The trial functions which recommend themselves are $\psi_a^{(+)}(\mathbf{r},t)=\Phi_a(\mathbf{r},t)$ and $\psi_b^{(-)}(\mathbf{r},t)=\Phi_b(\mathbf{r},t)$, and where the $\Phi_a(\mathbf{r},t)$ and $\Phi_b(\mathbf{r},t)$ are further specialized to be the unperturbed stationary states. From analogy with stationary state applications[5] it would appear legitimate to expect this choice of trial solution to provide amplitudes superior to those obtained in the second Born approximation.

PHYSICAL REVIEW VOLUME 95, NUMBER 2 JULY 15, 1954

Integral Equations of Quantized Field Theory

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(Received February 15, 1954)

A systematic method is developed for the solution of the integral equations of quantized field theory. It is first shown how to generalize these equations for any number of fermions in interaction with a boson field, and then how to apply the renormalization procedure to the resulting equations. A method of approximation to the solution of the "renormalized" equations is described, which yields divergence-free covariant scattering amplitudes, analogous to the Tamm-Dancoff amplitudes. It is shown that the nth order Tamm-Dancoff amplitudes do not approach limiting values for large n, and that the procedure does not therefore converge; but that a somewhat modified procedure would probably do so. Singularities in the scattering amplitudes, regarded as functions of the coupling constant, are related to the existence of bound states.

1. INTRODUCTION

MOST problems arising in quantum electrodynamics can be satisfactorily solved by perturbation methods. As Dyson[1] among others has pointed out, this is not because the renormalized perturbation series for scattering amplitudes converge; it just happens that the fine-structure constant is small enough to give these series an asymptotic character. In the meson theory of nuclear forces, the coupling constant is much larger and the perturbation series give no reliable quantitative predictions. Attention has therefore been directed to the solution of the integral equations of field theory by other methods.

The integral equations for quantum electrodynamics were first described, though not explicitly formulated, by Dyson.[2] Schwinger[3] later gave a formal derivation of the integral equations which was independent of perturbation theory, which has been used recently by Edwards[4] to obtain a very approximate solution of one of the integral equations. No systematic method of solving the integral equations, apart from the perturbation method, however, has yet been described. Meanwhile an apparently quite different nonperturbation approach to the same problem has been made via what is generally known as the Tamm-Dancoff method. Originally this method suffered from the inconvenience that it was incapable of a completely covariant formulation, but Dyson[5] has recently established a connection between covariant amplitudes and the corresponding Tamm-Dancoff amplitudes, and Cini[6] and Fubini[7] have considered covariant modifications of Tamm-Dancoff equations.

In spite of their very different appearance, the Tamm-Dancoff equations are closely related to the integral equations of Dyson and Schwinger. They have hitherto suffered from the same defect that no way has been found for the application to them of the renormalization procedure. Methods so far proposed for the elimination from them of the divergences either do not work, or would not work in higher approximation. The author has therefore considered in the present paper the question of "renormalizing" the integral equations, and devised a method by which this can be explicitly carried out.

Unfortunately the integral equations in their "renormalized" form are hopelessly nonlinear, and an exact solution of them is impossible. The Tamm-Dancoff method of approximation, however, has the great merit that it produces only linear integral equations which can be solved by standard methods. Of these, Fredholm's method has the advantage that one knows in advance that the solution will converge. Hence, if one applies the Tamm-Dancoff method to the solution of the "renormalized" integral equations, one is certain of obtaining a meaningful result at every stage of approximation. This, of course, no more guarantees the convergence of the method than the efficacy of the renormalization procedure guaranteed the convergence of the perturbation method; and it will in fact be shown that Tamm-Dancoff amplitudes of arbitrarily large but finite order will not closely approximate the exact scattering ampli-

[1] F. J. Dyson, Phys. Rev. **85**, 631 (1952); Proc. Roy. Soc. (London) **A207**, 395 (1951).
[2] F. J. Dyson, Phys. Rev. **75**, 1736 (1949).
[3] J. S. Schwinger, Proc. Natl. Acad. Sci. U. S. **37**, 452, 455 (1951).
[4] S. F. Edwards, Phys. Rev. **90**, 284 (1953).
[5] F. J. Dyson, Phys. Rev. **91**, 1543 (1953).
[6] M. Cini, Nuovo cimento **10**, 526 (1953).
[7] S. Fubini, Nuovo cimento **10**, 851 (1953).

tudes. The nature of this calamity, however, suggests a remedy.

To make our method of determining scattering amplitudes quite general, we first derive a generalization of Schwinger's integral equations for an arbitrary number of fermions in interaction with a boson field. It is shown how to eliminate the divergences from these integral equations, and then how to apply a Tamm-Dancoff type of approximation to reduce them to linear form.

2. DERIVATION OF THE INTEGRAL EQUATIONS

Consider as a preliminary step the equation

$$[i\nabla - m - f\gamma\cdot a(x)]K(x,x_0)=\delta_+(x-x_0), \tag{1}$$

where $\nabla=\gamma_\lambda(\partial/\partial x_\lambda)$, m is a fermion mass, and $f\gamma\cdot a(x)$ denotes $e\gamma_\lambda a^\lambda(x)$, $g\gamma_5 a_5(x)$, or some similar expression. It determines the Green's function $K(x,x_0)$ for a single fermion in interaction with an external field; $a(x)$ is arbitrary but unquantized, so that no interaction is yet introduced of the fermion with itself or any other fermion. The subscript $+$ to the δ function may be dropped, according to Feynman,[8] if m is given an infinitesimal imaginary part. We shall work mainly in the momentum representation, and accordingly write

$$\begin{aligned} K(x,x_0)&=\textstyle\sum_p K(p)\exp[-ip\cdot(x-x_0)],\\ a(x)&=\textstyle\sum_k a_k\exp[-ik\cdot(x-x_0)], \end{aligned} \tag{2}$$

within any large but finite region of space *and time*. For convenience units will be chosen so that the velocity of light, Planck's constant $\hbar$, and also the volume of the 4-dimensional region considered are all unity. Then (1) becomes

$$(\not{p}-m-f\gamma\cdot\textstyle\sum_k a_k E_k)K(p)=1, \tag{3}$$

where $\not{p}=\gamma_\lambda p^\lambda$ and E_k is the incremental operator which satisfies $E_k p=(p+k)E_k$, $E_k E_l=E_{k+l}$, etc., so that $E_k K(p)=K(p+k)$. For a system of n fermions, Eq. (3) is replaced by

$$\prod_{r=1}^{n}[\not{p}^{(r)}-m-f\gamma^{(r)}\cdot\textstyle\sum_k a_k E_k^{(r)}]K_n[p^{(1)},\cdots p^{(n)}]=1, \tag{4}$$

which, since the fermions do not interact, has the solution

$$K_n[p^{(1)},\cdots p^{(n)}]=K[p^{(1)}]\cdots K[p^{(n)}]. \tag{5}$$

It should be noticed that, whereas $K(p)$ is represented by a simple Dirac matrix with 2 spinor suffixes, $K_n[p^{(1)},\cdots p^{(n)}]$ is a *direct* product of n such factors and its representative has therefore $2n$ spinor suffixes; in (4), $\not{p}^{(r)}$ and $\gamma^{(r)}$ form matrix products with $K[p^{(r)}]$ but not with the other factors of K_n.

