PHYSICAL REVIEW VOLUME 95, NUMBER 2 JULY 15, 1954

Susceptibility Tensor and the Faraday Effect in Ferrimagnetics

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(Received April 14, 1954)

The components of the susceptibility tensor have been calculated for the two-sublattice model of a ferrimagnetic in which damping is neglected, and the results related to the theory of the Faraday effect in these materials. It is shown that in most cases the only change necessary is the replacement of the electron gyromagnetic ratio by the effective gyromagnetic ratio of the material. The effect of the sublattice structure is more apparent when the material has a compensation point, for then the rotation can change sign two or more times as a function of temperature or composition. The rotation to be expected for an unmagnetized antiferromagnetic is also determined, and some of the simpler results are generalized to the case in which there are an arbitrary number of sublattices. Other possible sublattice effects are also briefly considered.

INTRODUCTION

If the electronic magnetic moments in a ferromagnetic material are subjected to a magnetic field having a constant component in the z direction and a small oscillating component in the xy plane, then it has been shown[1–4] that the gyroscopic behavior of the spins leads to expressions for the transverse components of the magnetization of the form

$$\left.\begin{aligned} M_x &= \xi_x H_x - i(\zeta/\omega)H_y, \\ M_y &= +i(\zeta/\omega)H_x + \xi_y H_y, \end{aligned}\right\} \tag{1}$$

where ω is the circular frequency of the oscillating field.

A consequence of this tensor character of the susceptibility is that the index of refraction for a circularly polarized plane wave propagated in the z direction depends upon the sense of rotation of the field vectors. If a linearly polarized wave is incident upon the material, its two components of opposite circular polarization will travel through the material with different phase velocities. Their resultant at any point is still a linearly polarized wave, but one whose plane of polarization has been rotated through an angle proportional to the distance traversed in the medium, a result which is familiar from the usual description of the Faraday effect. One finds the angle of rotation after a distance l to be given by

$$\theta = (2\pi\epsilon_{\text{eff}}^{\frac{1}{2}}/c)l\zeta,$$

provided that ξ_x, ξ_y, and (ζ/ω) are all small compared to unity, and where $\epsilon_{\text{eff}} = \frac{1}{2}(|\epsilon| + \epsilon_1)$ if one assumes the dielectric constant to be a complex scalar. In an appropriate set of units, therefore, the rotation per unit length of path is simply given by

$$\theta = \zeta. \tag{2}$$

In the common case in which the applied frequency ω is sufficiently large compared to the ferromagnetic resonance frequency, it is easily shown[4] that

$$\zeta = -\gamma M, \tag{3}$$

where γ is the gyromagnetic ratio of the ferromagnetic electrons and M is the equilibrium component of the magnetization in the direction of the constant field. A principal reason for the simplicity of the results given by (2) and (3) is that damping terms which affect the response of the magnetization to the oscillating field have been neglected; in general, if damping is included, then ζ, ξ_x, and ξ_y all become complex, as has been shown by Hogan,[3] with a resulting complication in the expression for the rotation. For our purposes, the explicit inclusion of damping terms in the equations of motion is not necessary, and for simplicity will be neglected, although it will occasionally become convenient to include damping in an implicit manner as is often done in the elementary theories of optical dispersion.

Experimental results[3] obtained by using various ferrites are in good agreement with the predictions given by (3). This can be the more readily understood since it has been shown[5] that the resonance frequency of ferrimagnetics can, in most cases, be quite accurately taken to be of the same form as that given by the standard theory of ferromagnetic resonance. It is necessary, however, that the gyromagnetic ratio of a single electron, γ, be replaced by the effective gyromagnetic ratio, γ_{eff}, defined as the ratio of the total magnetic moment to the total angular momentum of the specimen. In the simplest case in which the ferrimagnetic material can be adequately described as being comprised of two magnetic sublattices, we have

$$\gamma_{\text{eff}} = M/S = (M_1 + M_2)/[(M_1/\gamma_1) + (M_2/\gamma_2)]. \tag{4}$$

As a result, it is reasonable to expect that the expression (3) for the Faraday rotation should actually be written for ferrimagnetics as

$$\zeta = -\gamma_{\text{eff}} M, \tag{5}$$

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[1] D. Polder, Phil. Mag. 40, 99 (1949).

[2] F. F. Roberts, J. phys. et radium 12, 305 (1951).

[3] C. L. Hogan, Bell System Tech. J. 31, 1 (1952); Revs. Modern Phys. 25, 253 (1953).

[4] G. T. Rado, Phys. Rev. 89, 529 (1953).

[5] R. K. Wangsness, Phys. Rev. 91, 1085 (1953); 93, 68 (1954). Hereinafter, these papers will be denoted by I and II, respectively, and numbers in following parentheses will refer to the corresponding equation of the paper.

which reduces to (3) in the ferromagnetic case in which one assumes the sublattice gyromagnetic ratios to be equal.

