that we let

$$\Delta E_{B2}{}^{M}\cong-i\int\varphi_P{}^*(\mathbf{r})\gamma_1{}^0\gamma_2{}^0$$
$$\times(J_{B2}(xx')_{\text{two pair}}\,\varphi_P\,(\mathbf{r}'). \quad (4.26)$$

The contribution of (4.26) in the region $0<k<A$ is negligible, and so the integral may be extended from 0 to B. This expression may be simplified to (see S, Sec. VIB)

$$\Delta E_{B2}{}^{M}=-\frac{\alpha^2}{(2\pi)^4m_1m_2}\int\int_0^B d^3k\int_0^\infty d^3k'\varphi_P{}^*(\mathbf{r})$$
$$\times e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}}\frac{[1+(\mathbf{k}\cdot\mathbf{k}'/kk')^2]}{kk'(k+k')}\varphi_P(\mathbf{r}). \quad (4.27)$$

Because of the α^2 coefficient, α^3Ry parts of (4.27) will appear only when the initial and final relative momenta are small. Hence k and k' must be roughly equal. In $2S$ states we utilize this fact to make the replacement

$$\int_0^B d^3k\int_0^\infty d^3k'\cdots\cong\int_0^{2B}d^3k\int_0^{2B-k}d^3k'\cdots.$$

In $2P$ states, we merely note that values of k larger than B do not contribute to our order, so that

$$\int_0^B d^3k\int_0^\infty d^3k'\cdots\cong\int_0^\infty d^3k\int_0^\infty d^3k'\cdots.$$

With these changes in the regions of integration, the integrals may be evaluated exactly, although considerable computation is required. The final result is

$$(\Delta E_{B2}{}^{M})_{2S}=\frac{\alpha^5\mu^2}{m_1+m_2}\frac{1}{4\pi}\left[\ln\left(\frac{\mu\alpha}{2B}\right)+\frac{7}{4}+\frac{4}{3}(1-\ln2)\right], \quad (4.28a)$$

and

$$(\Delta E_{B2}{}^{M})_{2P}=-\frac{\alpha^2}{2\pi}\frac{1}{m_1m_2}\left\langle\frac{1}{r^3}\right\rangle_{2P}=-\frac{\alpha^5\mu^2}{48\pi(m_1+m_2)}. \quad (4.28b)$$

The evaluation of $\Delta E_{B2}{}^{M}$ can be briefly outlined as follows. One need only calculate the integral (4.27) for a $1S$ state in order to find the energy shifts for all S states. [This is also true for the S state corrections which arise from J_{B1} and are displayed in Eq. (4.21).] The integral for the nS state may then be determined by parametric differentiations of the result for the $1S$ state. To evaluate $\Delta E_{B2}{}^{M}$ for the ground state, we first perform the integral over $\mathbf{r}$ followed by the integration over the remaining angle variables. After making the transformations $x=(k+k')/\sqrt{2}$ and $xu=(k-k')/\sqrt{2}$, we perform several partial integrations with respect to u and rearrange terms. The integral over x and then the integral over u may be simply evaluated. Confirmation of our method and that of S is provided by the agreement of our result, $(4/3)(1-\ln2)=0.409$, with the answer of S, 0.411, obtained by numerical integration.

To conclude this section, we summarize the total energy correction from J_B:

$$(\Delta E_B)_{2S}=(\mathcal{E}_{0\infty})_{2S}+\frac{\alpha^5\mu^2}{m_1+m_2}\frac{1}{8\pi}\left\{\frac{8}{3}\left[\frac{25}{12}+\ln\frac{2Ry_\infty}{\alpha k_0(2,0)}\right]\right.$$
$$+2\left[\ln\alpha+\frac{7}{4}+\frac{4}{3}(1-\ln2)\right]+\frac{2}{m_1{}^2-m_2{}^2}\left[m_1{}^2\ln\eta_1\right.$$
$$\left.\left.-m_2{}^2\ln\eta_2-\langle\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\rangle m_1m_2\ln\frac{m_1}{m_2}\right]\right\}, \quad (4.29a)$$

and

$$(\Delta E_B)_{2P}=(\mathcal{E}_{0\infty})_{2P}+\frac{\alpha^5\mu^2}{m_1+m_2}\frac{1}{8\pi}\left[\frac{8}{3}\ln\left(\frac{Ry_\infty}{k_0(2,1)}\right)-\frac{7}{18}\right]. \quad (4.29b)$$