Our object in this section will be to determine how Eq. (4) should be modified to take into account not only the interactions of the n fermions with the external field, but also their mutual interactions and self interactions. We shall freely use the ideas of Feynman[8] in doing so, though one could no doubt obtain the same result by generalizing the rigorous field-theoretical method of Schwinger.[3]

The perturbation solution of (4) is

$$K_n=\prod_{r=1}^{n}\Big\{\sum_{s=0}^{\infty}\{[\not{p}^{(r)}-m]^{-1}f\gamma^{(r)}\cdot\textstyle\sum_k a_k E_k^{(r)}\}^s[\not{p}^{(r)}-m]^{-1}\Big\} \tag{6}$$

in which the sth term in the summation gives the probability amplitude that the rth particle will interact s times with the external field, within the region considered. Now consider the effect of applying the operator $-D_k(\partial^2/\partial a_k\cdot\partial a_{-k})$ to the right-hand side of (6), where D_k is the probability amplitude for the transmission of a boson with momentum k; this will remove factors a_k and a_{-k} in all possible ways, summing the results after multiplication by $-D_k$. So, according to Feynman, $-\frac{1}{2}\sum_k D_k(\partial^2 K_n/\partial a_k\cdot\partial a_{-k})$ must be the probability amplitude for the fermions to interact any number of times with the external field, and just once among themselves. Similarly, $(s!)^{-1}[-\frac{1}{2}\sum_k D_k(\partial^2/\partial a_k\cdot\partial a_{-k})]^s K_n$ is the probability amplitude for the fermions to interact any number of times with the external field, and just s times among themselves. Therefore,[9]

$$G_n=\exp[-\tfrac{1}{2}\textstyle\sum_k D_k(\partial^2/\partial a_k\cdot\partial a_{-k})]K_n \tag{7}$$

is the probability amplitude for any number of external, and any number of mutual interactions. This is the Green's function which we wished to obtain. Regarded as a function of the a_k's, it has the property of a generating function for all scattering amplitudes involving n fermions. For, if G_n is developed in powers of the a_k's, thus:

$$G_n=\sum_{b(k)} G_n^*[b(k)]\textstyle\prod_k[b(k)!]^{-1}(a_k)^{b(k)}, \tag{8}$$

$G_n^*[b(k)]$ will be the exact scattering amplitude for the interaction of the set of n fermions with a boson distribution containing $b(k)$ bosons with momentum k. This is what one will primarily wish to calculate.

It should be remarked at this point that, if one wishes to take account of fermion loops, it is not sufficient to set $D_k=(k^2-\mu^2)^{-1}$, where μ is the boson mass; as shown by Schwinger,[3] D_k is a function of the a_k's, defined by

$$\{k^2-\mu^2+\mathrm{spur}[f\gamma\cdot E_{-k}\textstyle\sum_p(\partial/\partial a_k)G_1(p)]\}D_k=1. \tag{9}$$

Like m, μ is given an infinitesimal imaginary part to ensure that all bosons will carry positive energy.

The expression (7) for G_n is very formal, and in practice it is better to make use of the equation, analogous to (9), which is satisfied by G_n; this is obtained by applying

[8] R. P. Feynman, Phys. Rev. **76**, 769 (1949).

[9] The author has received a preprint of a paper entitled "Field Equations in Functional Form" by S. F. Edwards and R. E. Peierls, which contains, in effect, a derivation of (7) from (10), for the special case $n=1$. Mention is made of a reverse derivation by T. H. R. Skyrme.

the operator $\exp\{-\frac{1}{2}\sum_k D_k(\partial^2/\partial a_k\cdot\partial a_{-k})\}$ to the Eq. (4) for K_n. Bearing in mind the operational form of Taylor's theorem $f(x+h)=\exp(hd/dx)f(x)$, the result is clearly

$$\prod_{r=1}^{n}[\not{p}^{(r)}-m-f\gamma^{(r)}\cdot\sum_k(a_k-D_k\partial/\partial a_{-k})E_k^{(r)}]G_n=1. \quad (10)$$

This will be written, for future convenience, in the form

$$\prod_{r=1}^{n}\{\not{p}^{(r)}-m-f[\gamma_k{}^{(r)}\cdot a_k-D_k\gamma_{-k}{}^{(r)}\cdot\partial/\partial a_k]\}G_n=1, \quad (11)$$

where $\gamma_k{}^{(r)}$ is short for $\gamma^{(r)}E_k^{(r)}$, and summation over repeated momentum suffixes is understood (as for tensor suffixes in general relativity theory).

A perturbation solution of Eq. (11) is readily obtained, thus:

$$G_n=\prod_{r=1}^{n}[\sum_{s=0}^{\infty}\{[\not{\mathbf{p}}^{(r)}-m]^{-1}f[\gamma_k{}^{(r)}\cdot a_k-D_k\gamma_{-k}{}^{(r)}\cdot\partial/\partial a_k]\}^s(\not{p}^{(r)}-m)^{-1}] \quad (12)$$

and one can verify directly that this solution generates the Feynman probability amplitudes in their usual form. The validity of the operational procedures used to derive (11) is thereby checked, but the solution itself has not much application except in quantum electrodynamics, owing to the fact that it is at best only semiconvergent. There do exist, however, various ways of transforming semiconvergent series to convergent series, and our main hope of obtaining meaningful results from field theories with medium or strong coupling seems to depend on finding such a way which is applicable to (12). An alternative way of posing the same problem is to demand a nonperturbation solution of (11). The task is complicated by the fact that even the individual terms on the right-hand side of (12) are not meaningful until they have been "renormalized." It is obviously desirable, if not necessary, to adapt the renormalization procedure to Eq. (11) as it stands, to have reasonable hope of obtaining any other solution in a meaningful form. This will be attempted in the following section.

