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Nuclear Scattering of High-Energy Electrons*

J. H. SMITH
Knolls Atomic Power Laboratory,† Schenectady, New York
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The Born approximation is used to calculate matrix elements for the elastic and inelastic scattering of high-energy electrons by nuclei. Curves and simple numerical relations are given to describe the manner in which scattered amplitudes depend upon nuclear characteristics.

INTRODUCTION

THE highly energetic beams of electrons now available from accelerators offer promise in the examination of nuclear structure. Indeed, Hofstadter, Fechter, and McIntyre[1] have investigated the scattering of electrons of about 100 Mev by various nuclei and have obtained not only a very accurate value for the nuclear radius but also more detailed information about the charge distribution in the nucleus, especially near its surface. This is to be expected since $\lambda\!\!\!^{-}$ of these electrons is about 2×10^{-13}, just about the right order for the exploration of nuclear structure.

For an adequate analysis of these experiments, extensive numerical calculations including exact solution of the Dirac equation are necessary. Such calculations have been made by Yennie, Ravenhall, and Wilson[2] with very interesting results. However, these calculations require much time and expensive high-speed computing machines, and it will be difficult to explore in this manner the influence on the scattering of all parameters in the charge distribution.

It was therefore felt that a thorough investigation by means of the Born approximation would still have value. We fully realize that the results of this paper can be expected to be directly valid only for very light nuclei. There is some hope however that for heavy nuclei the Born approximation might give at least a *part* of the scattered amplitude: Baranger[3] has shown that to a certain approximation, the scattered amplitude may be represented as the Born approximation amplitude plus a correction depending primarily on $Ze^2/\hbar c$. Some aspects of the Born approximation have been discussed by Schiff[4] after this thesis was written but before its publication.

It is shown that analysis of the cross section for elastic scattering can provide information on nuclear radii and charge distribution. The latter is described herein by two parameters, one directly related to possible changes in proton population from nuclear center to "edge," and the second related to the sharpness of definition of a nuclear boundary.

The differential cross section for inelastic scattering summed over nuclear energy levels, is found to depend on the relative location of pairs of particles. Information on possible regularities in the internal "construction" of nuclei might be obtained from this quantity. Models chosen for the calculation of the inelastic effects include several kinds of crystal lattice patterns for proton pair distributions and a box in which nucleons are correlated due only to the action of the Pauli principle.

ELASTIC SCATTERING

In the Born approximation, the differential cross section for the elastic scattering of an unpolarized beam of electrons of energy E and charge e by a nucleus of charge Ze into the solid angle $d\Omega(=\sin\theta d\theta d\phi)$ is

$$d\sigma=\left(\frac{e^2\cos\frac{1}{2}\theta}{2E\sin^2\frac{1}{2}\theta}\right)^2 \times\left|\int\Phi_0{}^*\Phi_0\sum_{j=1}^{Z}\exp(i\mathbf{q}\cdot\mathbf{R}_j)d\mathbf{R}_1\cdots d\mathbf{R}_Z\right|^2 d\Omega, \quad (1)$$

where $|\mathbf{q}|=(2E/\hbar c)\sin\frac{1}{2}\theta$, $\mathbf{R}_j$ is the jth spatial coordinate in the nuclear system, and Φ_0 is the ground state wave function for the target nucleus. The electron-nucleus interaction has been taken to be Coulomb in form:

$$V=-\sum_{j=1}^{Z}\frac{e^2}{|\mathbf{R}_j-\mathbf{r}_e|}, \quad (2)$$

where $\mathbf{r}_e$ is the electronic coordinate.

The first factor in $d\sigma$ describes the Rutherford scattering by a single proton; its value is 10^{-30} cm^2 per unit solid angle for $E=100$ Mev and $\theta=90°$. The second factor gives interference effects amongst wavelets scattered by various protons. Were the probability distribution known for the nucleus, it would be possible to predict exactly the angular distribution of scattered particles. In lieu of this, there are hypothesized spatial nucleon configurations from which are calculated the corresponding scattered amplitudes.

$|\Phi_0|^2$ in the integral for $d\sigma$ contains all proton configuration coordinates symmetrically, so that the square

* Based on Ph.D. thesis submitted to Cornell University' February, 1951.