Ordinarily, this change can be expected to have no great effect upon the observed rotation since the γ_i of the sublattices are of the same magnitude, and also the magnetization of one sublattice is dominant over the complete range of temperature or composition considered. An interesting situation arises, however, when the ferrimagnetic in question possesses a compensation point for magnetization, that is, a temperature or composition at which the magnetizations of the two sublattices are equal and oppositely directed. This means that the dominant magnetization changes from one sublattice to the other as the compensation point is passed through and the net magnetization will change sign provided that any constant external magnetic field is sufficiently small. Both of these possibilities have been realized experimentally.[6,7] Further, one can expect the γ_i to be different so that there will also exist a compensation point for angular momentum which, in general, will not coincide with that for magnetization. Thus, in the range between these two points, M and S will be parallel rather than antiparallel as normally occurs, and we see from (4) that the effective gyromagnetic ratio of the material will have a sign opposite to that of the magnetic electrons themselves. Such an effect would not be observed by the usual absorption type resonance experiments, but (5) indicates that it should be detectable by a change in sign of the Faraday rotation.

We shall now illustrate the effect of these qualitative considerations by considering a definite example. In Fig. 1 are shown values of magnetization and γ_{eff} which are similar to those which have been observed as a function of temperature for $\text{Li}_{0.5}\text{Fe}_{1.25}\text{Cr}_{1.25}\text{O}_4$.[6,8] The temperatures m and s are the compensation temperatures for magnetization and angular momentum, respectively; the magnetization is taken to be positive for all temperatures since the constant field which is used in the Faraday effect is assumed to be large enough to ensure that the net magnetization will always be aligned parallel to this field. The negative sign of γ_{eff} makes ζ positive in the normal regions and a reversal in sign is then inferred from (4) for the temperature range between m and s although it is known[5] that such a definition of γ_{eff} in this range is not exact. Then, according to (5), the rotation can be obtained by a simple multiplication of the curves; the resulting curve is also shown in the figure. We note that (4) and (5) together would make the rotation infinite and discontinuous at s; actually, of course, this cannot be the case, and we should expect that the qualitative effect of any real damping would be to join these branches in a smooth manner (as in the case of optical dispersion) with the result that the actual rotation at s would be zero as shown. The general trend is clear, however, and we see that a consequence of the concept of effective gyromagnetic ratios is that as the temperature is varied the observed rotation should change sign near m and again near s.

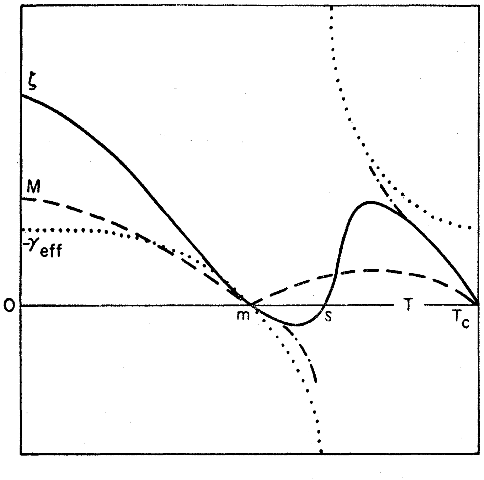

FIG. 1. Qualitative dependence of the magnetization, effective gyromagnetic ratio, and Faraday rotation upon temperature. m and s are compensation points for magnetization and angular momentum, respectively.

In the following sections, we shall justify these simple considerations by a more rigorous discussion. First, we shall calculate the exact expressions for the components of the susceptibility tensor for a two-sublattice system, and then consider the expressions obtained for the rotation in the special case in which the material apparently behaves in resonance experiments as a true ferromagnetic. After this we shall consider in some detail the effects that the existence of compensation points would have on the curve shown in Fig. 1. The rotation to be expected for an antiferromagnet will also be obtained from the results for the general two-sublattice case. Finally, we shall generalize our results as far as possible for the case in which we are dealing with an arbitrary number of sublattices.

When it is necessary to discuss a specific example, we shall restrict ourselves to a material exemplified by the temperature dependence shown in Fig. 1, although the discussion can be easily adapted to other cases, such as may occur as a function of composition.

THE SUSCEPTIBILITY TENSOR

We shall use the notation of I and II, except that the resonance frequencies discussed there will now be denoted by ω_0, so that ω represents the frequency of the external field.

If H_x and H_y are assumed to be proportional to $e^{i\omega t}$, then we assume that M_x and M_y also vary as $e^{i\omega t}$, so

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[6] E. W. Gorter and J. A. Schulkes, Phys. Rev. **90**, 487 (1953).

[7] L. R. Maxwell and S. J. Pickart, Phys. Rev. **92**, 1120 (1953).