5. RADIATIVE CORRECTIONS

Radiative corrections arise from self-energy (vertex) and vacuum polarization effects. The change in self-energy of the second particle in the Coulomb field

[first term of Eq. (2.15)] can be approximated by

$$\Delta E_{L2}{}^{C}\cong -4\pi\alpha\int \varphi_{K^C}{}^{*}\delta(x_0)\gamma_2{}^{0}\gamma_2{}^{\nu}\times\{[G^{0}-G_1{}^{0}G_2{}^{0}]D\}_{K}\gamma_2{}^{\nu}\gamma_1{}^{0}\delta(x_0')\varphi_{K^C}. \quad (5.1)$$

In the above expression we have replaced Λ in Eq. (2.8) by $G_1{}^{0}\gamma_1{}^{0}\delta_2$, thereby neglecting the intrinsically higher-order pair terms. After some integrations this becomes

$$\Delta E_{L2}{}^{C}=-i\alpha(2\pi)^{-3}\int \varphi_{K^C}{}^{*}(\mathbf{r}0)\gamma_2{}^{0}\gamma_2{}^{\nu}[G^{C}{}_{-\mathbf{k},K_0'}(\mathbf{r}0,\mathbf{r}'0)-(G_1{}^{0}G_2{}^{0})_{-\mathbf{k},K_0'}(\mathbf{r}0,\mathbf{r}'0)]e^{-i\eta_1\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\gamma_2{}^{\nu}\gamma_1{}^{0}\times e^{i(K_0-K_0')(X_0-X_0')}e^{-ik|X_0-X_0'|}k^{-1}\varphi_{K^C}(\mathbf{r}0). \quad (5.2)$$

Once more, the separation of the high- and low-quantum regions is tantamount to a separation of G^C into high- and low-energy parts. In the high-energy region all the terms of (2.15) recombine to yield

$$\Delta E_{L2}{}^{H}=-4\pi\alpha\int \varphi_{K^C}{}^{*}(\mathbf{r}0)e^{-iK(X-X')}\gamma_2{}^{0}\times\Big\{4\pi i\alpha\gamma_2{}^{\nu}G_2{}^{0}(x_2x_2')\gamma_1{}^{\mu}\gamma_2{}^{\mu}G_2{}^{0}(x_2'x_4)\gamma_2{}^{\nu}\delta(x_1-x_3)\times D(x_2'x_1)\frac{1}{(2\pi)^4}\int_{k>\Lambda}\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{k_\mu{}^2}d^4k\,d^4x_2'-\frac{\alpha}{2\pi}B_2\gamma_1{}^{\mu}\gamma_2{}^{\mu}\times\delta(x_1-x_3)\delta(x_2-x_4)D(x_1x_2)\Big\}\times\gamma_1{}^{0}\varphi_{K^C}(\mathbf{r}'0)d^4xd^4x'd^4X'. \quad (5.3)$$

In the foregoing expression, X' and x' are the center-of-mass and relative coordinates associated with x_3 and x_4 in the same way as X and x are related to x_1 and x_2. We have ignored the retardation corrections to the wave function in obtaining (5.3) since, to our accuracy,

$$(G_1{}^{0})_K(xx')\delta(x_0)\varphi_{K^C}(\mathbf{r}'0)\cong\varphi_{K^C}(\mathbf{r}0). \quad (5.4)$$

After integration over configuration space, we get

$$\Delta E_{L2}{}^{H}=-2\alpha(2\pi)^{-5}\int \varphi_{K^C}{}^{*}(\mathbf{r}0)e^{i\mathbf{p}'\cdot\mathbf{r}}\delta(\eta_2K_0{}^{C}+p_0')\times\delta(\eta_2K_0{}^{C}+p_0'')\delta(p'-p''+q)(q_\mu{}^2)^{-1}\gamma_2{}^{0}\gamma_1{}^{\nu}\times L_2{}^{\nu}(p'p'')\gamma_1{}^{0}e^{-i\mathbf{p}''\cdot\mathbf{r}''}\varphi_{K^C}(\mathbf{r}'0), \quad (5.5)$$

where

$$L_2{}^{\nu}(p'p'')=\frac{2i\alpha}{(2\pi)^3}\int_{k>\Lambda}d^4k\times\frac{\gamma_2{}^{\mu}[m_2-\gamma_2(p'+k)]\gamma_2{}^{\nu}[m_2-\gamma_2(p''+k)]\gamma_2{}^{\mu}}{k_\mu{}^2[m_2{}^2+(p'+k)^2][m_2{}^2+(p''+k)^2]}-\frac{\alpha}{2\pi}\gamma_2{}^{\nu}B_2.$$

