Canonical Transformation for an Electron-Positron Field Coupled to a Time-Independent Electromagnetic Field. II*

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A prescription for constructing well-defined field operators from a single particle operator is given in a form in which electrons and positrons appear in a symmetric way. Such a prescription removes certain trivial infinities arising from the conventional "hole" theory approach. The canonical transformation derived in a previous paper is used to calculate the vacuum expectation values of various operators to the second order in electron charge. It is shown that, despite the fact that the scattering operator does not exist when the electromagnetic field is constant in time, one obtains the usual results for current and charge densities. In addition, the striking result is proved that for certain purely electrostatic fields the expectation values of the number operator are finite but that when a magnetostatic field is also present the expectation values are infinite.

1. INTRODUCTION

IN a previous paper[1] we obtained an integro-difference equation for the canonical transformation for an electron-positron field, coupled to a time-independent electromagnetic field in terms of single-particle transformations or eigenfunctions. The canonical transformation for the field was solved to the second order in the electron charge by means of a perturbation procedure. The usual treatment of field problems is weighted with implicit assumptions on the character of the operators involved. It was the objective to present more explicitly than is usually done the various assumptions made on the spectra of the operators involved. That is, the canonical transformation was found under the assumption that the spectrum of the single-particle Hamiltonian interacting with the electromagnetic field was the same as that for the field-free Hamiltonian, and that the electron-positron field interacting with the electromagnetic field could be described with the aid of annihilation and creation operators of the same character as those of the free electron-positron field.

This paper is to be regarded as a direct extension of reference 1. We shall calculate the vacuum expectation values of various operators to the second order. It will be shown that, despite the fact that the scattering operator does not exist, the expectation value of the charge and current operators is the same as that obtained from a scattering operator formalism, for example, that of Schwinger.[2] In addition, we shall prove the surprising result, mentioned in reference 1, that for electromagnetic fields which consists of suitable electrostatic fields only, the vacuum expectation of the number operator is finite and the canonical transformation exists rigorously. It will be shown, however, that when a magnetic field is also present the vacuum expectation of the number operator is infinite. A form of this proof is also given by Friedrichs[3] who treats the same problem from a different point of view.

2. FIELD OPERATORS

We shall now obtain expressions for field operators in terms of the annihilation and creation operators $A^{\pm}(s)$ and $B^{\pm}(v)$. The field operators which we want can be obtained from operators which occur in the single-particle theory.

Let us introduce matrices $L^s(s;s')$ for single-particle operators L as given in the $H^{(0)}$ representation, i.e.,

$$L^s(s;s')=\sum_{\mu}\int \bar{\chi}^{(0)}(X,\mu;s)L^{X,\mu}\chi^{(0)}(X,\mu;s')dX, \quad (1)$$

where $L^{X,\mu}$ is the single particle operator L in the X-representation.

In the case of a simple fermion field a well-defined field operator $[L]_A$ can be constructed from single particle operators L by means of the formula,

$$[L]_A=\int\int L^s(s;s')A^+(s)A^-(s')dsds'; \quad (2)$$

(see, e.g., Friedrichs[4]).

It is essential that the creation and annihilation operators appear in the order shown in (2) for operators such as $[L]_A$ to be defined. This fact motivates the following definition for field operators which was suggested by Friedrichs[5] for the case of the electron-positron field.

$$[L]_A=\int\int L^s(s;s')\epsilon'[A^{\epsilon}(s)A^{-\epsilon'}(s')]_{\epsilon'}dsds', \quad (3)$$

where

$$[A^{\epsilon}(s)A^{-\epsilon'}(s')]_{\epsilon'}=A^{\epsilon}(s)A^{-\epsilon'}(s') \quad \text{for} \quad \epsilon'>0$$
$$=A^{-\epsilon'}(s')A^{\epsilon}(s) \quad \text{for} \quad \epsilon'<0. \quad (4)$$

* This paper is based, in part, on the report IMM-NYU No. 179, Institute for Mathematics and Mechanics, New York University, March, 1952 (unpublished).

[1] H. E. Moses, Phys. Rev. **89**, 115 (1953).

[2] J. Schwinger, Phys. Rev. **75**, 651 (1949).

[3] K. O. Friedrichs, *Mathematical Aspects of the Quantum Theory of Fields* (Interscience Publishers, Inc., New York, 1953), Part V, Sec. 28.

[4] K. O. Friedrichs, reference 3, Part II.

[5] K. O. Friedrichs (private communication).