[8] J. S. van Wieringen, Phys. Rev. **90**, 488 (1953).

that the equations of motion become

$$
\begin{aligned}
&i\omega M_{1x}-\tfrac{1}{2}(A_y+B_y+C_y+D_y)M_{1y}\\
&\qquad +\tfrac{1}{2}(A_y-B_y+C_y-D_y)M_{2y}=-\gamma_1 M_1 H_y,\\
&i\omega M_{2x}+\tfrac{1}{2}(A_y-B_y-C_y+D_y)M_{1y}\\
&\qquad -\tfrac{1}{2}(A_y+B_y-C_y-D_y)M_{2y}=-\gamma_2 M_2 H_y,\\
&i\omega M_{1y}+\tfrac{1}{2}(A_x+B_x+C_x+D_x)M_{1x}\\
&\qquad -\tfrac{1}{2}(A_x-B_x+C_x-D_x)M_{2x}=\gamma_1 M_1 H_x,\\
&i\omega M_{2y}-\tfrac{1}{2}(A_x-B_x-C_x+D_x)M_{1x}\\
&\qquad +\tfrac{1}{2}(A_x+B_x-C_x-D_x)M_{2x}=\gamma_2 M_2 H_x.
\end{aligned}
$$

If we introduce the new variables

$$M_j=M_{1j}+M_{2j},\quad \Delta_j=M_{1j}-M_{2j}, \tag{6}$$

where $j=x,y$ and the notation

$$\gamma_\pm=\gamma_1 M_1\pm\gamma_2 M_2, \tag{7}$$

the equations become

$$
\left.
\begin{aligned}
i\omega M_x-B_yM_y-C_y\Delta_y&=-\gamma_+H_y,\\
i\omega\Delta_x-D_yM_y-A_y\Delta_y&=-\gamma_-H_y,\\
B_xM_x+C_x\Delta_x+i\omega M_y&=\gamma_+H_x,\\
D_xM_x+A_x\Delta_x+i\omega\Delta_y&=\gamma_-H_x.
\end{aligned}
\right\} \tag{8}
$$

When we solve Eqs. (8) for M_x and M_y, we find them to be of the form given in (1), where

$$
\left.
\begin{aligned}
\xi_x&=\mathfrak{D}^{-1}(\mathfrak{L}_x\mathfrak{R}_y-\mathfrak{K}_y\omega^2),\\
\xi_y&=\mathfrak{D}^{-1}(\mathfrak{L}_y\mathfrak{R}_x-\mathfrak{K}_x\omega^2),
\end{aligned}
\right\} \tag{9}
$$

$$\zeta=-(\omega^2/\mathfrak{D})(\gamma_+\omega^2-\mathfrak{P}), \tag{10}$$

and where

$$
\begin{aligned}
\mathfrak{P}&=A_x(\gamma_+A_y-\gamma_-C_y)+C_x(\gamma_+D_y-\gamma_-B_y)\\
&=A_y(\gamma_+A_x-\gamma_-C_x)+C_y(\gamma_+D_x-\gamma_-B_x)\\
&=S(\gamma_1\gamma_2P)^2+\gamma_1\gamma_2M_1M_2\{P[\gamma_1(n_{1x}+n_{1y})\\
&\qquad+\gamma_2(n_{2x}+n_{2y})]+\gamma_1M_1n_{1x}n_{1y}+\gamma_2M_2n_{2x}n_{2y}\},
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
\mathfrak{K}_j=\gamma_+B_j+\gamma_-C_j&=(\lambda-N_j)(\gamma_1-\gamma_2)^2M_1M_2\\
&\quad+(\gamma_1^2M_1+\gamma_2^2M_2)Q_j+\gamma_1^2M_1^2n_{1j}+\gamma_2^2M_2^2n_{2j},
\end{aligned} \tag{12}
$$

$$\mathfrak{L}_j=\gamma_+A_j-\gamma_-C_j=\gamma_1\gamma_2[PM+M_1M_2(n_{1j}+n_{2j})], \tag{13}$$

$$
\begin{aligned}
\mathfrak{R}_j=A_jB_j-C_jD_j&=\gamma_1\gamma_2\{(\lambda-N_j)(M_1^2n_{1j}+M_2^2n_{2j})\\
&\quad+Q_j(P+M_1n_{1j}+M_2n_{2j})+M_1M_2n_{1j}n_{2j}\},
\end{aligned} \tag{14}
$$

and $\mathfrak{D}$, the determinant of the coefficients of the homogeneous equations corresponding to (8), is given by

$$\mathfrak{D}=(\omega^2-\omega_0^2)(\omega^2+\omega_0^2-2F). \tag{15}$$

If desired, Eqs. (11)–(14) can be written in terms of the macroscopic effective demagnetizing factors due to anisotropy by using II(9), but for many of our purposes it will be sufficient simply to know the components of the susceptibility tensor when anisotropy is neglected. Thus, if we set $n_{ij}=0$, we find from (9) and (10) that

$$\zeta=-(\omega^2/\mathfrak{D})[(\gamma_1M_1+\gamma_2M_2)\omega^2-S(\gamma_1\gamma_2P)^2], \tag{16}$$

and

$$
\begin{aligned}
\xi_j=\mathfrak{D}^{-1}\{(\gamma_1\gamma_2)^2P^2MQ_j-\omega^2[(\lambda-N_j)(\gamma_1-\gamma_2)^2M_1M_2\\
+Q_j(\gamma_1^2M_1+\gamma_2^2M_2)]\}.
\end{aligned} \tag{17}
$$

