We can make some estimates of the errors involved in using this method. Inside the turning point (which is at $x \cong j$), any small errors made in an integration step will bring in a small amount of irregular function. However, in this region the regular function increases rapidly with increasing x, while the irregular function decreases rapidly. Thus, in this region, the effect of a small error tends to be damped quickly. Of course the normalization near the origin may differ from that outside the turning point because of these errors, but that does not affect the present calculations since we need only the ratio (F_j/G_j) at the fitting-on radius.

Outside the turning point the error in the phase can be estimated by replacing the Eqs. (21) by the equations

$$g'=-f, \quad f'=+g, \tag{A28}$$

which have the solutions

$$g=\cos x, \quad f=\sin x.$$

Equations (A19–A21) have the solution[31]

$$g=\cos\lambda x, \quad f=\sin\lambda x,$$

where

$$\lambda=1-h^2/12,$$
$$\lambda=1-h^4/720,$$
$$\lambda=1-h^6/100\,800,$$

respectively. The total phase error in a distance L is therefore

$$\delta\phi=h^2L/12,$$
$$\delta\phi=h^4L/720,$$
$$\delta\phi=h^6L/100\,800,$$

for the three approximations.

If we require $\delta\phi<10^{-4}$, for $L=8$ we find in the respective cases:

$$h<10^{-2}, \quad h<0.3, \quad h<1.$$

The interval that would be required using Eq. (A19) is prohibitively small. With Eq. (A20) there is more computing work at each step of the integration, but the interval is reasonable. One could not use such a large interval with Eq. (A21) as estimated here because of the variation of A_n with x. With a smaller interva Eq. (A21) may be useful in obtaining greater accuracy

[31] This was pointed out to us by Professor L. I. Schiff.

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High-Energy Electron Scattering and Nuclear Structure Determinations. II*†‡

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Elastic scattering measurements have been carried out with electrons in Au^{197} at energies of 84, 126, 154, and 183 Mev and in Pb^{208} at 84, 153, and 186 Mev. Diffraction effects are observed which appear to vary with momentum and angular position as if a fundamental parameter $p\sin(\theta/2)$ were equal to a constant for a given diffraction feature. Such a behavior would be predicted by the Born approximation. A comparison of the scattering in Au^{197} and Pb^{208} suggests that inelastic scattering does not materially influence the scattering curves presented. The appearance of diffraction effects indicates a model more nearly uniform in charge density than early tentative conclusions based on Born approximation calculations.

I. INTRODUCTION

IN the first paper of this series with the above title[1] experimental electron scattering curves were presented for several materials at 125 Mev. Elastic profiles were shown, the apparatus was described, various checks on the experimental information were discussed, and other relevant information was given. A preliminary attempt to explain the at-that-time unexpected absence of prominent diffraction peaks was made in terms of a first-order Born approximation calculation for various nuclear charge distributions.[1,2] These approximate calculations led to a tentative interpretation which indicated a smooth decrease of charge density from the center to the outer regions of heavy nuclei such as gold and lead. It must be borne in mind that the conventional values of nuclear radius (for example, root-mean-square values) were retained in this interpretation.

It has recently been shown by Yennie, Wilson, and Ravenhall[3] that an accurate phase shift calculation for

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† Aided by a grant from the Research Corporation.
‡ This material was presented in part at the New York Meeting of the American Physical Society in January, 1954 [Phys. Rev. 94, 773 (1954)].
§ Visiting Post Doctoral Research Fellow of the Schweizerische Arbeitsgemeinschaft in Mathematik und Physik, Switzerland.
[1] Hofstadter, Fechter, McIntyre, Phys. Rev. 92, 978 (1953). We shall refer to this paper as I.
[2] L. I. Schiff, Phys. Rev. 92, 988 (1953).
[3] Yennie, Wilson, and Ravenhall, Phys. Rev. 92, 1325 (1953).

a more conventional nuclear model with uniform charge density and a sharp edge provides elastic scattering curves with washed-out minima and maxima for nuclei with $Z \cong 80$. For copper ($Z=29$) the minima and maxima are pronounced and the Born approximation is more pertinent although not completely reliable even in this case. The calculations of Yennie *et al.*, and also those of Baranger,[4] carried out by a less accurate method, therefore implied that perhaps the experimental data can be fitted by a uniform, or nearly uniform, model. In any case, it became apparent immediately that more experimental and theoretical information would be required before definite conclusions on the type of charge distribution could be made. The tentative interpretation which led to a strong taper at the center has to be abandoned although the early suggestion of a taper at the edge is confirmed (see below).