The symmetric way in which electrons and positrons were introduced and this prescription for obtaining field operators makes it unnecessary to consider a positron as being a "hole" in a "sea" of electrons. Furthermore, field operators obtained in this way have the desirable property that the expectation value in a vacuum of any such field operator $[L]_A$ is given by

$$(\Psi_0,[L]_A\Psi_0)=0,$$

where Ψ_0 is the vacuum state with respect to the operator N, i.e., $N\Psi_0=0$. It should be mentioned that Snyder[6] has a similar prescription for obtaining well-defined field operators, and Belinfante[7] has a similar procedure for avoiding the "hole" theory of the positron.

One can also construct field operators using the annihilation and creation operators $B^{\pm(v)}$. The field operator $[L]_B$ is defined by

$$[L]_B=\int\int L^v(v;v')\iota'[B^\iota(v)B^{-\iota'}(v')]_{\iota'}dvdv', \quad (5)$$

where $L^v(v;v')$ is the matrix element of the single-particle operator L in the H representation, i.e.,

$$L^v(v;v')=\sum_\mu\int\bar{\chi}(X,\mu;v)L^{X,\mu}\chi(X,\mu;v')dx. \quad (6)$$

We shall now give expressions for several field operators, though we shall not use all of them. (See earlier report by the author[8] for details of the derivations.)

1. The unperturbed Hamiltonian $\mathfrak{H}^{(0)}$ of the electron-positron field is obtained by using Eq. (3), where L is taken to be the unperturbed single-particle Hamiltonian $H^{(0)}$. That is,

$$\mathfrak{H}^{(0)}=[H^{(0)}]_A=\int|E(s)|A^+(s)A^-(s)ds, \quad (7)$$

where $E(s)$ is the eigenvalue of $H^{(0)}$ correspondlng to the case where the complete set of variables introduced in reference 1 have the values collectively denoted by s. It is useful to note that for any state $\Phi\underset{N}{\leftrightarrow}\{\Psi_n(s)_n\}$, we have

$$\mathfrak{H}^{(0)}\Phi\underset{N}{\leftrightarrow}\{\sum_{i=1}^{n}|E(s_i)|\Psi_n(s)_n\},$$

as is required in the more conventional treatment of the problem.

2. The number operator N is obtained from Eq. (3) by taking L to be EI, where I is the identity operator and E is the operator $H^{(0)}/|H^{(0)}|$, whose eigenvalue is ϵ. Thus,

$$N=[EI]_A=\int A^+(s)A^-(s)ds, \quad (8)$$

as required.

3. The charge operator Q is obtained from Eq. (3) by taking L to be the single-particle operator eI, where e is the electron charge. Then,

$$Q=[eI]_A=\int A^+(s)A^-(s)\delta_{\epsilon,+}ds-\int A^+(s)A^-(s)\delta_{\epsilon,-}ds$$
$$=e[N_+-N_-], \quad (9)$$

as required.

4. The current operator J_i is obtained from Eq. (3) where L is taken as $L=e\alpha_i$:

$$J_i=[e\alpha_i]_A=e\int\int\alpha_i{}^s(s;s')A^\epsilon(s)A^-(s')\delta_{\epsilon',+}dsds'$$
$$-e\int\int\alpha_i{}^s(s;s')A^+(s')A^\epsilon(s)\delta_{\epsilon',-}dsds', \quad (10)$$

where

$$\alpha_i{}^s(s;s')=\sum_\mu\int\bar{\chi}^{(0)}(X,\mu;s)\alpha_i{}^\mu\chi^{(0)}(X,\mu;s')dX. \quad (11)$$

In Eq. (11), $\alpha_i{}^\mu$ is the Dirac operator as expressed in the familiar matrix form.

5. The field charge density operator $\rho(X_0)$ for the charge density at X_0 is obtained by using for L the single-particle charge density at X_0, namely, $L^{X,\mu}=e\delta(X-X_0)$. That $L^{X,\mu}=e\delta(X-X_0)$ is the single-particle charge density is clear from the fact that if we have the single-particle state $\Psi(X,\mu)$, the expectation value of L is

$$\sum_\mu\int\bar{\Psi}(X,\mu)L^{X,\mu}\Psi(X,\mu)dX=e\sum_\mu\bar{\Psi}(X_0,\mu)\Psi(X_0,\mu),$$

which is just the charge density at X_0 usually assumed in the single-particle theory.