The computation follows standard techniques[20,21] from Eq. (5.5) on. We have merely to note that to our order of accuracy, $\eta_2K_C{}^{0}\cong m_2$, and so we can calculate $L_2{}^{\nu}(p'p'')$ as if we were dealing with free particles. In other words, we set

$$(\gamma_2p''+m_2)\gamma_1{}^{0}\varphi_{K^C}(\mathbf{r}'0)\delta(\eta_2K_0{}^{C}+p_0'')=\varphi_{K^C}{}^{*}(\mathbf{r}0)\delta(\eta_2K_0{}^{C}+p_0')\gamma_2{}^{0}(\gamma_2p'+m_2)=0. \quad (5.6)$$

Computation yields

$$L_2{}^{\nu}(p'p'')=\frac{\alpha}{2\pi}\Big\{\gamma_2{}^{\nu}\frac{q^2}{m_2{}^2}\Big(\frac{11}{36}+\frac{2}{3}\ln\frac{m_2}{2A}\Big)-\frac{1}{4m_2}q^{\mu}[\gamma_2{}^{\mu},\gamma_2{}^{\nu}]\Big\}. \quad (5.7)$$

Using the approximate φ of Eq. (4.3), we obtain

$$\Delta E_{L2}{}^{H}=\frac{4}{3}\frac{\alpha^2}{m_2{}^2}|\varphi(0)|^2\Big(\frac{5}{6}+\ln\frac{m_2}{2A}\Big)+\langle T\rangle, \quad (5.8)$$

where

$$T=\frac{\alpha}{4}\Big\{\frac{\boldsymbol{\sigma}_2\cdot L}{r^3}\Big[\frac{\alpha}{\pi}\frac{1}{m_2{}^2}+\frac{\alpha}{2\pi}\frac{2}{m_1m_2}\Big]-\frac{\alpha}{2\pi}\Big[\frac{\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2}{r^3}-\frac{3(\boldsymbol{\sigma}_1\cdot\mathbf{r})(\boldsymbol{\sigma}_2\cdot\mathbf{r})}{r^5}-\frac{8\pi}{3}(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\delta(r)\Big]\Big\},$$

and $\mathbf{L}=\mathbf{r}\times\mathbf{p}$. The operator T represents the effects of the anomalous magnetic moment of the particles. The correction to the interaction between spin and orbit of the same particle appears as an α/π instead of an $\alpha/2\pi$, because this interaction consists of the purely kinetic Thomas term and a magnetic term. The latter, which is twice as large as the total self spin-orbit effect, is corrected by a factor of $\alpha/2\pi$.[17]

The low energy part is treated by methods analogous to those in Sec. 4. Since renormalization terms have been eliminated, it is convenient to return to a non-covariant form. Low-frequency contributions of the required order *cannot arise* from vertices in which the *self-energy* part is due to the *Coulomb* interaction, since this interaction requires the existence of pairs. Such a process is, therefore, inevitably a high-energy one. Furthermore, non-negligible contributions *only arise* from *Coulomb* interactions *between* the particles. This leaves only one term, which by procedures used in

[20] R. Karplus and N. M. Kroll, Phys. Rev. **77**, 536 (1950), Eq. (27).

[21] F. J. Dyson, "Notes on Advanced Quantum Mechanics," Cornell University lecture notes, 1952 (unpublished).