3. THE "RENORMALIZATION" OF THE INTEGRAL EQUATIONS

It will be found that for the "renormalization" of Eq. (11), one needs to consider, in the first instance, only the special case with $n=1$. We shall therefore consider first

$$[\not{p}-m-f(\gamma_k a_k-D_k\gamma_{-k}\cdot\partial/\partial a_k)]G(p)=1. \quad (13)$$

According to Dyson[2] and Ward,[10] the divergences can be eliminated from this equation by writing ZG for G, YD_k for D_k, $Y^{\frac{1}{2}}a_k$ for a_k, $Y^{-\frac{1}{2}}f$ for f, and $m-\delta m$ for m, where Y, Z, and δm are suitably chosen constants. The result is

$$Z[\not{p}-m+\delta m-f(\gamma_k\cdot a_k-D_k\gamma_{-k}\cdot\partial/\partial a_k)]G(p)=1. \quad (14)$$

To make Z and δm precise, it is necessary to impose two conditions on $G(p)$, which can be formulated as follows. Let

$$\Gamma(p)=\{[G(p)]^{-1}\}_0 \quad (15)$$

denote the value of $[G(p)]^{-1}$ when all the a_k's are given the value zero; it is the reciprocal of Dyson's "renormalized" $S_F'(p)$. Also let

$$\Gamma_k(p)=-\{(\partial/\partial a_k)[fG(p)]^{-1}\}_0 \quad (16)$$

denote as usual the vertex function. Then δm and Z are to be chosen so that

$$\Gamma(q)=\not{q}-m, \quad \Gamma_0(q)=\gamma, \quad (17)$$

if q is a momentum satisfying $q^2=m^2$.

The elimination of Z and δm from (14) was attempted first by Utiyama, Sunakawa, and Imamura,[11] who, however, were compelled to resort to the perturbation expansion which it is the purpose of the method to avoid. A partial elimination has recently been effected by the author,[12] and this will now be carried to completion.

Equation (14) is first multiplied by $[G(p)]^{-1}$ from the right, the result being

$$Z[\not{p}-m+\delta m-f\gamma_k\cdot a_k+fD_k\gamma_{-k}\cdot(\partial G/\partial a_k)G^{-1}]=G^{-1}. \quad (18)$$

On account of the free operators E_k which are concealed in γ_k and $\gamma_{-k}(\partial G/\partial a_k)$, this step is not as trivial as it might appear; strictly it involves the expansion of the left-hand side of (14) in powers of the a_k's, separating the terms proportional to $\prod_k(a_k)^{b(k)}$, which are then multiplied from the right by $\{G[p+\sum_k b(k)k]\}^{-1}$, and finally reforming the equation. If

$$V(p)=[G(p)]^{-1}, \quad V_k(p)=-(\partial/\partial a_k)[fG(p)]^{-1}, \quad (19)$$

Eq. (18) becomes

$$V=Z(\not{p}-m+\delta m-f\gamma_k\cdot a_k)+Af^2D_k\gamma_{-k}\cdot GV_k. \quad (20)$$

Hence,

$$V_j=Z\gamma_j+Z\gamma_{-k}\cdot W_{jk}, \quad (21)$$

where

$$W_{jk}=-f(\partial/\partial a_j)(D_kGV_k). \quad (22)$$

[No summation over k is implied on the right-hand side of (22)]. In contrast with Γ_j, V_j represents a vertex at which any number of interactions with the external field may occur. The significance of W_{jk} is best described by saying that $(W_{jk}-f^2D_kGV_kGV_j)G$ is the amplitude for Compton scattering in an external field.

[10] J. C. Ward, Proc. Phys. Soc. (London) **A64**, 54 (1951); J. C. Ward, Phys. Rev. **78**, 182 (1950).

[11] Utiyama, Sunakawa, and Imamura, Progr. Theoret. Phys. (Japan) **8**, 77 (1952).

[12] H. S. Green, Proc. Phys. Soc. (London) **A66**, 873 (1952). A formal method applicable also to theories with scalar or pseudoscalar coupling was given by J. C. Ward, Phys. Rev. **84**, 897 (1951).

The elimination of Z from the second term on the right-hand side of (21) can be carried out in the following way. A new operator X_{jk} is defined in terms of W_{jk} by the equation

$$X_{jk}=W_{jk}-W_{-ik}X_{ji}. \tag{23}$$

This is formally an integral equation, with kernel W_{-ik} to determine X_{jk} when W_{jk} is known.[13] In practice, however, it never has to be solved because X_{jk} represents the same Feynman diagrams as W_{jk}, with the exception of all those which are reducible, in the sense that they can be obtained by joining simpler graphs end to end. In any approximate method of solving the integral equations, the solution X_{jk} of (23) will therefore be obtainable by inspection, being in fact a simpler expression than W_{jk} itself. Now, if one multiplies Eq. (23) on the left by $Z\gamma_{-k}$ and uses (21), one has

$$Z\gamma_{-k}\cdot X_{jk}=Z\gamma_{-k}\cdot W_{jk}-(V_{-i}-Z\gamma_{-i})X_{ji},$$

or, changing i to k in the last term,

$$Z\gamma_{-k}\cdot W_{jk}=V_{-k}X_{jk}. \tag{24}$$

Hence, (21) may be written

$$V_j=Z\gamma_j+V_{-k}X_{jk}. \tag{25}$$

This has the same form as an equation derived by Edwards,[4] who, however, did not define X_{jk} except in terms of perturbation series. The observation that X_{jk} represents a class of *irreducible* Compton-type graphs makes it clear that the summation (integration), implied by the double occurrence of the suffix k in (25), will not introduce any divergence which is not removable by the subtraction of a single constant $(1-Z)\gamma_j$.

The final elimination of Z from (25) involves the use of the second of Eqs. (17), from which it follows that

$$V_j(p)=\gamma_j+[V_j(p)-\Gamma_0(q)E_j](m+\boldsymbol{q})/(2m)+[V_j(p)-\Gamma_0(-q)E_j](m-\boldsymbol{q})/(2m). \tag{26}$$

Into the right-hand side of this equation one substitutes

$$V_j(p)-\Gamma_0(\pm q)E_j=V_{-k}(p)X_{jk}(p)-\Gamma_{-k}(\pm q)F_{0k}(\pm q)E_j, \tag{27}$$

where $F_{jk}(p)=[X_{jk}(p)]_0$ is what $X_{jk}(p)$ reduces to when the a_k's are set equal to zero. The result is a version of (21) from which the divergences have been explicitly cancelled.