From Eq. (3), and Eqs. (15) and (15a) of reference 1, we have

$$\rho(X_0)=e[\delta(X_{op}-X_0)]_A$$
$$=e\sum_\mu\int\int\bar{\chi}^{(0)}(X_0,\mu;s)\chi^{(0)}(X_0,\mu;s')\times\epsilon'[A^\epsilon(s)A^{-\epsilon'}(s')]_{\epsilon'}dsds'$$
$$=\tfrac{1}{2}e\sum_\mu[\Xi^+(X_0,\mu)\Xi^-(X_0,\mu)-\Xi^-(X_0,\mu)\Xi^+(X_0,\mu)]$$
$$+\tfrac{1}{2}e\Big[\sum_\mu\int\bar{\chi}^{(0)}(X_0,\mu;s)\chi^{(0)}(X_0,\mu;s)\delta_{\epsilon,+}ds$$
$$-\sum_\mu\int\bar{\chi}^{(0)}(X_0,\mu;s)\chi^{(0)}(X_0,\mu;s)\delta_{\epsilon,-}ds\Big]. \quad (12)$$

[6] H. S. Snyder, Phys. Rev. **78**, 98 (1950).

[7] F. J. Belinfante, preprint "A Positon Theory Without Holes" (1953). The author is grateful to Professor Belinfante for making this preprint available to him. See also Phys. Rev. **93**, 935 (1954).

[8] H. E. Moses, Report No. IMM-NYU No. 179, Institute of Mathematical Sciences, New York University, New York, 1952 (unpublished).

The first expression for $\rho(X)$ can be used to show

$$\int \rho(X)dX=Q,$$

as required. The second expression shows that the charge density is substantially the Heisenberg charge density, except that here the usually troublesome constant term has been eliminated.

6. The field current density operator at X_0 which we denote by $j_i(X_0)$ is similarly obtained by choosing L as $L^{X,\mu}=e\alpha_i{}^{\mu}\delta(X-X_0)$. An expression analogous to that for $\rho(X)$ is obtained.

7. The total energy $\mathfrak{H}$ is defined using expression (5) with L equal to the single-particle Hamiltonian H. One obtains

$$\mathfrak{H}=[H]_B=\int |E(v)|B^+(v)B^-(v)dv. \tag{13}$$

We have

$$\mathfrak{H} \underset{M}{\leftrightarrow} \{\sum_{i=1}^{n}|E(v_i)|\zeta_n(v)_n\},$$

where

$$\Phi \underset{M}{\leftrightarrow} \{\zeta_n(v)_n\}.$$

THE SCATTERING OPERATOR

In the scattering problems the total Hamiltonian is split up into the sum of two parts one of which is the unperturbed Hamiltonian and the other is the perturbation. In terms of the field Hamiltonians this splitting up is written

$$\mathfrak{H}=\mathfrak{H}^{(0)}+\mathcal{V}. \tag{14}$$

If there exists a class of solutions $\Phi(t)$ of the Schrödinger equation

$$i\frac{\partial}{\partial t}\Phi(t)=\mathfrak{H}\Phi(t) \tag{15}$$

such that the limits $\Phi_{\pm}$ exist where

$$\Phi_{\pm}=\lim_{t\to\pm\infty}\exp(it\mathfrak{H}^{(0)})\Phi(t), \tag{16}$$

then the scattering operator S is defined by

$$\Phi_+=S\Phi_-. \tag{17}$$

When the scattering operator exists one usually formulates an initial value problem by prescribing Φ_-, the state of the system in the infinite past, in the N representation and by finding Φ_+, the state in the infinite future, also usually in the N representation.

We shall show, however, that in our problem the limits $\Phi_{\pm}$ do not exist and, therefore, that the scattering operator cannot be defined. We proceed as follows: The solution of Eq. (15) is

$$\Phi(t)=\exp(-it\mathfrak{H})\Phi(0), \tag{18}$$

or in terms of the M representation of the field,

$$\zeta_n(v)_n(t)=\exp[-it\sum_{i=1}^{n}|E(v_i)|]\zeta_n(v)_n, \tag{18a}$$

where

$$\Phi(0) \underset{M}{\leftrightarrow} \zeta_n(v)_n.$$

From Eq. (35) of reference 1, together with the orthogonality relations (36), we can find $\Phi(t)$ in the N representation, namely,

$$\Psi_n(s)_n(t)=\sum_{n'}\int \bar{T}_{n':n}(v)_{n'}:(s)_n\zeta_{n'}(v)_{n'}(t)(dv)_{n'}. \tag{19}$$

