The Sachs Exchange Moment*

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A derivation of the Sachs exchange moment is given which clearly shows how this exchange moment is uniquely determined by the exchange potential. The relation between this and the treatments of Sachs and of Osborne and Foldy is discussed.

It has long been recognized[1,2] that the presence of exchange forces between nucleons implies the presence of currents in the space about interacting nucleons, currents which will generally contribute to the magnetic properties of nuclei. In fact, a contribution of this origin has long been recognized[3] in the magnetic moments of H^3 and He^3; these currents appear to play an important role in the n-d capture process[4] and their contribution to the n-p capture process has recently been established.[5] Physically, these currents are due to the charged mesons which are being exchanged between the nucleons, giving rise to the exchange force between them, and their contributions to the exchange moments have frequently been calculated according to various meson theories.[6] However, as was pointed out, first by Sachs[2] and more recently by Osborne and Foldy,[7,8] one term of this exchange moment has a phenomenological origin, independent of the particular meson theory, but in neither of these discussions does it appear clearly that this term is defined unambiguously. The purpose of the present note is to derive this term again, showing clearly its origin, and to discuss the relationship of these previous treatments.

Consider first two nucleons with position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ (with respect to an origin O) and denote the current density at the point $\mathbf{r}$ by $\mathbf{J}(\mathbf{r})$. Then from the equation of charge conservation,

$$\operatorname{div}\mathbf{J} = -\frac{1}{c}\frac{\partial\rho}{\partial t} = \frac{i}{\hbar c}[\rho, H], \qquad (1)$$

where $\rho(\mathbf{r})$ is the charge density and H the Hamiltonian for the two nucleons. If, as is usual, the nucleons are regarded as point charges, then

$$\rho(\mathbf{r}) = \tfrac{1}{2}e\{(1+\tau_3{}^1)\delta(\mathbf{r}-\mathbf{r}_1) + (1+\tau_3{}^2)\delta(\mathbf{r}-\mathbf{r}_2)\}. \qquad (2)$$

Lack of commutativity of ρ and H arises first from the kinetic energy terms and from any velocity dependent potentials, and also from the $\boldsymbol{\tau}$-dependent exchange potential. The currents arising from the first of these are those due to the translation of the charge of the nucleon and give the usual orbital contribution to the magnetic moment. The second terms come from the currents which are recognized when the replacement $\mathbf{p}\rightarrow\mathbf{p}-(e/c)\mathbf{A}$ is made in the velocity-dependent potentials. We will denote by $\mathbf{J}_{12}(\mathbf{r})$ the total current excluding these two current contributions, so that if the $\boldsymbol{\tau}$-dependent potential is $\boldsymbol{\tau}^1\cdot\boldsymbol{\tau}^2 V_{12}$, then,[7] from (1) and (2),

$$\operatorname{div}\mathbf{J}_{12} = \frac{e}{\hbar c}(\boldsymbol{\tau}^1\times\boldsymbol{\tau}^2)_3 V_{12}[\delta(\mathbf{r}-\mathbf{r}_1)-\delta(\mathbf{r}-\mathbf{r}_2)]. \qquad (3)$$

The magnetic moment of the two nucleons which arises from $\mathbf{J}_{12}(\mathbf{r})$ may then be broken up in the following way:

$$\int \mathbf{r}\times\mathbf{J}_{12}d\tau = \frac{(\mathbf{r}_1+\mathbf{r}_2)}{2}\times\int\mathbf{J}_{12}(\mathbf{r})d\tau + \int\left(\mathbf{r}-\frac{\mathbf{r}_1+\mathbf{r}_2}{2}\right)\times\mathbf{J}_{12}(\mathbf{r})d\tau. \qquad (4)$$

Now, since the current $\mathbf{J}_{12}(P)$ can depend only on the relative positions of P and the two nucleons, the second term of (4) does not depend on the origin O, where as the first term does depend on the choice of O. Using Green's theorem and the physical fact that the currents $\mathbf{J}_{12}$ vanish exponentially at large distances from the nucleons, the integral in the first term of (4) becomes

$$\int\mathbf{J}_{12}(\mathbf{r})d\tau = -\int\mathbf{r}\operatorname{div}\mathbf{J}_{12}(\mathbf{r})d\tau. \qquad (5)$$

According to (3), this integral (5) is known uniquely and its contribution to the magnetic moment (4) is unambiguously

$$\frac{(\mathbf{r}_1+\mathbf{r}_2)}{2}\times\left[-(\mathbf{r}_1-\mathbf{r}_2)\frac{e}{\hbar c}(\boldsymbol{\tau}^1\times\boldsymbol{\tau}^2)_3 V_{12}\right] = \frac{e}{\hbar c}(\boldsymbol{\tau}^1\times\boldsymbol{\tau}^2)_3(\mathbf{r}_1\times\mathbf{r}_2)V_{12}. \qquad (6)$$

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Therefore the magnetic moment terms which depend on the origin O are uniquely determined, whereas (3) imposes no restrictions on those last terms of (4) which are independent of the origin chosen. The possible forms for these latter terms, allowed after all invariance requirements are met, have been discussed by Osborne and Foldy,[7,8] by Kynch,[9] and by Austern and Sachs[2] and those which are velocity-independent are very limited in number, and all spin-dependent. For a system of nucleons the exchange-moment contributions for each pair of nucleons are to be added, three-particle interaction effects being neglected, and the origin is clearly to be chosen at the mass center of the nucleus.