A similar procedure can now be applied to Eq. (20). One first uses (25) to rewrite it in the form

$$V=Z(\boldsymbol{p}-m+\delta m)-fV_ka_k+f^2(V_{-k}-V_{-j}X_{-kj})D_kGV_k+fV_{-j}X_{kj}a_k. \tag{28}$$

Such divergences as arise from the summations in the last two terms of (28), taken together, can be removed by the subtraction of a single linear function of p, $(1-Z)(\boldsymbol{p}-m+\delta m)$. To carry out the subtraction explicitly, use is made of the formula

$$V(p)=\boldsymbol{p}-m+\{V(p)-[1+(p_\lambda-q_\lambda)(\partial/\partial q_\lambda)]\Gamma(q)\}\times(m+\boldsymbol{q})/(2m)+\{V(p)-[1-(p_\lambda+q_\lambda)\times(\partial/\partial q_\lambda)]\Gamma(-q)\}(m-\boldsymbol{q})/(2m), \tag{29}$$

which follows directly from the first of Eqs. (17). Into the right-hand side of this equation, one substitutes from (28) and the equation

$$\Gamma(\pm q)=Z(\pm q-m+\delta m)+f^2[\Gamma_{-k}(\pm q)-\Gamma_{-j}(\pm q)F_{-kj}(\pm q)]D_kS(\pm q)\Gamma_k(\pm q), \tag{30}$$

to which it reduces, where all the a_k's vanish and $p=\pm q$. The divergences then cancel, leaving an equation explicitly free of infinities.

It should be noticed that the freedom from divergences of the equations finally obtained is dependent on two or more terms, which separately contain divergent integrals, being evaluated together in such a way that the divergences cancel. There are various ways of doing this without risk of ambiguity in solving the equations; when the interaction $f\gamma\cdot a(x)$ contains a term of the type $e\gamma_\lambda\cdot a^\lambda(x)$, as in electrodynamics, one can most simply reduce (29) with the help of the identity[12]

$$\Gamma(p)-\Gamma(q)=(q-p)\cdot\Gamma_{q-p}(p). \tag{31}$$

In other cases, a relativistic cut-off procedure may be used consistently in the confidence that the cut-off constant cannot appear in any solution obtained.

The elimination of the infinities from the Eq. (9) after renormalization is complicated in meson theories because one has to introduce a direct interaction between the mesons to compensate for divergences introduced by the indirect interaction via pair production and annihilation. This complication will be avoided by ignoring fermion loops in the subsequent discussion, though of course they should be taken into account. The approximation involved amounts to setting $(k^2-\mu^2)$ for D_k. The generalization of the theory of this section to the many fermion problem will be discussed in Sec. 6. Meanwhile, we shall consider the principles involved in solving the equations already obtained.

4. TAMM-DANCOFF EQUATIONS AND THEIR TRANSFORMATION TO DIVERGENCE-FREE FORM

We shall now consider what can be done to obtain a solution of (14). There is little hope of solving this equation explicitly, involving as it does the infinitely many variables a_k, since this is not possible even for the much simpler Eq. (3). But from a formal point of view, (14) is a partial differential equation in the a_k's and requires a set of boundary conditions for the complete specification of its solution. The only obvious condition to impose is that G should be a regular function of a_k for

[13] X_{-jk} and $-W_{-jk}$ are reciprocal functions of the boson momenta in the sense of Volterra: see E. T. Whittaker and G. N. Watson's *Modern Analysis* (Cambridge University Press, London, 1940), Sec. 11.22.

$a_k=0$, and this makes a solution in the form

$$G=S-fS_ka_k+\tfrac{1}{2}f^2S_{kl}a_ka_l-\tfrac{1}{6}f^3S_{jkl}a_ja_ka_l+\cdots,$$
$$S=(G)_0,\quad S_k=-f^{-1}(\partial G/\partial a_k)_0, \tag{32}$$
$$S_{kl}=f^{-2}(\partial^2G/\partial a_k\partial a_l)_0,$$

a practical necessity. When this is substituted into (14), one obtains the infinite chain of equations

$$Z(\not p-m+\delta m)S=Zf^2D_j\gamma_{-j}\cdot S_j+1, \tag{33}$$

$$Z(\not p-m+\delta m)S_k+Z\gamma_kS=Zf^2D_j\gamma_{-j}\cdot S_{jk} \tag{34}$$

$$Z(\not p-m+\delta m)S_{kl}+Z(\gamma_kS_l+\gamma_lS_k)=Zf^2D_j\gamma_{-j}\cdot S_{jkl}, \tag{35}$$

etc. In order to solve this sequence of equations, it must be terminated by introducing some approximation on account of which one of the terms on the right-hand side either vanishes or can be simplified. The result will be a set of integral equations with the great advantage that it is linear in the unknowns and is therefore soluble in principle by existing methods. The Tamm-Dancoff method proceeds in this way.

We shall introduce here not the usual Tamm-Dancoff method, but an obviously covariant modification of it which may be described as follows. In the coordinate representation, an open 3-dimensional surface drawn through any point on the fermion trajectory will intersect a number of meson lines. The minimum number of meson lines intersected by such a variable surface depends only on the point on the fermion line selected. Configurations in which this minimum number exceeds a certain value n will be left out of consideration in the nth order approximation of the method adopted here. Thus, for example, S_{jkl} will certainly vanish in the second-order approximation.

To apply the method, it is necessary to know whether a particular meson is emitted or absorbed; this is possible if the sign of the 4th component of its momentum vector is given. We shall, therefore, henceforth adopt the convention that the 4th component of a momentum vector is always positive, replacing k by $-k$, etc., where necessary in the foregoing. The nth order approximation to any expression A_{-jkl} involving momentum suffixes will be denoted by $A_{-jkl}{}^{(n)}$. It is now easy to formulate rules concerning the nth order approximation to products of such expressions. One has, for instance, $(A_{-jkl}B_{ij})^{(n)}=A_{-jkl}{}^{(n-1)}B_{ij}{}^{(n)}$. The general rule is that the difference between the order of a factor and the order of the term in which it occurs is the sum of the number of unrepeated positive suffixes to the right of the factor and the number of unrepeated negative suffixes to the left of the factor. (The suffix to a factor D_j is ignored for this purpose, and a suffix to the right or left of the factor is repeated if it occurs also in the factor itself).

The approximation procedure just described affects the value of the renormalization constants Z and δm. Since $S_j{}^{(0)}=0$, it follows from (33) that $Z^{(0)}=1$ and $\delta m^{(0)}=0$, and

$$S^{(0)}=(\not p-m)^{-1}. \tag{36}$$

Similarly, from (34) one has

$$S_k{}^{(1)}=-(\not p-m)^{-1}\gamma_kS^{(1)}, \tag{37}$$

and by substitution in (33),

$$Z^{(1)}(\not p-m-\delta m^{(1)})S^{(1)}=-f^2D_j\gamma_{-j}(\not p-m)^{-1}\gamma_jS^{(1)}+1. \tag{38}$$

It is fairly easy to eliminate the divergent constants $Z^{(1)}$ and $\delta m^{(1)}$ from (38), but in the higher order approximations, the elimination of the divergences is impossible with the Eqs. (33)–(35) etc., in their original form. Looking over the considerations of the previous section, the reason is obvious: it is G^{-1}, rather than G, which is easily "renormalized."