† Operated by the General Electric Company for the U. S. Atomic Energy Commission.

1 Hofstadter, Fechter, and McIntyre, Phys. Rev. 91, 422, (1953); 92, 978 (1953).

2 Yennie, Wilson, and Ravenhall, Phys. Rev. 92, 1325 (1953); Phys. Rev. (to be published).

3 E. U. Baranger, Cornell University thesis (1954). Part of her results were published in Phys. Rev. 93, 1127 (1954).

4 L. I. Schiff, Phys. Rev. 92, 988 (1953).

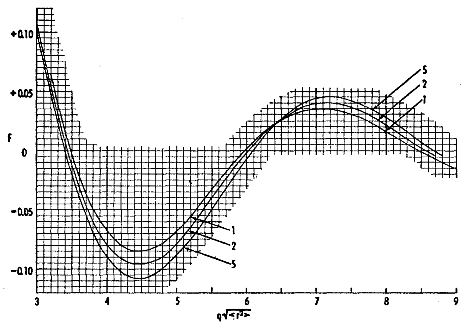

FIG. 1. Nuclear form factor for elastic scattering: models 1, 2, and 5.

of the matrix element is

$$Z^2\left|\int \phi_0{}^*\phi_0 \exp(i\mathbf{q}\cdot\mathbf{r})d\mathbf{r}\right|^2, \tag{3}$$

where

$$\phi_0{}^*\phi_0=\int \Phi_0{}^*\Phi_0\prod_{j=1}^{Z}{}' d\mathbf{R}_j,$$

i.e., $|\phi_0|^2 d\mathbf{r}$ is the probability that there is one proton in $d\mathbf{r}$, with the positions of all other protons arbitrary. It is convenient to compare the scattering by an extended nucleus with Z protons to that by a point charge of strength Ze. The relevant ratio is just

$$\left|\int \phi_0{}^*\phi_0 \exp(i\mathbf{q}\cdot\mathbf{r})d\mathbf{r}\right|^2, \tag{4}$$

the first power of which may quite naturally be referred to as a "nuclear form-factor" F.

Possible variation in charge density in the nucleus is allowed through use of the model described by

$$|\phi_0|^2=N^{-1}\begin{cases}1+f(r^2/a^2), & r\leq a,\\ (1+f)\exp[-(r-a)/da], & r>a,\end{cases} \tag{5}$$

where the normalization constant is

$$N=4\pi a^3\{\tfrac{1}{3}(1+\tfrac{3}{5}f)+d(1+f)(1+2d+2d^2)\}.$$

Of the parameters in $|\phi_0|^2$, f and d obviously determine the shape of the proton density, -- the average internal

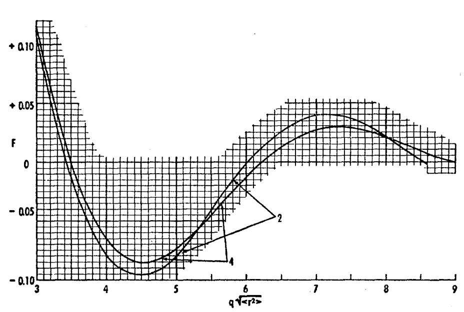

FIG. 2. Nuclear form factor for elastic scattering: models 2 and 4.

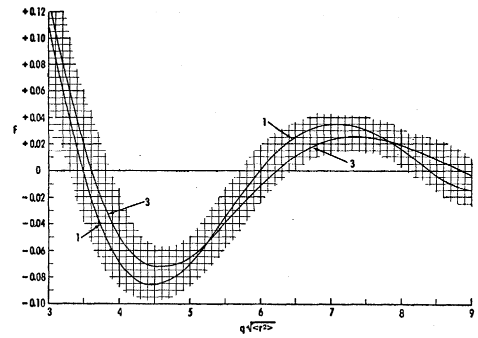

FIG. 3. Nuclear form factor for elastic scattering: models 1 and 3.

charge behavior, and the sharpness of a boundary. Even though adoption of a picture with spherical symmetry renders impossible any analysis of quadrupole moments, there nevertheless is ample opportunity for study of the two features mentioned.