It has been a major goal of our program to carry out elastic scattering at different energies in order to provide more than one "fix" on a given nucleus. For this reason we have presently continued the earlier studies and in this paper we give the experimental curves for Au^{197} and Pb^{208} at several energies.[5]

Since the first energy level in the Pb^{208} nucleus is at 2.6 Mev it is possible with the energy resolution of our spectrometer magnet to select only elastically scattered electrons.[6] In contrast, in the Au^{197} nucleus there is an energy level below 100 kev, and electrons exciting this level could not be rejected as inelastically scattered electrons by our apparatus. If one then assumed that inelastic scattering events significantly modified the gold angular distribution, it would be very unlikely that the gold and lead angular distributions would be similar. Experimental evidence of such a similarity would thus be evidence for a negligible contribution of inelastic scattering from gold.

II. APPARATUS

The main features of the experimental apparatus have been described in I (pp. 979 and 980). Some recent additions and improvements in the apparatus are noted below. A remotely controlled uranium slit at the exit of the analyzing magnet has been added. A beam "sniffer" which indicates small horizontal shifts in the position of the beam emerging from the second deflecting (refocusing) magnet has also been added. This is an ionization chamber split vertically in two with the readings of each half balanced against the other, and the slit being located on the beam line. The sniffer is used to control the beam manually from time to time as the occasion demands. The analyzing magnet current

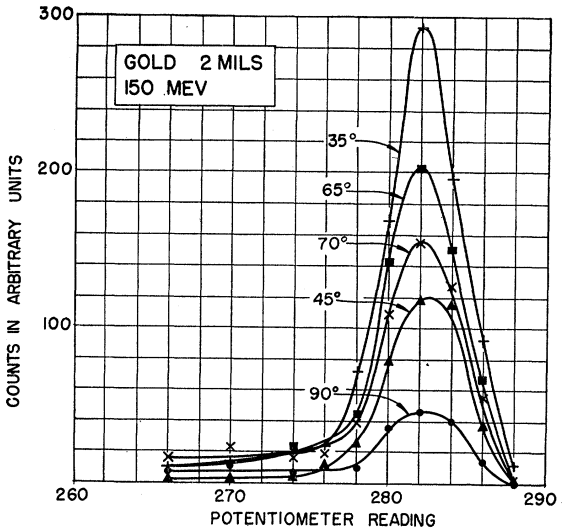

FIG. 1. Representative elastic scattering curves at 150 Mev in gold. The ordinates of the individual curves are unrelated.

is now regulated to better than 0.1 percent by means of electronic control of the generator field winding.

The analyzing magnet has also been rewound with hollow conductor square copper rod so that its upper bending limit is now 195 Mev, whereas it was previously about 150 Mev. The focusing properties at the highest energies have not been studied carefully although they are presumably not bad judging by the quality of the elastic curves. However, the magnet shows increasing saturation at the high energies so that the current scale is no longer proportional to energy. In order to deter-

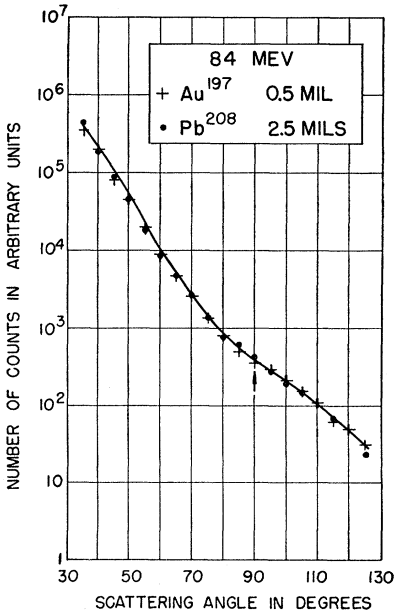

FIG. 2. Elastic scattering in Au^{197} and Pb^{208} at 84 Mev. The arrow marks an estimate of the angular position of a diffraction washed-out minimum. Curves normalized arbitrarily.

[4] E. Baranger, Phys. Rev. **93**, 1127 (1954). We wish to thank Mrs. Baranger for early communication of her results.

[5] The 96-percent pure Pb^{208} sample was obtained from the Isotopes Division, U. S. Atomic Energy Commission, Oak Ridge, Tennessee.

[6] We have recently observed inelastic electron scattering peaks in beryllium and other materials at 190 Mev.

TABLE I. Summary of data on minima in gold.

Energy (Mev)	θ_I (degrees)	θ_{II} (degrees)	$E \sin(\theta_I/2)$	$E \sin(\theta_{II}/2)$
84	90	...	59	...
126	51	109	55	103
154	45	90	58	109
183	...	70	...	105

mine the energy scale, a calibration curve was prepared by measuring the magnetic field (at the half-way point of the electron trajectories) against the current through the magnet windings.