THE APPARENT FERROMAGNETIC CASE

It was shown in II that when the product of the molecular field coefficient and the net magnetization is large compared to the components of the applied and anisotropy fields the resonance frequency has essentially the same form as that given by the theory of ferromagnetic resonance; namely, $\omega_0^2=\gamma_{\text{eff}}^2Q_xQ_y$. In this approximation, $P=\lambda M$ and $2F=(\lambda\gamma_1\gamma_2S)^2$ according to II(2). Generally we will also have $|2F|\gg\omega^2$, ω_0^2 so that

$$\mathfrak{D}\simeq(\omega^2-\omega_0^2)(-2F),$$

and then we find that (16) and (17) become

$$\zeta_f=-\omega^2\gamma_{\text{eff}}M/(\omega^2-\gamma_{\text{eff}}^2Q_xQ_y), \tag{18}$$

$$\xi_{jf}=-\gamma_{\text{eff}}^2MQ_j/(\omega^2-\gamma_{\text{eff}}^2Q_xQ_y). \tag{19}$$

In most of the experiments which have been performed on the Faraday effect in ferrites, the external field has been kept small enough so that ω is large compared to the ferromagnetic resonance frequency. In addition, it is usually a good approximation to assume that $|\gamma_{\text{eff}}M|\ll\omega$.[3,4] Under these conditions, Eq. (18) is seen to reduce to the usual expression given in (5) and that $|\xi_{jf}|\ll 1$ as expected. Thus, the results of this section verify our previous qualitative derivation of the formula (5) which has been amply tested by experiment. Equation (5) does differ from the usual formula in that it involves γ_{eff} which, however, under these conditions cannot be expected to differ much from the gyromagnetic ratio of the sublattice constituents and hence will be close to that of a free electron. Equation (18) also predicts the change in sign to be expected of the rotation as the field is increased to make the resonant frequency finally larger than ω; this effect is familiar from the study of the paramagnetic Faraday effect and has been experimentally observed in ferrites.[9]

EFFECT OF A COMPENSATION POINT

It is quite difficult to discuss the dependence of (16) upon temperature both exactly and in very general terms because no quantitative description of the temperature variation of the individual sublattice magnetizations seems to exist. What we shall do instead is to extrapolate our previous results as well as possible through the region of the compensation points and augment this by more precise discussions at the compensation points where exact calculations are more easily done.

I. Qualitative Discussion

The usual experimental procedure is to represent ω_0 as the product of the appropriate function of the ob-

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[9] Second article in reference 3.

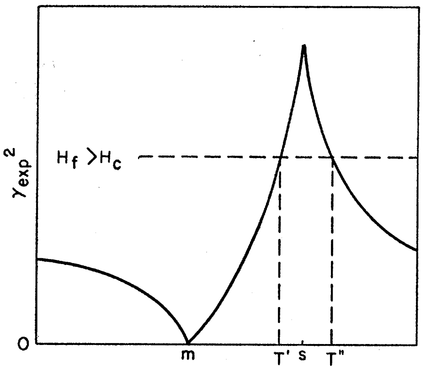

FIG. 2. Dependence of squared experimental gyromagnetic ratio upon temperature. Dashed lines locate two additional temperatures at which rotation will vanish for large enough external field.

served field needed for resonance and an experimental gyromagnetic ratio which we shall write as γ_{exp}; thus, $\omega_0^2=\gamma_{\mathrm{exp}}^2 Q_x Q_y$. In contrast to the prediction of infinity given by (4) at the point where $S=0$, the limited experimental data and the theoretical results given by I(22) and II(18–19) indicate that γ_{exp}^2 is large but finite at the compensation point for angular momentum. The general behavior to be expected of γ_{exp}^2 is shown in Fig. 2.

It is, therefore, probably not too unreasonable to expect that a fairly accurate representation of the rotation as a function of temperature can be given by the following generalization of (18),

$$\zeta_q=-\omega^2\gamma_{\mathrm{exp}}M/(\omega^2-\gamma_{\mathrm{exp}}^2 H_f^2), \tag{20}$$

where we have put $H_f^2=Q_xQ_y$.

There are now essentially two cases to be considered corresponding to whether or not the constant field is large enough to make H_f greater than a critical field H_c which is determined by the maximum value of γ_{exp} and is defined by the equation

$$H_c^2=\omega^2/\gamma_{\mathrm{exp\ max}}^2. \tag{21}$$

Using (21), we can then write (20) as

$$\zeta_q=-\gamma_{\mathrm{exp}}M/[1-(H_f\gamma_{\mathrm{exp}}/H_c\gamma_{\mathrm{exp\ max}})^2]. \tag{22}$$

Equation (22) indicates in general that there will be an infinite discontinuity in ζ_q when $\gamma_{\mathrm{exp}}^2=(H_c/H_f)^2\gamma_{\mathrm{exp\ max}}^2$. Accordingly, when damping is taken into account, we would actually expect the rotation to pass through zero at this point.