The nuclear form factor, F, is examined against the dimensionless quantity $q\bar{r}\equiv q\sqrt{\langle r^2\rangle}$. $q\bar{r}$ is taken as more appropriate than qa since the latter places undue emphasis on a single aspect of the models used for the density.

The variations employed in "internal" and "external" charge distributions are given in Table I. The numbers in the left-hand column are model numbers to which reference will be made in later discussions, and f and d have already been mentioned. Two models calculated, 6 and 9, are written explicitly since no choice for f and d describes them.

For nearly all models employed, a characteristic diffraction pattern is found in the scattered beam, maxima and minima occurring in the product of the Rutherford scattering and the square of the form factor. With model 9, the exponential density which lies closest to the shape adopted by Stanford workers,[1,2,4] the calculated form factor displays no diffraction effects. F is plotted *versus* $q\bar{r}$ for a representative selection of these models in Figs. 1, 2, 3, and 4. Careful numerical analysis of F^2 *versus* $q\bar{r}$ shows that d, $\bar{r}$, and f affect the scattering in ways which are independent of one another

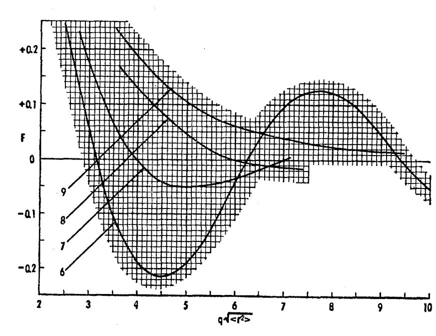

FIG. 4. Nuclear form factor for elastic scattering: models 6, 7, 8, and 9.

or coupled in a calculable way. These features of the scattering will now be discussed.

First, the ratio of the two values of F^2 taken at its first and second maxima depends significantly on d, but hardly at all on f. This is illustrated in Table II. The constraint that $dF/d(q\bar{r})=0$ serves to express F^2 at maxima as a function of d and f alone, so the variation of the above ratio with $q\bar{r}$ need not be considered. It is clear that a fuzzy boundary tends to decrease the second maximum relative to the first.

Secondly, the values of $q\bar{r}$ at which the first maximum in F^2 occurs may be found once an assignment of d is made. As mentioned, the condition $dF^2/d(q\bar{r})=0$ expresses $q\bar{r}$ (first maximum) as a function of f and d. However, it is seen from Table III that the variation of $q\bar{r}$ (first maximum) with f is of no importance.

Finally, there is the evaluation of the internal parameter f. For this, one can refer to the absolute value of F^2 at first maximum. A large value of f, implying a sharp peak in charge density near the nuclear surface, has the effect of increasing the form factor. (See the

TABLE I. Nuclear density models and their characteristic parameters.

Model number	f	d
1	0.0	0.0
2	0.4	0.0
3	0.0	0.1
4	0.4	0.1
5	1.0	0.0
6	$\lvert\phi_0\rvert^2=\delta(r-a)/4\pi a^2$	
7	0.0	0.2
8	0.0	0.5
9	$\lvert\phi_0\rvert^2=(b^3/8\pi)\exp(-br)$	
10	0.4	0.2
11	1.0	0.1
12	1.0	0.2

form factor in Fig. 4 for model 6, a simple shell of charge.) Empirical formulae which have been found to fit the characteristics of F^2 at extrema as just discussed are:

$$F^2(\text{first maximum})/F^2(\text{second maximum}) = 5.86+(18.6d)^{1.43}, \quad (6)$$

$$q\bar{r}(\text{first maximum})=4.48+(3.9d)^{2.6}, \quad (7)$$

$$F^2(\text{first maximum})=F_{00}^2\frac{\{1+[(1+d)^5+0.56]f\}^{\frac{1}{2}}}{1+(6.65d)^{2.12}}, \quad (8)$$

where

$$F_{00}^2=(0.0862)^2.$$

Equations (6), (7), and (8) fit a large set of detailed calculational results to less than 2 percent in most cases. It may be of interest to give examples of the theoretical accuracy with which they indicate d, F, and f can be measured, and to point out experimental difficulties which would be met. Of course, for any information to be obtained from experiment, it is necessary to measure $d\sigma/d\Omega$ as a function of angle of scattering, and calculate

TABLE II. Ratios of maxima in F^2 and characteristics of corresponding nuclear density.