III. PROCEDURE

The angular distributions have been obtained from elastic profiles taken at the various angular positions. In most cases "complete" elastic curves have been taken at all angular settings and the area under each elastic curve has been plotted as an individual point in the angular distribution. As an example, a few members in a set of elastic curves are shown in Fig. 1. In this case the energy spread in the incident 150-Mev beam was 0.5 percent, the thickness of the gold foil was 2 mils. The exit slit of the spectrometer was set at a width corresponding to 0.5 percent in energy and the entrance slit was $\frac{3}{4}$ in. wide and $\frac{3}{4}$ in. high at a distance of 11 in. from the center of the scattering foil. In the case of Pb^{208} the measurements have usually been made with five points straddling the center of the peak instead of a complete elastic curve. The best energy resolution realized to date with the analyzing magnet is 0.28 percent at 125 Mev.

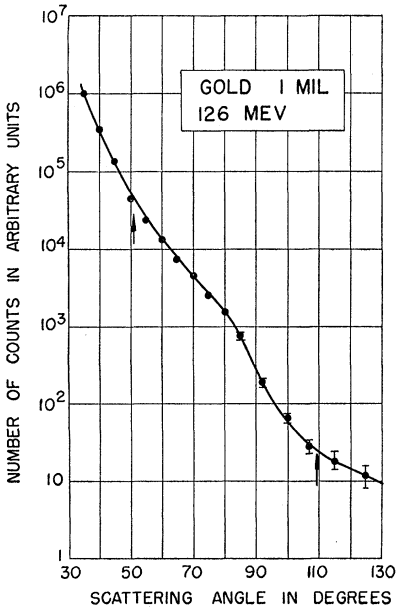

FIG. 3. Elastic scattering in Au^{197} at 126 Mev. The arrows mark an estimate of the angular position of diffraction washed-out minima.

IV. RESULTS

Experiments were carried out at 84, 126, 154, and 183 Mev in Au^{197} and at 84, 153, and 186 Mev in Pb^{208}. The angular distributions so obtained are shown in Figs. 2, 3, 4, and 5. In Figs. 2, 4, and 5 both the lead and gold data are plotted, while in Fig. 3 only the gold data appear. The statistical errors are also shown in Fig. 3 and are typical of those obtained at the other energies. It is clear from these curves that the angular distributions for lead and gold are identical within the accuracy of the experiments, and it may therefore be presumed that inelastic processes contribute a negligible amount to the angular distributions. The newer data are also more consistent internally than the data in paper I and the study in gold at 126 Mev has therefore been repeated. The agreement is quite good and the slight differences observed are within the old experimental errors.

The data for gold and lead show clearly the signs of washed-out diffraction structure. The 84-Mev data for

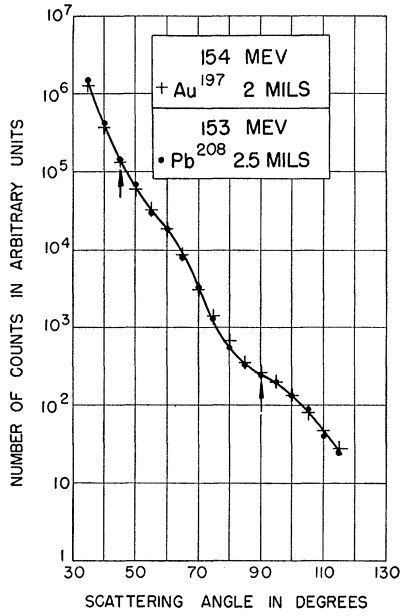

FIG. 4. Elastic scattering in Au^{197} and Pb^{208} at 154 and 153 Mev, respectively. The arrows mark estimates of the angular positions of diffraction washed-out minima. Curves normalized arbitrarily.

gold and lead show a washed-out minimum near 90°. At 126 Mev in gold there is a similar point at 51° and a second washed-out minimum at 109°. At 154 Mev the first "minimum" appears near 45° and a second more pronounced one at 90°. The second "minimum" at 154 Mev falls at 90° where we found the first "minimum" at 84 Mev, an energy close to one-half of 154 Mev. At 183 Mev the first "minimum" has moved near 35° and is not clearly visible. The second one has now moved to about 70°. Hence it is now possible to trace the diffraction structure through many scattering

curves at different energies. The diffraction peaks and valleys are not prominent but nevertheless are definite, as shown by a "form factor" plot of Fig. 6 in which the point charge calculations have been divided into the experimental curves at the various energies. It is to be noted that a uniform charge distribution with square edge[3] gives an appearance somewhat like the experimental data but with more pronounced diffraction features.