Now when $H_f<H_c$, the denominator in (22) will never vanish, and if we assume that the sign of γ_{exp} has the same temperature dependence as shown in Fig. 1 for γ_{eff}, then the rotation predicted by (22) is as shown by the solid curve in Fig. 3. This, of course, is similar to that found in Fig. 1.

On the other hand, when $H_f>H_c$, there are two temperatures T' and T'' at which we can expect the denominator of (22) to vanish. The method of finding T' and T'' is illustrated in Fig. 2. This result means that ζ_q will change sign a total of four times: at m, T', s, and T''. This leads to a temperature dependence of ζ_q similar to that shown by the solid curve in Fig. 4. As H_f decreases to H_c, the points T' and T'' approach each other and eventually coincide so that there will be a smooth transition from the curve of Fig. 4 to that of Fig. 3.

II. $M=0$

With $M_2=-M_1$, we know that, with the neglect of anisotropy, $\omega_0^2=0$, and $-2F\simeq-[(\gamma_1-\gamma_2)\lambda M_1]^2\gg\omega^2$ from II(14) and II(12), so that (16) becomes

$$\zeta_m=\frac{-(\gamma_1-\gamma_2)M_1(\omega^2+\gamma_1\gamma_2H^2)}{\{\omega^2-[(\gamma_1-\gamma_2)\lambda M_1]^2\}}. \tag{23}$$

Since in most cases the applied field is small and the molecular field ($|\lambda M_1|$) large compared to the field needed for resonance at the applied frequency, Eq. (23) can generally be approximated further as

$$\zeta_m=-\omega^2/[\lambda^2(\gamma_2-\gamma_1)M_1]. \tag{24}$$

III. $S=0$

For this case, it simplifies matters to assume that we are dealing with the usual experimental case in which the sample has axial symmetry about the direction of the constant field so that $Q_x=Q_y=Q$. Then we find from II(16, 18, 19) that

$$\omega_0^2=-\lambda\gamma_2(\gamma_1-\gamma_2)M_1Q=F$$

and therefore $\mathfrak{D}=(\omega^2-F)^2$. Equation (16) is then found to reduce to

$$\zeta_s=\frac{-(\gamma_1^2-\gamma_2^2)\omega^4M_1}{\gamma_1[\omega^2+\lambda\gamma_2(\gamma_1-\gamma_2)M_1Q]^2}. \tag{25}$$

The approximate form of (25) corresponding to (24) is

$$\zeta_s=\frac{-(\gamma_1+\gamma_2)\omega^4}{\lambda^2\gamma_1\gamma_2^2(\gamma_1-\gamma_2)M_1Q^2}. \tag{26}$$

IV. The General Aspect of the Rotation

Using the results of the last two sections, we can now get an estimate of the relative signs and magnitudes

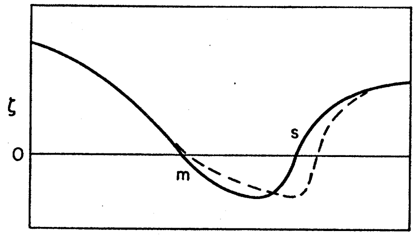

FIG. 3. Qualitative plot of rotation *vs* temperature for small applied field. Dashed line shows the effect of the more exact calculations for the compensation points upon the shape of the curve.

of ζ at the two compensation points. It will be sufficient for our purposes to use (24) and (26). We find that

$$\zeta_s/\zeta_m = -(\gamma_1+\gamma_2)\omega^2/(\gamma_1\gamma_2^2Q^2). \qquad (27)$$

It is immediately evident from (27) that ζ has opposite signs at these two points so that the rotation has changed sign at least once between m and s. A satisfactory estimate of the relative magnitudes is obtained by taking $\gamma_1 \simeq \gamma_2 \simeq \gamma$, so that

$$|\zeta_s/\zeta_m| = (\omega/\gamma Q)^2;$$

however, we shall continue to write

$$Q = H + (N-N_z)M = H + (N-N_z)(\gamma_1-\gamma_2)(M_1/\gamma_1).$$

If we substitute as fairly representative the values $H=500$, $N-N_z=5$, $(\gamma_1-\gamma_2)/\gamma_1=0.1$, $M_1=600$, and $\omega/\gamma=3000$, we find a value for the ratio of about 14.