One is thus led to consider the expansion

$$V=G^{-1}=\Gamma-f\Gamma_ka_k+\tfrac{1}{2}f^2\Gamma_{kl}a_ka_l-\cdots, \tag{39}$$

in preference to (32). There certainly exists a set of simple relations between the coefficients of the two series:

$$S=\Gamma^{-1},\quad S_k=-S\Gamma_kS,$$
$$S_{kl}=-S(\Gamma_{kl}-\Gamma_kS\Gamma_l-\Gamma_lS\Gamma_k)S, \tag{40}$$

etc., but the Γ's are the only quantities to which the renormalization procedure can be directly applied. A particular case of this can be seen in the work of Fubini,[7] who proposed using the relation

$$S_{-kl}{}^{(2)}=-S^{(1)}[\Gamma_{-kl}{}^{(2)}-\Gamma_{-k}{}^{(1)}S^{(0)}\Gamma_l{}^{(1)}-\Gamma_l{}^{(2)}S^{(2)}\Gamma_{-k}{}^{(2)}]S^{(1)} \tag{41}$$

to calculate the second-order approximation to the amplitude for meson-nucleon scattering, as defined above. [The method he suggested for obtaining a divergence-free $\Gamma_l{}^{(2)}$, however, does not work; the Eq. (47) below must be used].

The equations satisfied by the coefficients Γ, Γ_k, Γ_{kl}, etc., of (39) can be obtained either by substituting (40) in (33)–(35) etc., or more directly by substituting (39) in (20). They are

$$\Gamma=Z(\not p-m+\delta m)+Zf^2D_k\gamma_{-k}\cdot S\Gamma_k, \tag{42}$$

$$\Gamma_k=Z\gamma_k+Zf^2D_j\gamma_{-j}\cdot S(\Gamma_{jk}-\Gamma_kS\Gamma_j), \tag{43}$$

$$\Gamma_{kl}=Zf^2D_j\gamma_{-j}\cdot S(\Gamma_{jkl}-\Gamma_{kl}S\Gamma_j-\Gamma_kS\Gamma_{jl}-\Gamma_lS\Gamma_{jk}+\Gamma_kS\Gamma_lS\Gamma_j+\Gamma_lS\Gamma_kS\Gamma_j), \tag{44}$$

etc. All the equations of this section have an obvious geometrical interpretation.

In proceeding from (33)–(35) to (42)–(44) the equations have lost their linear character. Fortunately this is restored by the method of approximation which has been suggested. Thus, (42) and (43) yield

$$\Gamma^{(n)}=Z^{(n)}[\not p-m+\delta m^{(n)}]+Z^{(n-1)}f^2D_k\gamma_{-k}\cdot S^{(n-1)}\Gamma_k{}^{(n)}, \tag{45}$$

and

$$\Gamma_k^{(n)}=Z^{(n-1)}\gamma_k+Z^{(n-2)}f^2D_j\gamma_{-j}\cdot S^{(n-2)}[\Gamma_{jk}^{(n)}-\Gamma_k^{(n-1)}S^{(n-1)}\Gamma_j^{(n)}], \quad (46)$$

which are linear in the unknowns if $S^{(n-1)}$ and $\Gamma_k^{(n-1)}$ [and $S^{(n-2)}$] are supposed to be already determined.

For small values of n, no difficulty arises in eliminating the divergence from the equations thus obtained. As an example, (46) reduces for $n=2$ to

$$\Gamma_k^{(2)}=Z^{(1)}\gamma_k-f^2D_j\gamma_{-j}\cdot S^{(0)}\Gamma_k^{(1)}S^{(1)}\Gamma_j^{(2)}, \quad (47)$$

since, as may be seen from (44), $\Gamma_{kl}^{(2)}$ vanishes. The cancelling of the divergence $[Z^{(1)}-1]\gamma_k$ from a corresponding divergence in the second term is now readily effected. But in higher approximations it is necessary to adhere closely to the method of the previous section. Setting the a_k's equal to zero, the function W_{jk} defined by (22) reduces to

$$R_{jk}=f^2D_jS(\Gamma_{jk}-\Gamma_kS\Gamma_j), \quad (48)$$

and X_{jk} to F_{jk}, where, according to (23),

$$F_{jk}=R_{jk}-R_{-ik}X_{ji}. \quad (49)$$

Instead of the Eq. (46), one will use

$$\Gamma_j^{(n)}=Z^{(n-1)}\gamma_j+\Gamma_{-k}^{(n)}F_{jk}^{(n)}, \quad (50)$$

in which $F_{jk}^{(n)}$ is supposed to be already known. Also, from (28) with the a_k's set equal to zero, one has

$$\Gamma^{(n)}=Z^{(n)}(\not{p}-m+\delta m)+f^2[\Gamma_{-k}^{(n)}-\Gamma_{-j}^{(n)}F_{-kj}^{(n)}]D_kS^{(n-1)}\Gamma_k^{(n)}. \quad (51)$$

The exact order in which the various functions can be determined will be indicated in the next section.

5. SOLUTION OF THE EQUATIONS

To discuss systematically the principles involved in the solution of the system of equations already obtained, it will be necessary to introduce a new general class of scattering amplitudes, typified by $L(j_1\cdots j_r,k_1\cdots k_s)S$, where the factor S, inserted for convenience, is defined in (32). Here $j_1\cdots j_r$ denote the energy-momenta of external bosons absorbed by the fermion considered, in the order of their absorption along the fermion line; $k_1\cdots k_s$ denote the energy-momenta of external bosons emitted by the fermion, in the *reverse* order of their emission along the fermion line. Since the j's and k's represent boson momenta, they would appear in the foregoing discussion as suffixes, the k's prefixed by a negative sign. But to obtain functions of the type hitherto considered, one would have to sum $L(j_1\cdots j_r,k_1\cdots k_s)$ over all permutations of the j's and k's.