Sec. 4, reduces to

$$\Delta E_{L2}{}^{L}=-\frac{\alpha}{(2\pi)^2}\int_0^A\frac{d^3k}{k^2}\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)$$
$$\sum_n\frac{K_n-K_0{}^C}{k+K_n-K_0{}^C}\varphi_K{}^*(\mathbf{r}0)\alpha_2{}^ie^{-i\mathbf{k}\cdot\eta_1\mathbf{r}}\varphi_{-\mathrm{k},n}(\mathbf{r}0)$$
$$\times\varphi^*{}_{-\mathrm{k},n}(\mathbf{r}'0)\alpha_2{}^ie^{i\mathbf{k}\cdot\eta_2\mathbf{r}'}\varphi_K(r'0). \quad (5.9)$$

Equation (5.9) yields

$$(\Delta E_{L2}{}^{L})_{2S}=\frac{1}{6\pi}\frac{\alpha^5\mu^2}{m_1+m_2}\frac{m_1}{m_2}\ln\left[\frac{A}{k_0(2,0)}\frac{Ry_\infty}{Ry_\mu}\right], \quad (5.10a)$$

and

$$(\Delta E_{L2}{}^{L})_{2P}=\frac{1}{6\pi}\frac{\alpha^5\mu^2}{m_1+m_2}\frac{m_1}{m_2}\ln\frac{Ry_\infty}{k_0(2,1)}. \quad (5.10b)$$

The constant A can be eliminated between Eqs. (5.8) and (5.10a). The first particle can be treated in the same way, to give finally for the contribution of all vertex terms,

$$(\Delta E_L)_{2S}=\frac{\alpha^5\mu^2}{6\pi(m_1+m_2)}\left[\frac{m_1{}^2+m_2{}^2}{m_1m_2}\left(\frac{5}{6}-\ln\frac{\alpha^2k_0(2,0)}{Ry_\infty}\right)\right.$$
$$\left.-\frac{m_1}{m_2}\ln\eta_1-\frac{m_2}{m_1}\ln\eta_2+\frac{1}{2}\langle\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\rangle\right], \quad (5.11a)$$

and

$$(\Delta E_L)_{2P}=-\frac{\alpha^5\mu^2}{6\pi(m_1+m_2)}\frac{m_1{}^2+m_2{}^2}{m_1m_2}\ln\frac{k_0(2,1)}{Ry_\infty}$$
$$+\frac{\alpha^2}{2\pi m_1m_2}\left\langle\frac{1}{2}\left(\frac{m_2}{m_1}\boldsymbol{\sigma}_1+\frac{m_1}{m_2}\boldsymbol{\sigma}_2\right)\cdot\frac{\mathbf{L}}{r^3}+\frac{\mathbf{S}\cdot\mathbf{L}}{r^3}\right.$$
$$\left.-\frac{1}{2}\left(\frac{\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2}{r^3}-\frac{3(\boldsymbol{\sigma}_1\cdot\mathbf{r})(\boldsymbol{\sigma}_2\cdot\mathbf{r})}{r^5}\right)\right\rangle; \quad (5.11b)$$
$$\mathbf{S}=\tfrac{1}{2}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2).$$

The other radiative correction is that arising from the effect of vacuum polarization on the exchange of a single photon between the particles. This term is not altered by the two-body formalism since the correction arises from a change in the photon Green's function and has no direct connection with the two particles. The interaction kernel may be shown to be[5,20]

$$J_{V\neq}(12,34)=\frac{2\alpha^2i}{(2\pi)^4}\gamma_1{}^\nu\gamma_2{}^\nu\delta(x_1-x_3)\delta(x_2-x_4)$$
$$\times\int d^4ke^{ik(x_1-x_2)}\int_0^1 dv2v^2(1-\tfrac{1}{3}v^2)$$
$$\times\{[4m_1{}^2+k^2(1-v^2)]^{-1}+[4m_2{}^2+k^2(1-v^2)]^{-1}\}. \quad (5.12)$$

This kernel will yield no contributions larger than α^3Ry and so it is sufficient to set

$$\Delta E_{V\neq}=-i|\varphi(0)|^2\int\gamma_1{}^0\gamma_2{}^2(J_{V\neq})_K(xx')d^4xd^4x'$$
$$=-\frac{4}{15}\alpha^2|\varphi(0)|^2\left(\frac{1}{m_1{}^2}+\frac{1}{m_2{}^2}\right). \quad (5.13)$$

For positronium, the result is

$$\Delta E_{\mathrm{pos}}=-\frac{4}{15}\alpha^2|\varphi(0)|^2\frac{1}{m^2}. \quad (5.13a)$$