We have yet to determine the magnitude and sign of, say, ζ_m, relative to the normal value ζ_f given in (18). If we assume that the applied frequency is large compared to the ferromagnetic resonance frequency we find from (18) and (24) that

$$\zeta_m/\zeta_f = \omega^2/[(\gamma_2-\gamma_1)\gamma_{\text{eff}}\lambda^2 M_1 M]. \qquad (28)$$

In order to make further use of (28), we must choose which of the sublattice magnetizations is to be regarded as being dominant at a given temperature, although it is easily seen that our final results are independent of our particular choice. We shall assume that M_1 is the dominant magnetization from $T=0$ to the compensation point for magnetization. In other words, for $T<m$, we assume that $|M_1|>|M_2|$, and for $T>m$, that $|M_1|<|M_2|$. Now the condition for $S=0$ is that $|M_2/M_1|=|\gamma_2/\gamma_1|$. If $S=0$ is to occur at a temperature below that for which $M=0$, then we see that $|\gamma_2|<|\gamma_1|$. On the other hand, if $S=0$ is to occur at a temperature above that for which $M=0$, then we must have $|\gamma_2|>|\gamma_1|$. Since it is this latter case which corresponds to the situation pictured in Fig. 1, we see that we have $(\gamma_2-\gamma_1)\gamma_{\text{eff}}>0$, so that ζ_m and ζ_f always have the same sign according to (28).

In order to estimate the relative magnitudes of ζ_m and ζ_f, we shall assume that $\gamma_{\text{eff}} \simeq \gamma$ and $M \simeq \frac{1}{2}M_1$ in addition to the numerical estimates used in the last paragraphs. Then we find that

$$\zeta_m/\zeta_f \simeq 20(\omega/\gamma H_E)^2 \simeq 0.02,$$

where we have put $-\lambda M_1 = H_E \simeq 10^4$ oersted.

In summary, we have found that ζ_m is very small although not yet zero and shows no sign change, while ζ_s is of the same order of magnitude as the usual rotation but is always of the opposite sign. We are now able to redraw Figs. 3 and 4 somewhat more accurately. The first crossover near m will occur at a slightly higher temperature while the crossovers shown at s in Figs. 3 and 4 should occur at somewhat higher and lower temperatures, respectively. Hence, the general aspect of the rotation for the two cases should be more like that shown by the dashed lines in Figs. 3 and 4. The curves are meant to be purely qualitative and there is no necessary quantitative significance in the values of the relative maxima and minima.

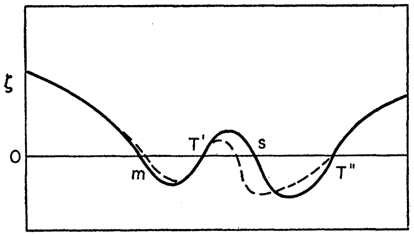

FIG. 4. Qualitative plot of rotation *vs* temperature for large applied field. Dashed lines shows the effect of the more exact calculations for the compensation points upon the shape of the curve.

As yet, there are no substantial experimental data which can be compared with these predictions, although preliminary results obtained on a sample of $Li_{0.5}Fe_{1.25}Cr_{1.25}O_4$ by Brown and Hebbert of this Laboratory[10] apparently show a behavior similar to that of Fig. 3. Some of the difficulties encountered in measuring the rotation of waves propagated in a wave guide rather than as plane waves and which complicate the interpretation of the results have been discussed by Hogan.[9]

THE ANTIFERROMAGNETIC CASE

It is easy to adapt our general results to the case in which the material is antiferromagnetic; however, if we were simply to take $\gamma_1=\gamma_2$ and $M=0$, the rotation would be zero by (23). Actually, we must now take anisotropy into account since it is known to have so important a role in antiferromagnetic resonance. We shall assume, as in II, that $\gamma_i=\gamma$ and $n_{ij}=n_j$. According to II(9), this last assumption not only implies equal anisotropy constants but equal values of M_i^2, i.e., $M=0$. As a result, we are omitting corrections of higher order in H since M is not exactly zero when $H=0$; this will be seen from our results below which will still give a value of ζ proportional to H in spite of our assumption that $M=0$.

With these assumptions, we easily find from (10) that

$$\zeta_a = -2\omega^2\gamma^3 M_1^2(n_x+n_y)H/\mathfrak{D}, \qquad (29)$$

where $\mathfrak{D}$ can be evaluated by the use of II(20) and the preceding equation for F.

For convenience, we shall simplify this equation somewhat by introducing the anisotropy and exchange fields in the usual way by setting $n_j=H_a/M_1$, and $-\lambda M_1=H_E$; we shall also assume that $H_E \gg H$.

If we now keep only the largest terms in II(20) and preceding equations, we find as usual that

$$\omega_0^2 = 2\gamma^2 H_E H_a = F,$$

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[10] R. W. Brown and R. S. Hebbert (private communication).