We shall wish also to consider the function $L_m^{(n)}(j_1\cdots j_r,k_1\cdots k_s)$ which represents a contribution to $L(j_1\cdots j_r,k_1\cdots k_s)$ from certain configurations: namely, those for which (in the coordinate representation) the minimum number of boson lines intersected by an open 3-dimensional surface through any point on the fermion line is never less than m nor greater than n. It accordingly represents an nth order approximation, of the type already considered, to a function $L_m(j_1\cdots j_r,k_1\cdots k_s)$ representing configurations in which at least m bosons are always "present." It should be noticed that $L_m(j_1\cdots j_r,k_1\cdots k_s)$ vanishes if $m>r$ or $m>s+1$. For $m=0$, L_m reduces to the total scattering amplitude L, and can be expressed in the form

$$L_0(j_1\cdots j_r,k_1\cdots k_s)=L_1(j_1\cdots j_r,k_1\cdots k_s)+L_1(j_1\cdots j_r,-)SL_1(-,k_1\cdots k_s). \quad (52)$$

Since $S^{(n)}=[\Gamma^{(n)}]^{-1}$ can be expressed in terms of $L_1^{(n)}(j,-)$, $L_2^{(n)}(j,k)$, $L_1^{(n)}(-,k)$ and $S^{(n-1)}$ with the help of (51), the problem of obtaining $L_0^{(n)}$ is reduced in principle to that of obtaining $L_1^{(n)}$ and $L_2^{(n)}$, when $L_m^{(n-1)}$ ($m=0$, 1, 2 etc.) are assumed to be already determined.

The equations satisfied by the newly defined functions can be inferred from those of the previous section. It follows from (40), (43), and (44), for example, that

$$L_1^{(2)}(j,k)=S^{(1)}[\gamma_{-k}S^{(0)}\gamma_j-f^2D_l\gamma_{-l}S^{(0)}\gamma_jL_1^{(2)}(l,k)], \quad (53)$$

and (47) may be written

$$L_1^{(2)}(j,-)=S^{(1)}[Z^{(1)}\gamma_j-f^2D_l\gamma_{-l}S^{(0)}\gamma_jL_1^{(2)}(l,-)]. \quad (54)$$

Any of these equations has an adjoint, obtained by reversing the order of the factors in any product, and interchanging the j's and k's in $L_m^{(n)}(j_1\cdots j_r,k_1\cdots k_s)$. Equations of this type can of course be written down at once from graphical considerations similar to those used in formulating the Bethe-Salpeter equation.[14] In fact, analytical relations can be obtained in this way which one would not easily discover otherwise, though they are relatively easy to prove analytically when known. For example, one has

$$L_2^{(3)}(j_1j_2,k_1k_2)=L_1^{(2)}(j_1,k_2)L_1^{(2)}(j_2,k_1)-f^2L_1^{(2)}(j_1,l)D_lL_2^{(3)}(j_2l,k_1k_2), \quad (55)$$

$$L_2^{(3)}(j_1j_2,k)=L_1^{(2)}(j_1,k)L_1^{(2)}(j_2,-)-f^2L_1^{(2)}(j_1,l)D_lL_2^{(3)}(j_2l,k), \quad (56)$$

and

$$L_1^{(3)}(j_1j_2,k_1k_2)=L_0^{(2)}(j_1,k_2)L_0^{(2)}(j_2,k_1)-f^2L_0^{(2)}(j_1,l)D_lL_1^{(3)}(j_2l,k_1k_2), \quad (57)$$

$$L_1^{(3)}(j_1j_2,k)=L_0^{(2)}(j_1,k)L_0^{(2)}(j_2,-)-f^2L_0^{(2)}(j_1,l)D_lL_1^{(3)}(j_2l,k), \quad (58)$$

$$L_1^{(3)}(j,k)=L_0^{(2)}(-,k)L_0^{(2)}(j,-)-f^2L_0^{(2)}(-,l)D_lL_1^{(3)}(jl,k). \quad (59)$$

To continue, one has

$$L_3^{(4)}(j_1j_2j_3,k_1k_2k_3)=L_2^{(3)}(j_1j_2,k_2k_3)L_1^{(2)}(j_3,k_1)-f^2[L_2^{(3)}(j_1j_2,k_3l)+L_2^{(3)}(j_1j_2,lk_3)]D_lL_2^{(3)}(j_3l,k_1k_2)+f^4[L_2^{(3)}(j_1j_2,il)+L_2^{(3)}(j_1j_2,li)]\times D_iD_lL_3^{(4)}(j_3il,k_1k_2k_3), \quad (60)$$

[14] E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951).

etc. Though the equations become more complicated as one proceeds, their mode of formation is fairly obvious.

The computation of successively higher approximations to L_m in this way involves nothing worse than the computation of convergent integrals and the solution of integral equations of the general type

$$\varphi_\lambda(j,l)=f_\lambda(j,l)+\int K_{\lambda\mu}(j,k)\varphi_\mu(k,l)dk, \quad (61)$$

where j, k, and l represent sets of momenta $j_1,\cdots j_r$, $k_1\cdots k_r$, and $l_1\cdots l_s$, the integration $\int\cdots dk$ denotes integration over all components of the 4-vectors $k_1\cdots k_r$, and λ, μ are spinor suffixes summed from $1\cdots 4$ when repeated in any term. This is a somewhat generalized form of Fredholm's equation, the solution of which is

$$\varphi_\lambda(j,l)=f_\lambda(j,l)+D^{-1}\int D_{\lambda\mu}(j,k)f_\mu(k,l)dk, \quad (62)$$

where

$$D=1-\int K_{\lambda\lambda}(k,k)dk+(2!)^{-1}\times\int\int\begin{vmatrix}K_{\lambda\lambda}(k,k)K_{\mu\lambda}(k',k)\\K_{\lambda\mu}(k,k')K_{\mu\mu}(k',k')\end{vmatrix}dkdk'-(3!)^{-1}\int\int\int\cdots, \quad (63)$$

and

$$D_{\lambda\mu}(j,k)=K_{\lambda\mu}(j,k)-\int\begin{vmatrix}K_{\lambda\mu}(j,k)K_{\nu\mu}(k',k)\\K_{\lambda\nu}(j,k')K_{\nu\nu}(k',k')\end{vmatrix}dk'+(2!)^{-1}\int\int\cdots. \quad (64)$$

The series in (63) and (64) are *absolutely convergent*, provided the integral $\int\int|K_{\lambda\mu}(k,k')|^2dkdk'$ exists; and the only difficulty in satisfying this condition arises in connection with Eqs. (53) and (54), which will be discussed presently. It follows that the only singularities in $\varphi_\lambda(j,l)$, which are not already present in $f_\lambda(j,l)$ or the kernel, arise from the vanishing of D. When $D=0$, there is no solution of (61), though there does exist a solution of the corresponding homogeneous equation

$$\varphi_\lambda(j,l)=\int K_{\lambda\mu}(j,k)\varphi_\mu(k,l)dk, \quad (65)$$

which is of the type proposed by Salpeter and Bethe[14] for bound states.