6. THE ANNIHILATION INTERACTION

The energy contribution of this interaction J_A is calculated in KK-III.[22] We agree with its result, which is

$$(\Delta E_A)_{2S}=\frac{\alpha^5m}{64\pi}\{-4(1-\ln2)$$
$$+\langle S^2\rangle[2(1-\ln2)-4-8/9]\}, \quad (6.1a)$$

and

$$(\Delta E_A)_{2P}=0. \quad (6.1b)$$

7. SUMMARY AND DISCUSSION

The total α^3Ry energy shift for the $2S$ and $2P$ states of positronium is determined by adding the α^3Ry parts of (3.10), (4.29), (5.11), (5.13a), and (6.1). In $2S$ states the result is

$$(\Delta E_B)_{2S}=\frac{\alpha^5m}{64\pi}\left\{-\frac{4}{3}+\frac{8}{3}\left[\frac{25}{12}+\ln\frac{2}{16.646\alpha}\right]\right.$$
$$+2\left[\ln\alpha+\frac{7}{4}+\frac{4}{3}(1-\ln2)\right]$$
$$\left.+(1-2\ln2)-\langle 2S^2-3\rangle\right\}$$
$$=\frac{\alpha^5m}{64\pi}(8.784-2\langle S^2\rangle), \quad (\langle S^2\rangle=S(S+1)) \quad (7.1)$$

for the recoil and retardation corrections;

$$(\Delta E_{L,V})_{2S}=\frac{\alpha^5m}{64\pi}\left[\frac{8}{3}\left(\frac{5}{6}-\frac{1}{10}-\ln\frac{16.646\alpha^2}{2}\right)+\frac{2}{3}\langle 2\mathbf{S}^2-3\rangle\right]$$
$$=\frac{\alpha^5m}{64\pi}\left(20.546+\frac{4}{3}\langle\mathbf{S}^2\rangle\right) \quad (7.2)$$

for the radiative contributions; and

$$(\Delta E_A)_{2S}=-\frac{\alpha^5m}{64\pi}(1.228+4.275\langle\mathbf{S}^2\rangle) \quad (7.3)$$

[22] There are several typographical errors in Sec. V of KK-III. We agree with the final answer, however.

for the annihilation terms. The total energy shift is

$$(\Delta E)_{2S}=\frac{\alpha^5 m}{64\pi}(28.101-4.942\langle \mathbf{S}^2\rangle). \tag{7.4}$$

The contribution in $2P$ states is

$$(\Delta E_B)_{2P}=-\frac{\alpha^5 m}{64\pi}\left(\frac{7}{18}+\frac{8}{3}\ln 0.9704\right)=-\frac{\alpha^5 m}{64\pi}(0.309) \tag{7.5}$$

for the recoil and retardation corrections, and

$$(\Delta E_{L,V})_{2P}=\frac{\alpha^5 m}{64\pi}\left(-\frac{8}{3}\ln 0.9704+\frac{1}{3}\langle \mathbf{S}\cdot\mathbf{L}\rangle\right)$$
$$-\frac{\alpha^2}{m^2}\frac{1}{4\pi}\left(\frac{(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)}{r^3}-\frac{3(\boldsymbol{\sigma}_1\cdot\mathbf{r})(\boldsymbol{\sigma}_2\cdot\mathbf{r})}{r^5}\right) \tag{7.6}$$

for the radiative corrections. There are no contributions from the annihilation interaction. Numerically, we find for the total $2P$ energy shift:

$$(\Delta E)_{2P}=\frac{\alpha^5 m}{64\pi}\left[-0.229+\begin{bmatrix}0\\0.300\\-0.167\\-1.000\end{bmatrix}\ \text{for}\ \begin{matrix}{}^1P_1\\{}^3P_2\\{}^3P_1\\{}^3P_0\end{matrix}\right]. \tag{7.7}$$

Our energy unit $(64\pi)^{-1}\alpha^5 m$ is equal to 12.72 Mc/sec.

In positronium, unlike hydrogen, there is no $\alpha^2 Ry$ degeneracy. Hence the $\alpha^3 Ry$ shifts must be added to the much larger $\alpha^2 Ry$ fine and hyperfine structures in order to obtain the level spacing. Table I gives the corrections ascertained by Ferrell,[23] together with the $\alpha^3 Ry$ contributions which we calculated.