so that $\mathfrak{D}=(\omega^2-\omega_0{}^2)^2\simeq\omega_0{}^4$. Substituting these results into (29) we get

$$\zeta_a=-\omega^2HM_1/(\gamma H_aH_E{}^2). \tag{30}$$

It is interesting to compare the magnitude of this expression with that for the ferromagnetic region given by (18). We find that

$$\zeta_a/\zeta_f=\omega^2HM_1/(\gamma\gamma_{\text{eff}}H_aH_E{}^2M)\approx10^{-3}$$

if we continue to use the same numerical approximations as before, and, in addition, take $H_a\approx10^3$. The expected value for the rotation in the antiferromagnetic case is thus much smaller than that for the ferromagnetic case. On the other hand, if we compare (30) with the value to be expected in the paramagnetic region above the Curie temperature where we can write $\zeta_p=-\gamma M=-\gamma\chi H$, we find that

$$\zeta_a/\zeta_p=\omega^2M_1/(\gamma^2\chi H_aH_E{}^2)\approx1$$

if we assume the value of the susceptibility to be about 6×10^{-4}. Since these last two quantities are of the same order of magnitude, one would not expect to observe an abrupt change in the rotation as the material is cooled below the Curie temperature in contrast to what is observed for the absorption.[11]

This interesting result that one could expect to find a Faraday rotation for an antiferromagnet even if $M=0$ is another case which illustrates how important is the anisotropy for the determination of the electrodynamic properties of the antiferromagnetic case. This is most easily illustrated by considering the explicit form taken by the general Eqs. (8) in this case. They are easily found to be

$$\begin{aligned}&i\bar{\omega}M_x-HM_y-H_a\Delta_y=0,\\&i\bar{\omega}\Delta_x-2H_EM_y-H\Delta_y=-2M_1H_y,\\&HM_x+H_a\Delta_x+i\bar{\omega}M_y=0,\\&2H_EM_x+H\Delta_x+i\bar{\omega}\Delta_y=2M_1H_x,\end{aligned}$$

where $\bar{\omega}=\omega/\gamma$ and where we have written $H_a+2H_E\simeq2H_E$. Upon inspection of these equations, we see that the existence of a nonzero H_a is the only thing that keeps us from treating M_x and M_y independently from Δ_x and Δ_y, and thus serves as the necessary coupling which brings the exchange field into the resonance frequency and also makes the transverse components of the magnetization depend on the oscillating components of the applied field. This is vividly illustrated when one calculates ξ and ζ and finds them both proportional to H_a, while the exchange field enters in only through the determinant of the coefficients.

AN ARBITRARY NUMBER OF SUBLATTICES

The calculation of detailed results equivalent to those of the last sections for the case in which we are dealing with an arbitrary number of sublattices would seem to be too complex to be possible. Instead, we shall show only that the simplest results as given in (18) and (19) are probably very accurately also true in this case.

For simplicity, we shall not include demagnetizing and anisotropy fields so that the equation of motion for the ith sublattice becomes

$$d\mathbf{M}_i/dt=\gamma_i\mathbf{M}_i\times(\mathbf{H}+\textstyle\sum_k\lambda_{ik}\mathbf{M}_k),$$

where the λ_{ik} are molecular field coefficients satisfying the relation $\lambda_{ik}=\lambda_{ki}$. Summing this equation over all sublattices, we find that

$$d[\textstyle\sum_i(\mathbf{M}_i/\gamma_i)]/dt=(\sum_i\mathbf{M}_i)\times\mathbf{H}, \tag{31}$$

since $\sum_{ik}\lambda_{ik}\mathbf{M}_i\times\mathbf{M}_k=0$. If we let $M=\sum_iM_{iz}=\sum_iM_i$, $M_j=\sum_iM_{ij}$, and assume as usual that M_j and H_j are small and proportional to $e^{i\omega t}$, we find from (31) that

$$\left.\begin{aligned}i\omega\textstyle\sum_i(M_{ix}/\gamma_i)&=-MH_y+HM_y,\\ i\omega\textstyle\sum_i(M_{iy}/\gamma_i)&=MH_x-HM_x.\end{aligned}\right\} \tag{32}$$

For small deviations from the z direction we can expect M_{ij} to be proportional to M_i, and, in the limit of strong molecular fields, the constants of proportionally ϑ_i will be the same for all sublattices.[12] Then we can write

$$\textstyle\sum_i(M_{ij}/\gamma_i)=\vartheta\sum_i(M_i/\gamma_i)=(\sum_iM_{ij})/\gamma_{\text{eff}},$$

so that Eqs. (32) become

$$\begin{aligned}&i\omega M_x=-\gamma_{\text{eff}}MH_y+\omega_0M_y,\\&i\omega M_y=\gamma_{\text{eff}}MH_x-\omega_0M_x,\end{aligned}$$

where $\omega_0=\gamma_{\text{eff}}H$.

When these last equations are solved for M_x and M_y, we find the results to have the form (1) where

$$\zeta=-\omega^2\gamma_{\text{eff}}M/(\omega^2-\gamma_{\text{eff}}{}^2H^2),$$

and

$$\xi=-\omega_0\gamma_{\text{eff}}M/(\omega^2-\omega_0{}^2),$$

which essentially agree with (18) and (19).

DISCUSSION

The results of our calculations in the previous sections have been seen to support in general the qualitative discussion given in the introduction. It would be possible, of course, to obtain better approximations to our very general results, but since most of the effects which have been predicted have not yet been observed experimentally, this is not really necessary at present.