We shall accordingly take the point of view that any zero of $D_n^{(n+1)}$—the value of D corresponding to a kernel K formed from $L_{n-1}^{(n)}$—is associated with a bound state of the fermion and n bosons (a hyperon in nuclear theory). This point of view is justified by the fact that amplitudes, solutions of (65), exist for the creation of such bound states through the successive absorption of n bosons by the "bare" fermion. The states so defined are highly idealized because the true amplitudes do not describe exclusively configurations in which the number of bosons "present" varies within such narrow limits. An actual hyperon will fluctuate continually between the ideal state and other ideal states; such fluctuations, however, are virtual and may be regarded as self-energy processes of the hyperon concerned. The suggestion is simply that all processes, both real and virtual, can be conveniently interpreted in terms of transitions of the fermion between states, in any particular one of which the number of virtual bosons "present" is either n or $n+1$.

It should be noticed that $D_n^{(n+1)}$ is a function of w^2, where w is the resultant energy momentum vector of the system of the fermion with its n or $n+1$ bosons, and that for sufficiently large values of w^2, $D_n^{(n+1)}$ is complex. The roots of the equation $D_n^{(n+1)}(w^2)=0$ are therefore of the form $w^2=(M_n^{(n+1)}+iN_n^{(n+1)})^2$, where $M_n^{(n+1)}$ may be regarded as the mass of the ideal hyperon and $N_n^{(n+1)}$ as its reciprocal lifetime—which may be very large. The effect of virtual transitions to other ideal states will be to modify the mass value and lifetime so determined. So far as transitions to states with fewer bosons "present" are concerned, one can allow for this by solving the equation $D_1^{(n+1)}(w^2)=0$ instead of $D_n^{(n+1)}(w^2)=0$; but to take into account transitions to states with more than n bosons "present," one would have to raise the order of approximation of the kernel from which $D_1^{(n+1)}$ is formed.

It is known[15] that the right-hand side of (63) can be obtained by expanding the exponential, and regrouping the terms so obtained, on the right-hand side of

$$D^*=\exp\Big[-\int K_{\lambda\lambda}(j,j)dj-\tfrac{1}{2}\int\int K_{\lambda\mu}(j,k)K_{\mu\lambda}(k,j)djdk-\tfrac{1}{3}\int\int\int K_{\lambda\mu}(j,k)K_{\mu\nu}(k,l)\times K_{\nu\lambda}(l,j)djdkdl-\cdots\Big]. \quad (66)$$

In fact, $D=D^*$ within the range in which the exponent of (66) is meaningful. The "first" zero of D therefore corresponds to the radius of convergence of the series in (66), and can be obtained by solving the equation

$$\int^{(n)}\cdots\int K_{\lambda\mu}(j_1,j_2)K_{\mu\nu}(j_2,j_3)\cdots\times K_{\omega\lambda}(j_n,j_1)dj_1\cdots dj_n=1 \quad (67)$$

[15] See E. Hellinger and O. Toeplitz, *Integralgleichungen* (Teubner, Leipzig, 1928), Sec. 11.

for sufficiently large values of n. The identity of D and D^*, within their common range of significance, enables one to see the connection between Fredholm's and Liouville's solutions of (61), for

$$D_{\lambda\mu}{}^*(j,k)=D^*\Big[K_{\lambda\mu}(j,k)+\int K_{\lambda\mu}(j,l)K_{\mu\nu}(l,k)dl+\cdots\Big] \quad (68)$$

defines a function which, where it exists, is equivalent to that defined by (64).

We return now to the consideration of the difficulties, additional to those resolved by ordinary renormalization procedures, which arise in the solution of (52) and (53), as well as similar equations of higher order. The kernel for both (52) and (53) is

$$K(j,k)=-f^2S^{(1)}(w-j)\gamma_{-k}D_kS^{(0)}(w-j-k)\gamma_j,$$

and the integrals $\mathrm{spur}\int K(j,j)dj$, $\mathrm{spur}\iint K(j,k)\times K(k,j)djdk$, which appear in Fredholm's solution of these equations are logarithmically divergent. Let the divergent parts of these integrals, constants chosen so that the convergent remainders vanish when $w^2=m^2$, be C_1, C_2, C_3, etc. Then it is clear from (66) that if $\exp(C_1+\frac{1}{2}C_2+\frac{1}{3}C_3+\cdots)D^*$ is expanded and the terms of the expansion suitably regrouped, the divergences will cancel identically. Also, if $\exp(C_1+\frac{1}{2}C_2+\frac{1}{3}C_3+\cdots)\times D_{\lambda\mu}{}^*(j,k)$ is expanded and the terms of the expansion regrouped, the divergences will cancel identically. In the solution of (53), but not of (52), additional divergences appear; these, however, can be eliminated by standard renormalization methods, in the course of which the divergent constant Z is reduced to 1.

One indisputable conclusion, which can be drawn from the above considerations, is that the singularities in the Tamm-Dancoff amplitudes are all associated with the existence of (not necessarily observable) bound states. If $L_m{}^{(n)}$ approached a limiting value L_m when n became very large, it could be concluded immediately that all singularities in the exact amplitudes were likewise associated with bound states of the type contemplated. Unfortunately $L_m{}^{(n)}$ does not approach a limiting value, as we shall immediately demonstrate.

It is sufficient for our purpose to notice a contribution to the amplitude for the scattering of a single boson by a fermion, namely $-f^2S\Gamma_j'S\Gamma_{-k}'S$, where Γ_j' is defined by

$$\Gamma_j'=Z'\gamma_j-f^2\gamma_{-l}D_lS^{(0)}\Gamma_j'S^{(0)}\gamma_l. \quad (69)$$

This is an equation of the type whose explicit solution was attempted by Edwards; he obtained a solution of the homogeneous equation for a particular case, but nothing corresponding to the general Fredholm solution —probably because of the divergences, not removed by ordinary renormalization procedure, which have already been noticed in connection with the similar Eq. (53). Now the nth order approximation to the solution of (69) is a polynomial of the nth degree in f^2. The general solution does not approach a limiting value for large n, as one can infer from the mere existence of a solution of the homogeneous equation with $Z'=0$.[16]

