Compton Scattering*

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The canonical transformation of Bloch and Nordsieck is used to examine the infrared divergences present in double Compton scattering and radiative corrections to single Compton scattering.

INTRODUCTION

SCHAFROTH[1] and subsequent authors[2,3] have demonstrated that the infrared catastrophes which exist in double Compton scattering and in the radiative correction to single Compton scattering (to order e^6) cancel each other identically. The divergence in the former case results from real photons and that in the latter from virtual photons. A quasi-philosophical argument based on the inability of a photon detector of finite resolution to distinguish between real and virtual photons of extremely low energy has been suggested.[2] It is the purpose of this note to point out that it is possible to find a canonical transformation which will remove those terms in the interaction Hamiltonian (nonrelativistic) between charged particles and the photon field which cause the individual divergences. The canonical transformation employed is a generalization of that discussed by Block and Nordsieck;[4] this transformation has been used successfully in dipole approximation to discuss such diversified phenomena as, for example, radiative corrections in electron scattering[5] and radiative effects in π-μ decay.[6] It seems reasonable to discuss such low-energy divergences using nonrelativistic theory; the high-energy divergences which result in such a treatment are both expected and unavoidable.

REMOVAL OF THE INFRARED CATASTROPHES

It is well-known that the Bloch-Nordsieck transformation when applied to the nonrelativistic Hamiltonian of nonbound charged particles interacting with a photon field removes all terms linear in *both* (p/m_0c) and e; it produces, as a by-product, a nonrelativistic analog of mass-renormalization of the charged particle.[7] (We adopt the notation, m_0 = observable mass of the charged particle.) The proof of the above contention is straightforward and is discussed elsewhere[7] in some detail; it is only sketched below.

The Hamiltonian for the system is chosen to be:

$$H=\frac{p^2}{2m_0}\left(1+\frac{\delta m}{m_0}\right)-\frac{(4\pi e^2)^{\frac{1}{2}}}{m_0c}\left(1+\frac{\delta m}{m_0}\right)\mathbf{p}\cdot\mathbf{A}+\frac{4\pi e^2}{2m_0c^2}\left(1+\frac{\delta m}{m_0}\right)A^2+H_{\text{rad}},\quad(1)$$

where

$$\mathbf{A}=\sum_k \mathbf{A}_k=\sum_k \hat{e}_k\left(\frac{\hbar c}{k\Omega}\right)^{\frac{1}{2}}\{\Pi_k\cos\mathbf{k}\cdot\mathbf{x}+Q_k\sin\mathbf{k}\cdot\mathbf{x}\},$$

$$H_{\text{rad}}=\tfrac{1}{2}\hbar c\textstyle\sum_k k\{\Pi_k{}^2+Q_k{}^2\},$$

$$\Pi_k=(2)^{-\frac{1}{2}}(a_k{}^*+a_k),\quad Q_k=(-2)^{-\frac{1}{2}}(a_k{}^*-a_k).$$

The quantity $\delta m/m_0$ is the mass renormalization parameter which is proportional, in first approximation, to e^2. [The remaining symbols in Eq. (1) have their conventional meaning.] A canonical transformation is now performed on the Hamiltonian of Eq. (1) in the following manner:

$$H'=\exp(-i\eta)H\exp(i\eta)=H+\exp(-i\eta)[H,\exp(i\eta)]\quad(2)$$

where

$$\eta=\sum_k\left(\frac{4\pi e^2}{\hbar ck^3\Omega}\right)^{\frac{1}{2}}\left(\hat{e}_k\cdot\frac{\mathbf{p}}{m_0c}\right)\times\{Q_k\cos\mathbf{k}\cdot\mathbf{x}-\Pi_k\sin\mathbf{k}\cdot\mathbf{x}\}=\sum_k\eta_k.\quad(3)$$

The transformation when applied to H_{rad} of Eq. (1) yields:

$$\exp(-i\eta)H_{\text{rad}}\exp(i\eta)=H_{\text{rad}}+\frac{4}{3\pi}\left(\frac{e^2}{m_0c^2}\right)\frac{p^2}{2m_0}\int_0^{k^{\max}}dk+\frac{(4\pi e^2)^{\frac{1}{2}}}{m_0c}\mathbf{p}\cdot\mathbf{A}.\quad(4)$$

Equation (4) is exact. The following commutation relationships are used in the derivation of Eq. (4) and

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[2] L. Brown and R. Feynman, Phys. Rev. **85**, 231 (1952).
[3] A. Mitra, Nature **169**, 1009 (1952).
[4] F. Bloch and A. Nordsieck, Phys. Rev. **52**, 54 (1937).
[5] A. Nordsieck, Phys. Rev. **52**, 59 (1937).
[6] H. Primakoff, Phys. Rev. **84**, 1255 (1951).
[7] See, for example, G. Morpurgo, Nuovo cimento, Suppl. Vol. **8**, 109 (1951).

subsequent expressions:

$$[\Pi_k, \exp(i\eta)]=\left(\frac{4\pi e^2}{\hbar c k^3\Omega}\right)^{\frac{1}{2}}\left(\hat{e}_k\cdot\frac{\mathbf{p}}{m_0 c}\right)\times\cos\mathbf{k}\cdot\mathbf{x}\exp(i\eta), \quad (5a)$$

$$[Q_k, \exp(i\eta)]=\left(\frac{4\pi e^2}{\hbar c k^3\Omega}\right)^{\frac{1}{2}}\left(\hat{e}_k\cdot\frac{\mathbf{p}}{m_0 c}\right)\times\sin\mathbf{k}\cdot\mathbf{x}\exp(i\eta). \quad (5b)$$