Hence, $\exp(it\mathfrak{H}^{(0)})\Phi(t)$ as given in the N representation is, on using Eq. (18a),

$$\exp(it\mathfrak{H}^{(0)})\Phi(t) \underset{N}{\leftrightarrow} \exp[it\sum_{i=1}^{n}|E(s_i)|]\Psi_n(s)_n(t)$$

$$=\sum_{n'}\int \bar{T}_{n':n}(v)_{n'}:(s)_n\exp[it(\sum_{i=1}^{n}|E(s_i)|$$

$$-\sum_{i=1}^{n'}|E(v_i)|)]\times\zeta_{n'}(v)_{n'}(dv)_{n'}. \tag{20}$$

If we let $t\to\pm\infty$ in expression (20), the exponential factor under the integral sign merely oscillates for terms of the sum in which n' does not equal n. Such terms arise because particles are being created or destroyed. Hence the limits of $\exp(it\mathfrak{H}^{(0)})\Phi(t)$ as $t\to\pm\infty$ do not exist, and thus the scattering does not exist in a true sense. Generally speaking, scattering operators will not exist in field-theoretical problems where the perturbation is independent of the time. In order to have scattering operators, the perturbation must be functions of time which die out sufficiently rapidly as $t\to\pm\infty$. This is the reason adiabatic switching on and off of the perturbations is usually assumed.

Since the scattering operator does not exist in a rigorous sense, we must resort to formulations of the initial value problem other than that used when the scattering operator exists. The most obvious formulation is to prescribe $\Phi(0)$ is the N representation and to obtain $\Phi(t)$ also in the N representation. For this purpose we can use the canonical transformation of reference 1. If, then, we have

$$\Phi(0) \underset{N}{\leftrightarrow} \{\Psi_n(s)_n\},$$

where the functions $\Psi_n(s)_n$ are prescribed, we have from Eq. (35) of reference 1 and Eqs. (18a) and (19) of the present paper:

$$\Psi_n(s)_n(t)=\sum_{n''}\int\Psi_{n''}(s'')_{n''}(ds'')_{n''}$$

$$\times\sum_{n'}\int\bar{T}_{n':n}(v')_{n'}:(s)_nT_{n':n''}(v')_{n'}:(s'')_{n''}$$

$$\times\exp\{-it\sum_{i=1}^{n'}|E(v_i')|\}(dv')_{n'}. \tag{21}$$

A second type of initial value problem is based on the assumption of the validity of the adiabatic theorem for quantized fields. If the theorem were valid, it would state that if we prescribe an arbitrary state Φ in the infinite past as being given in the N representation by

$$\Phi \underset{N}{\leftrightarrow} \{\Psi_n(s)_n\},$$

then at any finite time, i.e., after the electromagnetic field has been switched on infinitely slowly, the solution of the Schrödinger equation $\Phi(t)$ is given in the M representation by

$$\Phi(t) \underset{M}{\leftrightarrow} \{\exp[-it\sum_{i=1}^{n}|E(v_i)|]\Psi_n(v)_n\},$$

where the functions $\Psi_n(v)_n$ are the same as those used to describe Φ in the N representation.

The validity of the adiabatic theorem for fields has not been discussed very much, though its validity is often implicitly assumed. Friedrichs[9] has proved the validity of the theorem for the case of a neutral meson field coupled to a source.

If we assume the validity of the adiabatic theorem, then we prescribe the state in the infinite past as given above. Then, at any finite time, $\Phi(t)$ is given in the N representation by

$$\Psi_n(s)_n(t)=\sum_{n'}\int \bar{T}_{n':n}(v)_{n'}:(s)_n\Psi_{n'}(v)_{n'} \times\exp[-it\sum_{i=1}^{m'}|E(v_i)|](dv)_{n'}. \quad (22)$$

The solutions of the Schrödinger equation given by (21) or (22) will be described as belonging to the first and second initial conditions respectively. That is, the first and second initial conditions correspond to the prescription of the state at $t=0$ and $t=-\infty$, respectively.

4. VACUUM EXPECTATION OF VALUES OF OPERATORS

We shall restrict ourselves to the case in which we have a vacuum initially. That is, in (21) or (22) (corresponding to the first or second initial conditions) we shall set

$$\Psi_0=1, \qquad \Psi_n(s)_n=0 \quad \text{for} \quad n\neq 0. \quad (23)$$

We shall then indicate the results for the expectation values for various operators to the second order in electron charge e when $t\to\infty$. Such a calculation simulates the results one would obtain if a scattering operator existed. At finite time t, we have for the expectation value of N using the first initial condition

$$\langle N\rangle_t=-4e^2\int\int|u_{-+}^{(1)}(s;s')|^2 \times[\cos t(|E(s)|+|E(s')|)-1]dsds'. \quad (24)$$