To discover the physical reason for this failure of the Tamm-Dancoff method, it should be noticed that (69) is an equation of Fredholm's type, and the divergence of the solution when expressed as a power series in f^2 is a consequence of the existence of a zero of the Fredholm denominator corresponding to the kernel $-f^2D_j[\gamma_{-j}S^{(0)}]_{\lambda\mu}[S^{(0)}\gamma_j]_{\sigma\rho}$. The latter is precisely the lowest order approximation to the kernel of the original Bethe-Salpeter equation for bound states of a pair of fermions. The vanishing of the particular Fredholm denominator we are considering is therefore associated with the existence of such bound states. Fermion pairs in a bound state can be created in the vacuum, and these will influence the amplitudes for scattering of bosons by a single fermion. Quite generally the existence of bound states of a system of n fermions will spoil the convergence of the Tamm-Dancoff method applied to a single fermion. Very probably the existence of such bound states is also the *only* cause of failure of the approximation method we have examined; if this is conceded, two conclusions may be drawn. Firstly, the singularities in scattering amplitudes—which are the prime cause of the failure of the perturbation series—are all associated in one way or another with transitions, real or virtual, to bound states. Secondly, if account could be taken of bound states involving any number of fermions, the failure of the Tamm-Dancoff method of approximation applied to the single fermion problem would be removed. Without attempting a detailed solution of the problems thus raised, we shall therefore briefly examine the generalization of the previous considerations to systems of more than one fermion.

6. THE MANY-FERMION PROBLEM

The interaction of a system of n fermions with a boson field is completely described by the generating function G_n, which is the solution of Eq. (11). This equation, when formally renormalized, assumes the form

$$Z^n\prod_{r=1}^{n}\{p^{(r)}-m+\delta m-f[\gamma_k{}^{(r)}\cdot a_k-D_k\gamma_{-k}{}^{(r)}\cdot(\partial/\partial a_k)]\}G_n=1. \quad (70)$$

One can write down immediately a partial solution of (70), which effectively reduces the many-fermion problem to the single-fermion problem already considered; it is

$$Z\{p^{(n)}-m+\delta m-f[\gamma_k{}^{(n)}\cdot a_k-D_k\gamma_{-k}{}^{(n)}\cdot(\partial/\partial a_k)]\}G_n=G_{n-1}. \quad (71)$$

[16] At the particular value of the total energy for which Edwards solved the equation, the Fredholm denominator vanishes for a rather large value of the coupling constant, which would have to be exceeded before the perturbation series diverged. At other energies the divergence would occur sooner. See also J. S. Goldstein, Phys. Rev. **91**, 1516 (1953) and a paper by the author to appear soon in Proc. Phys. Soc. (London) **A**.

As a matter of interest, one can construct from this the Schrödinger-like equation

$$Z\sum_{r=1}^{n}\{p^{(r)}-m+\delta m-f[\gamma_k{}^{(r)}\cdot a_k - D_k\gamma_{-k}{}^{(r)}\cdot(\partial/\partial a_k)]\}G_n=\sum_{r=1}^{n}G_{n-1}{}^{(r)}, \quad (72)$$

where $G_{n-1}{}^{(r)}$ is the function G_{n-1} with a set of energy-momenta $p^{(1)}\cdots p^{(n)}$, excluding $p^{(r)}$, as its arguments. This last equation, though more symmetrical, is not fully equivalent to (71), which is therefore to be preferred.

In considering (71) further, we shall write p for $p^{(n)}$ and γ_k for $\gamma_k{}^{(n)}$ for the sake of simplicity and to avoid confusion with the notation developed in the previous two sections. It will be found convenient also to set

$$G_n=GG_{n-1}H_n, \quad (73)$$

where G is the function of p, i.e., $p^{(n)}$, considered exclusively in No. 3 and No. 4. When (73) is substituted into (72), and use is made of the equation satisfied by G, (72) reduces to

$$G_{n-1}H_n+ZfD_k\gamma_{-k}\cdot G[(\partial G_{n-1}/\partial a_k)H_n+G_{n-1}(\partial H_n/\partial a_k)]=G_{n-1}. \quad (74)$$

Dividing by G_{n-1} on the left and H_n on the right gives

$$H_n{}^{-1}=1-ZfD_k\gamma_{-k}\cdot G[(\partial G_{n-1}{}^{-1}/\partial a_k)G_{n-1}+H_n(\partial H_n{}^{-1}/\partial a_k)]. \quad (75)$$

Here $Z\gamma_{-k}$ may be replaced by $\Gamma_{-k}-\Gamma_{-l}X_{-kl}$, completing the elimination of the divergent constants.

The functions most easily expressed in a divergence-free form are clearly the coefficients U, U_k, etc., in the expansion

$$H_n{}^{-1}=U-fU_ka_k+\tfrac{1}{2}f^2U_{kl}a_ka_l-\cdots. \quad (76)$$

When these coefficients have been determined, it will be possible to determine G_n in the form

$$G_n=GG_{n-1}U^{-1}\{1+f[U_k+\tfrac{1}{2}f(U_kU^{-1}U_1+U_1U^{-1}U_k-U_{kl})a_l]a_kU^{-1}+\cdots\}. \quad (77)$$

The equations satisfied by U, U_k, etc., are easily written down from (75); one has

$$U=1+Zf^2D_k\gamma_{-k}\cdot S(P_k+U^{-1}U_k), \quad (78)$$

$$U_l=Zf^2D_k\gamma_{-k}\cdot S_lS(P_k+U^{-1}U_k)+Zf^2D_k\gamma_{-k}\cdot S[P_{kl}+U^{-1}(U_{kl}-U_lU^{-1}U_k)], \quad (79)$$

etc., if

$$(\partial G_{n-1}{}^{-1}/\partial a_k)G_{n-1}=P_k-fP_{kl}a_l+\cdots. \quad (80)$$

The Eqs. (78), (79), etc., are very closely analogous to those connecting Γ, Γ_l, etc., and this fact alone suggests that they should be soluble by an approximation procedure similar to that developed in No. 4. Thus, for example, (78) yields at the nth approximation

$$U^{(n)}=1+Zf^2D_k\gamma_{-k}\cdot S^{(n-1)}\{P_k{}^{(n)}+[U^{(n-1)}]^{-1}U_k{}^{(n)}]. \quad (81)$$

One has, accordingly, at one's disposal at least one method of solving the equations which does not depend on a perturbation expansion at any stage.

The method thus sketchily indicated for the solution of the many-nucleon problem leaves one still very far from an exact convergent solution to even the single-fermion problem. What is needed to carry one further is a system of classifying the bound states which arise in many-fermion-boson interactions, and a somewhat less restricted method of approximation which ultimately takes all such states into account. There is reasonable hope that these can be found; but until they are found, one will not be able to say with certainty whether the exact equations of field theory are soluble in principle, or not.

The author acknowledges the assistance of Mr. I. E. McCarthy in proving a few of the formulas of this paper, and discussions with Dr. J. C. Ward.