In addition:

$$\exp(-i\eta)\left[-\frac{(4\pi e^2)^{\frac{1}{2}}}{m_0 c}\mathbf{p}\cdot\mathbf{A}\right]\exp(i\eta)$$
$$=-\frac{(4\pi e^2)^{\frac{1}{2}}}{m_0 c}\mathbf{p}\cdot\mathbf{A}-\frac{8}{3\pi}\left(\frac{e^2}{m_0 c^2}\right)\frac{p^2}{2m_0}\int_0^{k^{\max}}dk$$
$$+i\frac{(4\pi e^2)^{\frac{1}{2}}}{m_0 c}\sum_{k\neq k'}\{\eta_k\mathbf{p}\cdot\mathbf{A}_{k'}-\mathbf{p}\cdot\mathbf{A}_{k'}\eta_k\}+O(e^3). \quad (6)$$

It is not possible to calculate the transformed term in $(\mathbf{p}\cdot\mathbf{A})$ exactly since the commutator

$$[\mathbf{p}, \exp(i\eta)]$$

cannot be expressed in any convenient closed form. A power series expansion in $(e^2/\hbar c)$ is required and results in Eq. (6). Finally, we evaluate

$$\exp(-i\eta)\left[\frac{4\pi e^2}{2m_0 c^2}A^2\right]\exp(i\eta)=\frac{4\pi e^2}{2m_0 c^2}A^2$$
$$+\frac{4\pi e^2}{2m_0 c^2}\frac{(4\pi e^2)^{\frac{1}{2}}}{m_0 c\Omega}\left\{\mathbf{A}\cdot\mathbf{J}+\mathbf{J}\cdot\mathbf{A}+\frac{(4\pi e^2)^{\frac{1}{2}}}{m_0 c\Omega}J^2\right\}, \quad (7)$$

where

$$\mathbf{J}=\sum_k\frac{1}{k^2}\hat{e}_k(\hat{e}_k\cdot\mathbf{p}).$$

The first three terms in Eq. (7) lead to

$$\frac{4\pi e^2}{2m_0 c^2}A^2+\frac{4}{3\pi}\left(\frac{e^2}{m_0 c^2}\right)\int_0^{k^{\max}}dk\cdot\frac{(4\pi e^2)^{\frac{1}{2}}}{m_0 c}\mathbf{p}\cdot\mathbf{A}. \quad (7a)$$

The last term in Eq. (7) is evaluated in a straight forward manner and yields

$$B(k^{\max})(e^2/\hbar c)^2(p^2/2m_0), \quad (7b)$$

the value of the constant B depending on the cut-off value, $k^{\max}$. We see that this term is interpretable as a mass renormalization proportional to e^4 and will be neglected. If we now set

$$\frac{\delta m}{m_0}=\frac{4}{3\pi}\left(\frac{e^2}{m_0 c^2}\right)\int_0^{k\max}dk, \quad (8)$$

and combine the results of Eqs. (2)–(8), we discover

$$H'=\frac{p^2}{2m_0}+\frac{4\pi e^2}{2m_0 c^2}A^2+i\frac{(4\pi e^2)^{\frac{1}{2}}}{m_0 c}\times\sum_{k\neq k'}\{\eta_k\mathbf{p}\cdot\mathbf{A}_{k'}-\mathbf{p}\cdot\mathbf{A}_{k'}\eta_k\}+H_{\text{rad}}$$
$$+\text{terms higher order in } e \text{ and/or } (p/m_0 c). \quad (9)$$

The Hamiltonian of Eq. (9) contains no terms which will lead to the long wavelength divergences in Compton scattering mentioned above.

To demonstrate the appearance of the separate long-wavelength divergences in the Compton processes in the more usual perturbation treatment, it is necessary to return to Eq. (7) and expand $\exp(\pm i\eta)$ in a power series in η.

$$\exp(-i\eta)\left[\frac{4\pi e^2}{2m_0 c^2}A^2\right]\exp(i\eta)$$
$$=\frac{4\pi e^2}{2m_0 c^2}\left\{A^2+i[\eta,A^2]+\frac{i^2}{2}[\eta,[\eta,A^2]]+\cdots\right\}. \quad (7c)$$

The first-order (in a perturbation sense) matrix element of the second term in Eq. (7c) yields a nonvanishing result in the double Compton process. The cross section calculated for the double process from this matrix element contains a term which is logarithmically divergent at long wavelengths:

$$d\sigma=-\frac{2}{3\pi}\left(\frac{e^2}{m_0 c^2}\right)^2\left(\frac{e^2}{\hbar c}\right)\left(\frac{\hbar k_0}{m_0 c}\right)^2\ln\left(\frac{\hbar k^{\min}}{m_0 c}\right)\cdot(1-\cos\theta)(1+\cos^2\theta)d\Omega, \quad (10)$$

where k_0 = wave number of incident photon, θ = angle of scattering, and $k^{\min}\rightarrow 0$. The first-order matrix element of the third term in Eq. (7c) yields a nonvanishing result in the radiative correction to the single Compton process; this matrix element is of fourth order in e and when combined with the first-order matrix element of the first term in Eq. (7c) yields the cross section of Eq. (10) with opposite sign.[1–3]

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