The exchange-moment term (6) was obtained by Sachs[2] from the interaction

$$U_{12}=\exp\left\{\frac{-ie}{2\hbar c}(\boldsymbol{\tau}_3{}^1-\boldsymbol{\tau}_3{}^2)\int_{r_1}^{r_2}\mathbf{A}(\mathbf{r})\cdot d\mathbf{r}\right\}\times V_{12}(1+\boldsymbol{\tau}^1\cdot\boldsymbol{\tau}^2), \quad (7)$$

which is a quite special gauge-invariant generalization of the exchange interaction $V_{12}(1+\boldsymbol{\tau}^1\cdot\boldsymbol{\tau}^2)$, the integration being taken along the straight line joining the nucleons. The term of U_{12} linear in $\mathbf{A}(\mathbf{r})$, which is all that is relevant in the present application, may be written

$$\frac{-ie}{2\hbar c}(\boldsymbol{\tau}_3{}^1-\boldsymbol{\tau}_3{}^2)(1+\boldsymbol{\tau}^1\cdot\boldsymbol{\tau}^2)V_{12}\int_{r_1}^{r_2}\mathbf{A}(\mathbf{r})\cdot d\mathbf{r}$$
$$=-\int\frac{e}{\hbar c}(\boldsymbol{\tau}^1\times\boldsymbol{\tau}^2)_3V_{12}\int_0^1 ds\delta(\mathbf{r}-s\mathbf{r}_1-(1-s)\mathbf{r}_2)\times(\mathbf{r}_1-\mathbf{r}_2)\cdot\mathbf{A}(\mathbf{r})d\tau,$$

so that the expression (7) given by Sachs involves the assumption that the current flow between the nucleons occurs only along the straight line joining them, the current density being

$$\mathbf{J}_{12}(\mathbf{r})=-\frac{e}{\hbar c}(\boldsymbol{\tau}^1\times\boldsymbol{\tau}^2)_3(\mathbf{r}_1-\mathbf{r}_2)V_{12}\times\int_0^1 ds\delta(\mathbf{r}-s\mathbf{r}_1-(1-s)\mathbf{r}_2). \quad (8)$$

Although this is clearly not the physical situation, this treatment gives the origin-dependent term (6) of the exchange moment correctly since all that is necessary for this is that the space integral of the current density should equal that implied by the Eqs. (5) and (3), and this is so for the current distribution (8). Sachs' treatment gives only the exchange moment (6) since the second term of (4) vanishes, this current flow having zero moment about the midpoint between the nucleons.

In the treatment of Osborne and Foldy,[7] it is proposed to split the current $\mathbf{J}_{12}$ into an irrotational part which is to be determined from (3) and an undetermined solenoidal part. The exchange moment (6) is considered to arise from this irrotational current flow, a paradoxical conclusion since it is well known that any irrotational current flow of finite extent has zero magnetic moment. In fact the integral (4) for this irrotational flow is quite indeterminate since, for large distances, the solenoidal current flow decreases only as $\sim(\mathbf{r}_{12}-3\mathbf{r}\mathbf{r}\cdot\mathbf{r}_{12}/r^2)/r^3$ [compare Eqs. (23) and (25) of Oxborne and Foldy[7]]. Since the physical currents are restricted to a distance of order $\hbar/\mu c$ about the nucleons, it is artificial (and, as we have seen, quite unnecessary) to split this current into irrotational and solenoidal parts each of which decreases only slowly at large distances, though their sum decreases exponentially.

To exemplify explicitly a typical meson-current distribution,[6] we consider finally the adiabatic limit of symmetric scalar meson theory. The exchange-current density due to the meson field is given by

$$\mathbf{J}_{12}(\mathbf{r})=\frac{-ie}{\hbar c}\sum_{\alpha,\beta}\epsilon_{3\alpha\beta}[\phi_\alpha(1)\nabla\phi_\beta(2)-\phi_\beta(2)\nabla\phi_\alpha(1)],$$

where $\phi_\alpha(i)=g\tau_\alpha{}^i\exp(-\kappa|\mathbf{r}-\mathbf{r}_i|)/|\mathbf{r}-\mathbf{r}_i|$, $\kappa=\mu c/\hbar$, and $\epsilon_{\gamma\alpha\beta}$ is the alternating tensor. It may be verified readily that this current distribution satisfies Eq. (3), V_{12} being $-g^2e^{-\kappa r_{12}}/r_{12}$, and that a direct calculation of the magnetic-moment integral (4) gives the result (6). In the adiabatic limit, the scalar meson field and currents do not depend on the nucleon spins, so that there can be no origin-independent exchange-moment term.

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