$\frac{F^2(\text{first maximum})}{F^2(\text{second maximum})}$	d	f
5.86	0	0.
5.90	0	0.4
6.06	0	1.0
8.28	0.1	0
8.41	0.1	0.4
12.3	0.2	0
12.9	0.2	0.4

$$(d\sigma/d\Omega)_{\text{exp}}(2e^2EZ\cos\tfrac{1}{2}\theta)^{-2}(\hbar cq)^4=(F^2)_{\text{exp}}. \quad (9)$$

If the ratio of F^2 at its first two maxima were known to 10 percent, d could be fixed to within ± 0.025. Since d is a rather fine detail of nuclear structure, this accuracy appears satisfactory. Even for a heavy nucleus, $d\sigma/d\Omega$ for elastic scattering is only about 10^{-30} cm^2. Thus, there is the problem of a measurement only barely within current accurate experiments—coupled with the possibility that inelastic effects and background may mask the elastically scattered beam.

Should d be found to be 0.1 to about 10 percent, $q\bar{r}$ would be calculable from (7) to 0.5 percent. But since $\bar{r}$ is known only in terms of $q[=(2E/\hbar c)\sin\frac{1}{2}\theta]$, it is likely that the experimental determination of the scattering angle at the peak of a rather broad maximum will limit the accuracy of $\bar{r}$.

For intermediate values of both f and d, a value of F^2 accurate to 10 percent would fix f to about ± 0.1.

Although the situation suggested by the preceding discussion is an agreeable one, it cannot be inferred that the measurements mentioned would necessarily establish numbers for the parameters. Other features of the elastic scattering, such as absolute values of maxima beyond the first, location of minima, etc., should be given accurately. Indeed, it may be impossible to satisfy all requirements with a three-parameter density function. Possible reasons for such failure would be lack of spherical symmetry in the charge distribution, an

TABLE III. $q\bar{r}$ at first maximum in F^2 for various values of f and d.

$q\bar{r}$ (first maximum)	f	d
4.46	0	0
4.46	0.4	0
4.50	1	0
4.58	0	0.1
4.57	0.4	0.1
4.56	1	0.1
5.05	0	0.2
5.01	0.4	0.2
4.98	1	0.2

internal structure of the nucleons which would modify the Coulomb interaction at very small distances, significantly large magnetic effects, and of course, breakdown of the Born approximation.

INELASTIC SCATTERING

The cross section for inelastic scattering in which the nucleus is left in an excited state characterized by Φ_α and the electron has a final energy E_β, is given by

$$\frac{d\sigma}{dq_\beta}=\frac{4}{\pi}\left(\frac{e^2}{\hbar c}\right)^2\frac{E_\beta}{E}\frac{\cos^2\frac{1}{2}\theta}{q_\beta{}^3}\times\left|\int\Phi_\alpha{}^*\Phi_0\sum_{j=1}^{Z}\exp(i\mathbf{q}_\beta\cdot\mathbf{R}_j)d\mathbf{R}_1\cdots d\mathbf{R}_Z\right|^2, \quad (10)$$

where

$$(\hbar c q_\beta)^2=E^2+E_\beta{}^2-2EE_\beta\cos\tfrac{1}{2}\theta. \quad (11)$$

It is possible to measure transitions to a definite excited state α of the nucleus by measuring the energy of the scattered electron. However, we shall find that more information on nuclear structure is revealed if we sum over all possible final states α. Theoretically, this is most easily done by keeping q_β fixed; experimentally, of course, it would be easiest to measure at constant angle θ.