On the other hand, a thorough discussion of ferrimagnetic resonance and allied effects requires for completeness that damping be included in as simple and accurate a manner as possible. When this is done, a possible sublattice effect to be considered is that the relaxation times for the various sublattices are very likely unequal because of the different nature of their

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[11] Trounson, Bleil, Wangsness, and Maxwell, Phys. Rev. **79**, 542 (1950); L. R. Maxwell and T. R. McGuire, Revs. Modern Phys. **25**, 279 (1953).

[12] J. Smit and H. P. J. Wijn, *Advances in Electronics* **VI** (to be published).

surroundings. These probably unlike quantitative interactions between the ions and the crystalline lattice as well as among the magnetic ions themselves may then result in different approaches to thermal equilibrium. In this connection one may be able to devise a method of measuring the relaxation times of the individual sublattices; a possible approach may consist of a careful study of line widths and saturation properties of a series of compounds of varying composition in which one alters the constituents of the sublattices in a known and regular manner.

The apparent inertial mass of a moving ferromagnetic domain wall, which was first discussed by Döring,[13] is known to be a consequence of the connection between the electronic angular momentum and magnetic moment and arises from the precessional motion of the spins in the internal demagnetizing fields of a moving wall. In fact, the apparent mass per unit area depends on γ and it is tempting to think that in a ferrimagnetic γ_{eff} should be involved instead. If this were actually the case, then it may be observable as a dependence of the natural resonance frequency due to wall motion upon the temperature or composition.

I wish to thank Dr. T. R. McGuire and Dr. G. T. Rado for some helpful discussions, and Mr. R. W. Brown and Mr. R. S. Hebbert for permission to mention their preliminary experimental work.

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[13] W. Döring, Z. Naturforsch. **3a**, 374 (1948); G. T. Rado, Phys. Rev. **83**, 821 (1951).

PHYSICAL REVIEW VOLUME 95, NUMBER 2 JULY 15, 1954

Backscattering of Kilovolt Electrons from Solids*

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(Received November 13, 1953)

The total number and energy distribution of backscattered electrons at 0.2–4 kev incident energy (V) have been measured for six elements using electrostatic retarding potential techniques. For atomic number $Z \lesssim 30$, backscattering was found to be essentially independent of V and almost linearly dependent on Z. For $Z \gtrsim 30$, backscattering decreases with decreasing V below 2–3 kev to values less than those for elements of $Z \simeq 30$, and it no longer is a simple function of Z. The ratio of the mean energy of the backscattered electrons to that of the primaries is found to be close to one-half for $Z=6$ and to increase only slightly for the heavier elements. These results are shown to indicate a dominant role of inelastic processes in the scattering of intermediate energy electrons, in contrast to scattering at very high energies, where elastic interactions control the phenomenon.

I. INTRODUCTION

PRESENT experimental and theoretical knowledge of the interaction of intermediate energy electrons with complex atoms is still very limited, in contrast to the situation at energies very large or very small relative to atomic binding energies. Measurements of the total number and energy distribution of electrons backscattered from solids such as reported on below offer a means of shedding new light on the relative importance of elastic and inelastic processes in the intermediate energy region. Such information is of importance in the formulation of a theory of backscattering more complete than existing theoretical treatments.[1] Aside from its practical interest, data on the backscattering process is also of value in the development of theories of bremsstrahlung production, cathode-luminescence, secondary emission, and bombardment-induced conductivity, where information on the primary beam spreading and total energy loss in the material is required. In particular, knowledge of the number of backscattered electrons at low energies would for the first time allow a separation of the true, low-energy secondary electrons from the total yield of emitted electrons for comparison with theory.[2]

At present, reliable data on the number of backscattered electrons[3] exist only for a few elements below

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* This paper constitutes part of a thesis submitted in partial fulfillment of the requirements for the Ph.D. degree at Cornell University. Portions of the work were carried out at the U. S. Naval Ordnance Laboratory, White Oak, Maryland, and at the Westinghouse Research Laboratories, East Pittsburgh, Pennsylvania. A report on preliminary work was presented at the November, 1950, meeting of the Electron Physics Division of the American Physical Society, by Sternglass, Frey, and Grannis [Phys. Rev. **85**, 391 (1952)].

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[1] W. Bothe, Ann. Physik **6**, 44 (1949); see also article in *Handbuch der Physik* (Springer, Berlin, 1933), Vol. 24, part 2, p.1 for summary of earlier work.

[2] See the review article on Secondary Emission by K. G. McKay, *Advances in Electronics* (Academic Press, New York, 1948), Vol. 1.

[3] The term "backscattered electrons" as used here refers to all electrons emitted from a bombarded target whose energy is greater than 50 ev in accordance with the usual convention. This definition is designed principally to eliminate the true, low-energy secondary electrons, more than 95 percent of which have energies less than 30 ev for all solids and which will be referred to simply as "secondary electrons." It must be remembered that, because of the indistinguishability of electrons, the so-called "backscattered electrons" defined in this manner include some high-energy secondaries formed in close electron-electron collisions.