Table I shows a summary of the data on "minima" in gold. From these it may be seen that the product

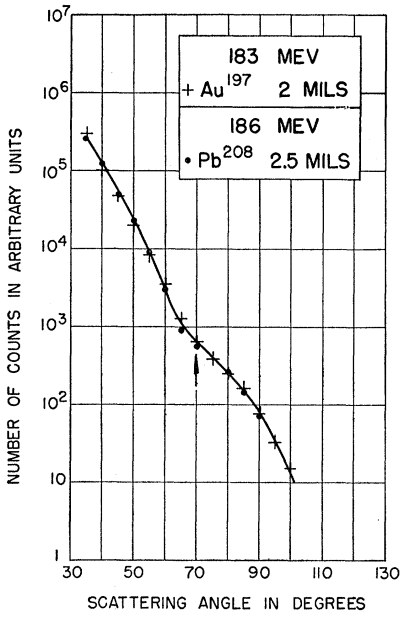

FIG. 5. Elastic scattering in Au^{197} and Pb^{208} at 183 and 186 Mev, respectively. The arrow marks an estimate of the angular position of a diffraction washed-out minimum. Curves normalized arbitrarily.

$E\sin\theta/2\cong 57$, where E is the energy in Mev ($E=pc$ at these energies), for the first "minimum" in all the gold and lead curves. The product $E\sin\theta/2\cong 106$ for the second "minimum." These figures imply a nuclear radius for a uniformly charged model, $R=r_0A^{\frac{1}{3}}$, where $r_0=(1.1\pm 0.1)\times 10^{-13}$ cm for both Au^{197} and Pb^{208} where we have used the curves of Yennie *et al.*[3] Therefore the charge density is considerably higher than that given by previous models where $r_0=1.45\times 10^{-13}$ cm. A rounded-off model gives a better fit than the uniform, but as yet not an exact fit of the experimental data.[7,8] Similar

[7] Yennie, Ravenhall, and Wilson, preceding paper [Phys. Rev. **95**, 500 (1954)].

[8] Brenner, Brown, and Elton (to be published). We wish to thank Prof. R. E. Peierls for early communication of these results.

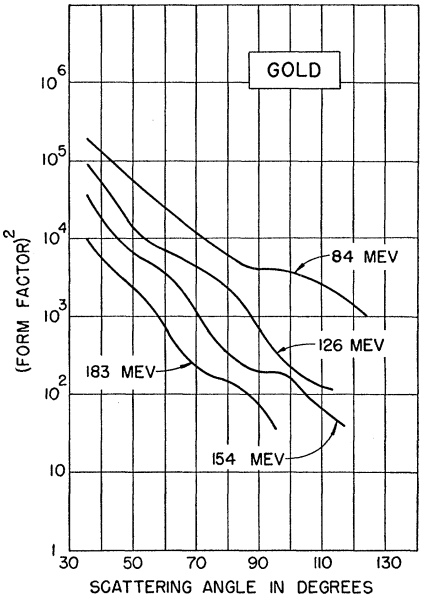

FIG. 6. The squares of form factors plotted against angle for gold at 84, 126, 154, and 183 Mev. The ordinate is obtained by plotting the quotient of the observed scattering by the point charge scattering curve. The ordinates of individual curves are unrelated.

conclusions concerning a smaller nuclear size have been drawn recently from the measurements of Fitch and Rainwater[9] and Pidd, Hammer, and Raka.[10]

A model such as a Gaussian or exponential will not explain simultaneously the scattering data at 126 and 183 Mev because the theoretical curves for such peaked distributions fall off too rapidly toward large angles at the higher energies. Furthermore, the observed diffraction structure implies a type of finite boundary rather than the smooth taper which predicts an entirely smooth scattering curve.[3,7]

ACKNOWLEDGMENT

We are grateful to Professor L. I. Schiff for many interesting and fruitful conversations. We wish to thank Dr. D. R. Yennie, Dr. D. G. Ravenhall, and Mr. R. N. Wilson for many stimulating discussions concerning their theoretical calculations. We should also like to express our great appreciation to Mr. B. Chambers, Mr. V. Prosper, and Mr. J. Fregeau for their help in building parts of our experimental apparatus.

[9] V. L. Fitch and J. Rainwater, Phys. Rev. **92**, 789 (1953).

[10] Pidd, Hammer, and Raka, Phys. Rev. **92**, 436 (1953).