The two types of summation are not identical as can be seen from (11); however, if E is several hundred Mev, the difference is not large. The transition probabilities (10), for given q_β, are closely related to those for the photoelectric effect; therefore, experimentally[5] the most important excitation energies will be of the order of 15–30 Mev. It can easily be shown from (11) that under these conditions, and with q_β held fixed, θ will not vary much with excitation energy within the range for which the cross section (10) is large. Therefore, although the sum is performed for q fixed, this is closely equivalent to keeping θ constant. Indeed, when the important energy changes in the scattering are small compared to the energy of the incident electron, q and θ are related by

$$|q|=(2E/\hbar c)\sin\tfrac{1}{2}\theta, \quad (12)$$

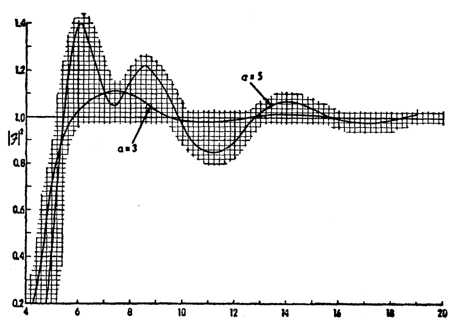

FIG. 5. Square of nuclear form factor for inelastic scattering (simple cubic structure model).

[5] K. Strauch, Ann. Rev. Nuclear Sci. 2, 105 (1953).

independent of the final energy of the electron. Moreover, for the purpose of actual summation of (11), both E_β and $\cos^2\frac{1}{2}\theta$ may be regarded as constant.

The sum rule applied to (10) gives then for the total inelastic scattering into $d\Omega$:

$$\sum_{\alpha\neq 0}d\sigma_\alpha=\left(\frac{e^2\cos\frac{1}{2}\theta}{2E\sin^2\frac{1}{2}\theta}\right)^2\left\{\int\Phi_0{}^*\Phi_0\zeta^*\zeta d\mathbf{R}-\left|\int\Phi_0{}^*\Phi_0\zeta d\mathbf{R}\right|^2\right\}d\Omega, \quad (13)$$

where

$$\zeta=\sum_{j=1}^{Z}\exp(i\mathbf{q}\cdot\mathbf{R}_j). \quad (14)$$

The second term in the curly bracket in (13) gives the elastic scattering, while the first term is the total scattering. From (14),

$$\zeta^*\zeta=Z+\sum_{j,k}^{Z}{}'\exp[i\mathbf{q}\cdot(\mathbf{R}_k-R_j)].$$

$|\Phi_0|^2$ integrated over all coordinates save one (or two) can be expected to have the same form regardless of which coordinate (or pair of coordinates) remains, so for the purpose of integration, one may write

$$\zeta=Z\exp(i\mathbf{q}\cdot\mathbf{R}_1),$$

and

$$\zeta^*\zeta=Z+Z(Z-1)\cos[\mathbf{q}\cdot(\mathbf{R}_1-\mathbf{R}_2)].$$

In combination with $\zeta^*\zeta$, $|\Phi_0|^2$ is integrated over all coordinates save two to give a "two-particle density" which characterizes the correlation in location of pairs of protons: this density is hereafter called $\Pi(\mathbf{R}_1,\mathbf{R}_2)$. Combined with ζ, $|\Phi_0|^2$ is integrated over all but one coordinate and, thus, gives the "single particle density," $\rho(\mathbf{R}_1)$, such as was used in the discussion of elastic scattering. Corresponding to the form factor F, there now exists a factor,

$$|\mathfrak{F}|^2=\int\int\{1+(Z-1)\cos\mathbf{q}\cdot(\mathbf{R}_1-\mathbf{R}_2)\}\times\Pi(\mathbf{R}_1,\mathbf{R}_2)d\mathbf{R}_1d\mathbf{R}_2-\left|\int\rho(R_1)\exp(i\mathbf{q}\cdot\mathbf{R}_1)d\mathbf{R}_1\right|^2, \quad (15)$$

which describes the effect of nucleon spatial distribution on the inelastic scattering.