A word about the accuracy of this calculation seems in order. We have included all $\alpha^3 Ry$ corrections. There are, of course, $\alpha^4 Ry$ terms with large coefficients; some of these may contain $\ln\alpha$. Such corrections may well amount to several Mc/sec. For example, in the expansion of G^C in (5.1) we have omitted the term, $\alpha_1{}^i G_1{}^0 G_2{}^0 I^C G_1{}^0 G_2{}^0 I^C G_1{}^0 G_2{}^0 \alpha_2{}^i D$, which is known to contribute 7 Mc/sec (an $\alpha^4 Ry$ term with a coefficient of eight) to hydrogen.[24,25] The factor $\frac{1}{4}$ converts the lowest-order Lamb shift correction from hydrogen to positronium. (This number comes from multiplying 2 by $\frac{1}{8}$. The multiplicand 2, comes from the equal contributions of electron and positron self-energies; the multiplier $\frac{1}{8}=(\mu_{\text{pos}}/\mu_H)^3$, represents the reduction of the density of the wave function at the origin.) If the same factor applies to the 7-Mc/sec correction to the Lamb shift, then this term alone would amount to about 2 Mc/sec.

TABLE I. Additions to the nonrelativistic $n=2$ level of positronium, Mc/sec.

Order	$^1S_0$	$^3S_1$	$^1P_1$	$^3P_2$	$^3P_1$	$^3P_0$
$\alpha^2 Ry$	−18 135	7413	−3536	−981	−5360	−10 835
$\alpha^3 Ry$	357	232	−3	1	−5	−16
Total	−17 778	7645	−3539	−980	−5365	−10 851

This calculation also determines corrections to the hydrogen Lamb shift. In the terminology of reference 1, and incorporating the change discussed in footnote 8, the theoretical shifts become

$$S_{\text{H}}=[1057.07-22\epsilon_\alpha+S_z{}^{(3)}+S^{(R)}+\epsilon_{st}\pm 0.13]\ \text{Mc/sec},$$
$$S_{\text{D}}=[1058.43-22\epsilon_\alpha+S_z{}^{(3)}+S^{(R)}\pm 0.13]\ \text{Mc/sec}, \tag{7.8}$$
$$S_{\text{D}}-S_{\text{H}}=[1.371-\epsilon_{st}\pm 0.035]\ \text{Mc/sec}.$$

The theoretical value of the difference $S_{\text{D}}-S_{\text{H}}$ is no longer in such close agreement with experiment, being just at the limit of the experimental error.

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APPENDIX

One of us (P.C.M.), with Professor J. Schwinger, has previously derived a method for treating binding which does not involve "selective resummation" of the free-particle expansion of the interacting Green's function. This independent derivation seems sufficiently short and illuminating to merit its inclusion despite articles by FK and by Eden.[10] We employ the variational derivative techniques used by Schwinger[2] in deriving the relativistic two-body equation. To avoid the problems of symmetrization, we use two distinguishable fermion fields. If the particles have masses m_1 and m_2, and charges e_1 and e_2, the Green's function equation is

$$\left(\gamma\pi+m+ie\gamma(\xi)\frac{\delta}{\delta J(\xi)}\right)_1 \times\left(\gamma\pi+m+ie\gamma(\xi')\frac{\delta}{\delta J(\xi')}\right)_2 G_{12}=1. \tag{A.1}$$

The form of the two-body equation, obtained also by Salpeter and Bethe,[3] is derived by writing Eq. (A.1) as

$$[(\gamma\pi+M)_1(\gamma\pi+M)_2-I_{12}]G_{12}=1, \tag{A.2}$$

[23] R. A. Ferrell, Phys. Rev. **84**, 858 (1951). We have recomputed the numerical results of Dr. Ferrell from his formula for the energy to suit the greater numerical accuracy demanded by the present work.

[24] Karplus, Klein, and Schwinger, Phys. Rev. **86**, 288 (1952).

[25] Baranger, Bethe, and Feynman, Phys. Rev. **92**, 482 (1953).

and using an expansion of I_{12} in powers of e and free-particle Green's functions. A perturbation theory based upon the solution of an equation containing some interaction, is then employed to determine energies, wave functions, and so forth. A more satisfactory and symmetrical approach than this one is to use, in the approximation of I_{12}, the Green's function satisfying the same unperturbed equation (more accurate than the free particle equation) as is subsequently employed for finding the energies and wave functions. Such a procedure is sketched below. It amounts to using a different first approximation in the equation derived by Schwinger for the interaction operator.