CRYSTAL MODELS

As the first model for illustration of possible effects of nuclear structure on inelastic scattering, consider the following normalized two-particle density,

$$\Pi(\mathbf{R}_1,\mathbf{R}_2)=[(Z-1)V_1b^3\pi^{\frac{3}{2}}]^{-1}\times\sum_{\mathbf{n}\neq 0}\exp[-(\mathbf{R}_{12}-a\mathbf{n})^2/b^2]. \quad (16)$$

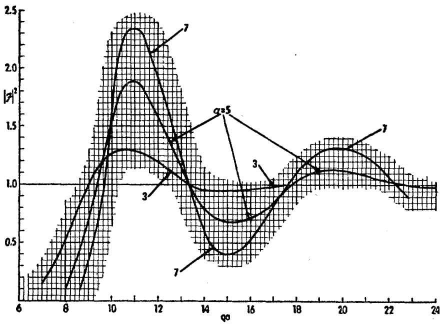

FIG. 6. Square of nuclear form factor for inelastic scattering (face-centered cubic structure model).

This is based on the assumption that the particles are arranged as in a regular crystal lattice, smeared out to some extent in a manner similar to thermal motion. In (16), V_1 = total nuclear volume; extends over $(Z-1)$ occupation points; $\mathbf{R}_{12}=\mathbf{R}_1-\mathbf{R}_2$; $\mathbf{n}$ = a numerical vector locating occupation points in a lattice, e.g., in a simple cubic structure, $\mathbf{n}$ is a triplet of integers; a = edge of basic cubic cell in lattice, and b = parameter determining certainty with which a proton can be found at or near an occupation point.

Apart from the assumption of a crystalline structure, this model may also be seriously in error due to the fact that it effectively fails to recognize the edge effects which arise for a finite nucleus. The parameter b is taken as proportional to a:

$$b=a|\mathbf{n}|/\alpha. \tag{17}$$

Introduction of $\mathbf{n}$ in (17) means that the sharpness of correlation between nucleons decreases as their distance $|\mathbf{n}|$ increases, thus making the assumed structure more similar to a liquid than to a solid. The additional parameter α permits adjustment of this sharpness independent of $|\mathbf{n}|=n$.

The scattered intensity in this case is found quite readily to vary as

$$|\mathfrak{F}|^2=1+\sum_{\mathbf{n}\neq 0}\frac{\sin nqa}{nqa}\exp\left[-\frac{(nqa)^2}{2\alpha}\right]-9Z\left[\frac{\sin qR_0-qR_0\cos qR_0}{(qR_0)^3}\right]^2, \tag{18}$$

where

$$R_0=a(3Z/4\pi\rho)^{\frac{1}{3}}, \tag{19}$$

and ρ is the average number of protons assigned to each unit cell in the lattice.

$|\mathfrak{F}|^2$ for Pb is plotted *versus* qa in Figs. 5 and 6. Figure 5 presents the data for a simple cubic lattice structure, while Fig. 6 corresponds to the assumption of a face-centered cubic structure. The curves are not given for small qa, since it is in this range that the calculation is least realistic in regard to edge effects. As is to be expected, large values of the parameter α consistently give the sharpest interference patterns. While separate peaks in $|\mathfrak{F}|^2$ for the simple cubic structure can be identified as constructive interference in scattering from specific pairs of lattice points, the same is not true for the face-centered lattice model. For this latter case, none of the "resolution factors" α was sufficiently large to prevent the smearing of adjacent peaks into each other.

Since the Born approximation is really not valid for such a heavy nucleus as Pb, it is not appropriate to discuss any quantitative details of $\mathfrak{F}$. The curves given are of value in demonstrating the kind of inelastic scattering which might be expected and in providing a form factor which can perhaps be corrected to give accurate results.

"FERMI PARTICLE" MODEL

The last model to be considered is one in which the Z protons are constrained to be within a cubic box of edge L and to be correlated only through the Pauli exclusion principle. For such a model, it can be shown[6] that the square of the inelastic form factor is

$$|\mathfrak{F}|^2=1-Q\int_0^X\frac{\sin\nu x}{\nu x}\left(\frac{\sin x-x\cos x}{x^2}\right)^2dx, \tag{20}$$

where

$$Q^{-1}=\int_0^X\left(\frac{\sin x-x\cos x}{x^2}\right)^2dx,$$

$$x=2\pi Nr_{12}/L,$$

$$\nu=qL/2\pi N,$$

$$N=Z^{\frac{1}{3}}/2,$$

$$X=2\pi NR_{12}/L,$$

where R_{12} is the upper limit for inter-particle distances in the nucleus. In Figs. 7 and 8, $|\mathfrak{F}|^2$ is plotted *versus* qr_0, where r_0 is the radius of a spherical volume con-

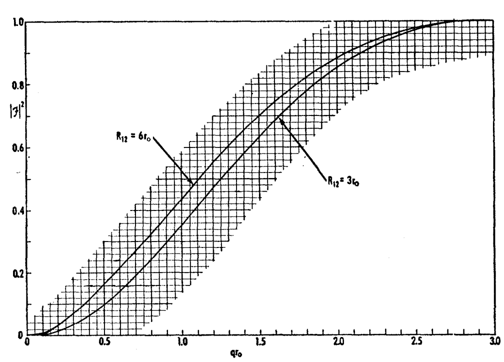

FIG. 7. Square of nuclear form factor for inelastic scattering (Fermi particle model).

[6] J. H. Smith, Ph.D. thesis, Cornell University, 1951 (unpublished).

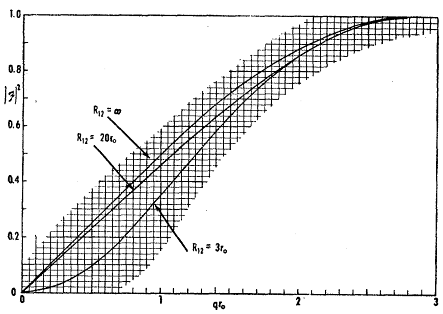

FIG. 8. Square of nuclear form factor for inelastic scattering (Fermi particle model).

taining one nucleon. Thus,

$$L^3=\tfrac{1}{3}4\pi A r_0^3.$$

For comparison with the crystalline model results, it may be said that a for the simple cubic structure is only slightly greater than $2r_0$.

The principal dependence of $|\mathfrak{F}|^2$ is on qr_0. In addition, there is a somewhat less important dependence on R_{12}/r_0. One possible choice is $R_{12}=A^{\frac{1}{3}}r_0$, and indeed, $4\pi R_{12}^3/3$ describes the nuclear volume correctly. The choice $R_{12}=2A^{\frac{1}{3}}r_0$ probably would overestimate the nuclear size. More generally, one can use $R_{12}=gr_0$, in which case a given value of g would mean that some nucleus with A between g^3 and $g^3/8$ was described. The values $g=3$, 6, and 12 were calculated, corresponding to either Pb with $g=6$ or 12, or to Al with $g=3$ or 6. The cases $X\rightarrow\infty$ for an infinite nucleus and $g=20$ to bridge the gap to finite dimensions are also included.

COMPARISON OF ELASTIC AND INELASTIC SCATTERING

In the limit of large q, the inelastic scattering is Z times the scattering from a single proton, as if the protons scattered independently. For small q, the elastic scattering is Z^2 times that of a proton. Both of these results are well known. It is also clear that at large q the inelastic scattering is much larger than the elastic.

Apart from the possibility of separating elastic and inelastic scattering by measuring the energy of the scattered electron, the interference phenomena for the two types of scattering can also be distinguished by the angles at which they occur. The interferences in elastic scattering are determined by qR, where R is the radius of the nucleus, because the entire nucleus participates in the phenomenon. The inelastic scattering, on the other hand, depends primarily on the correlation of neighboring protons, and hence on qr_0. The interference phenomena in elastic scattering, therefore, occur at smaller angles (smaller q) than those for inelastic. Taking Pb as an example—in spite of the inapplicability of the Born approximation—we have $R=6r_0$. Hence, the first maximum of elastic scattering occurs at $qr_0\approx 0.75$. At this value of q, the Fermi model gives an inelastic scattering of less than 40 percent of the asymptotic value, and only at much higher qr_0 is the asymptotic value approached. For the crystal model, even higher qr_0 are significant: the first maximum occurs at $qr_0>3$. There will thus be no confusion between elastic and inelastic maxima.

In conclusion, we should like to repeat our warning that the results of this paper must not be used for heavy nuclei because the Born approximation is not valid for these

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