Suppose that we employ as a first approximation to G_{12}, the solution to

$$[(\gamma\pi+m)_1(\gamma\pi+m)_2-O_{12}]G_{12}^0=1, \qquad \text{(A.3)}$$

where O_{12} is an arbitrary operator which may depend on the external current, J. If we use the relation

$$\left(ie\gamma(\xi)\frac{\delta}{\delta J(\xi)}\right)_1 G_{12}=-ie_1\gamma_1(\xi)G_{12}\frac{\delta G_{12}^{-1}}{\delta J(\xi)}G_{12}, \qquad \text{(A.4)}$$

and approximate the right-hand side of (A.4) by

$-ie_1\gamma_1(\xi)G_{12}^0[\delta(G_{12}^0)^{-1}/\delta J(\xi)]G_{12}$, we obtain

$$\begin{aligned}\left(ie\gamma(\xi)\frac{\delta G_{12}}{\delta J(\xi)}\right)_1 &\cong -ie_1\gamma_1(\xi)G_{12}^0\frac{\delta(G_{12}^0)^{-1}}{\delta A(\xi')}\frac{\delta A(\xi')}{\delta J(\xi)}G_{12}\\ &\cong -ie_1\gamma_1(\xi)G_{12}^0\left\{\frac{\delta}{\delta A(\xi')}[(\gamma\pi+m)_1\right.\\ &\quad\left.\times(\gamma\pi+m)_2-O_{12}]\right\}\mathcal{G}(\xi\xi')G_{12}. \qquad \text{(A.5)}\end{aligned}$$

As long as O_{12} is an operator independent on J, insertion of (A.5) into (A.1) yields the "improved" equation for G_{12},

$$\begin{aligned}&[(\gamma\pi+m)_1(\gamma\pi+m)_2\\ &\quad+ie_2^2(\gamma\pi+m)_1\gamma_2(\xi)G_{12}^0(\gamma\pi+m)_1\gamma_2(\xi')\mathcal{G}(\xi,\xi')\\ &\quad+ie_1^2(\gamma\pi+m)_2\gamma_1(\xi)G_{12}^0(\gamma\pi+m)_2\gamma_1(\xi')\mathcal{G}(\xi\xi')\\ &\quad+ie_1e_2(\gamma\pi+m)_1\gamma_2(\xi)G_{12}^0(\gamma\pi+m)_2\gamma_1(\xi')\mathcal{G}(\xi\xi')\\ &\quad+ie_1e_2(\gamma\pi+m)_2\gamma_1(\xi)G_{12}^0(\gamma\pi+m)_1\gamma_2(\xi')\mathcal{G}(\xi\xi')\\ &\quad-ie_1e_2\gamma_1(\xi)\gamma_2(\xi')\mathcal{G}(\xi\xi')+(ie\gamma(\xi)\delta/\delta J(\xi))_1\\ &\qquad\times(ie\gamma(\xi')\delta/\delta J(\xi'))_2]G_{12}^1=1. \qquad \text{(A.6)}\end{aligned}$$

The procedure of this paper has been to take $O_{12}=I_C$. To the order to which (A.6) is then valid,

$$(ie\gamma(\xi)\delta/\delta J(\xi))_1(ie\gamma(\xi')\delta/\delta J(\xi'))_2$$

does not contribute, and hence has not been evaluated in (A.6). Replacement of G_{12}^0 by $G_1^0G_2^0$ would convert (A.6) into the usual lowest-order two-body equation,

$$[(\gamma\pi+M^1)_1(\gamma\pi+M^1)_2-I^1]G_{12}^1=0. \qquad \text{(A.7)}$$

To derive a still better equation, we merely repeat the procedure of Eqs. (A.3)–(A.5), but with an operator given by (A.6). (O_{12} will now depend on J through both $\gamma\pi$ and G_{12}^0.) Needless to say, this iteration gives rise to many terms. However, to the degree of accuracy required by our calculation, G_{12}^0 may be replaced in some of these terms by its Born approximation. This results, for example, in the combination of seven different terms with plus signs and six with minus signs into the one "crossed transverse" term of the usual treatment.

We note in passing that if we knew the solution to (A.3) with $O_{12}=-ie_1e_2\gamma_1(\xi)\gamma_2(\xi')\mathcal{G}(\xi\xi')$, we would only need to proceed as far as (A.6) to include all processes relevant to α^3Ry electrodynamic corrections. In order to have solutions in which the relative time dependence is known, we had to restrict ourselves to an instantaneous interaction. Hence a second approximation to the Green's function equation is necessary to introduce two transverse photons.