If the interaction potentials are sufficiently smooth, the oscillatory term will vanish by the Riemann-Lebesgue theorem as $t\to\infty$. Hence, for $t=\infty$ the expectation value is given by

$$\langle N\rangle_\infty=4e^2\int\int|u_{-+}^{(1)}(s;s')|^2dsds'. \quad (25)$$

Using the second boundary condition, we have for any time t,

$$\langle N\rangle_t=2e^2\int\int|u_{-+}^{(1)}(s;s')|^2dsds'. \quad (26)$$

Since this expression is independent of the time, it will also hold when $t=\infty$. It is to be noted that the expectation value obtained using the second initial condition is just one-half that obtained using the first initial condition. Such factors also occur in a similar calculation of Friedrichs in his treatment of the neutral meson field.[9] This factor of one-half presumably occurs in many problems in field theory.

It is to be further noted that the expressions for $\langle N\rangle_\infty$ is within a factor just that for $T_{0:0}^{(2)}$, which to the second order is the only matrix element of the transformation that might not be finite. Hence if $\langle N\rangle_\infty$ is finite, the canonical transformation will exist to the second order.

For the unperturbed Hamiltonian $\mathfrak{IC}^{(0)}$ corresponding to the kinetic energy of the electron-positron field one obtains, using the first initial condition, the result

$$\langle\mathfrak{IC}^{(0)}\rangle_\infty=2e^2\int\int|u_{-+}^{(1)}(s;s')|^2 \times[|E(s)|+|E(s')|]dsds'. \quad (27)$$

Using the second initial condition, one obtains just one-half of the above result.

The expectation value of the charge density operator, using *either* initial condition is given by

$$\langle\rho(X)\rangle_\infty=2e^2R\sum_\mu\int\int u_{-+}^{(1)}(s;s')\chi^{(0)}(X,\mu;s) \times\bar{\chi}^{(0)}(X,\mu;s')dsds', \quad (28)$$

where R stands for the real part of what follows.

For the current density one obtains

$$\langle j_i(X)\rangle_\infty=2e^2R\sum_\mu\int u_{-+}^{(1)}(s;s') \times\chi^{(0)}(X,\mu;s)\bar{\alpha}^\mu\bar{\chi}^{(0)}(X,\mu;s')dsds'. \quad (29)$$

[9] K. O. Friedrichs, reference 3, Part III.

5. EXPRESSIONS FOR THE EXPECTATION VALUES

We shall use the expressions for $u^{(1)}$ and $\chi^{(0)}$ to obtain the expectation values as ordinary integrals instead of the symbolic integrals used above.

From Eqs. (32), (31), and (30) of reference 1 we find

$$u_{-+}^{(1)}(s;s')=-[|E(s)|+|E(s')|]^{-1}\times\sum_{\mu}\int\bar{\chi}^{(0)}(X,\mu;s)[\sum_{i}\alpha_i^{\mu}A_i(X)+\phi(X)]\chi^{(0)}(X,\mu;s')dX\cdot\delta_{\epsilon,-}\delta_{\epsilon',+}. \quad (30)$$

We shall first calculate $\langle\rho(X)\rangle_{\infty}$. Since it can be shown that the vector potential contributes nothing, let us assume $A_i(X)=0$. Substituting into (28) we have

$$\langle\rho(X)\rangle_{\infty}=-2e^2R\int dX'\phi(X')\sum_{\mu}\sum_{\mu'}\times\int\int\frac{dsds'}{[|E(s)|+|E(s')|]}\bar{\chi}^{(0)}(X',\mu';s)\times\chi^{(0)}(X',\mu';s')\chi^{(0)}(X,\mu;s)\times\bar{\chi}^{(0)}(X,\mu;s')\delta_{\epsilon,-}\delta_{\epsilon',+}. \quad (31)$$

To simplify this expression we shall use some properties of the single-particle eigenfunctions $\chi^{(0)}$. First of all, we now explicitly use the fact that the variable s consists of the triple set of eigenvalues consisting of the momentum vector P, the sign of the energy ϵ, and spin component in the direction of the momentum τ. Integrals over s are to be replaced by summations and integrations as follows:

$$\int ds=\sum_{\tau}\sum_{\epsilon}\int dP,$$

where dP is integration over the vector space P. Furthermore, we note

$$E(s)=\epsilon(m^2+P^2)^{\frac{1}{2}}=\epsilon\omega(P), \quad (32)$$

where

$$\omega(P)=(m^2+P^2)^{\frac{1}{2}}. \quad (32a)$$

We use

$$|E(s)|=(m^2+P^2)^{\frac{1}{2}}=\omega(P). \quad (32b)$$

Hence, $|E(s)|$ depends only on the vector P.

The eigenfunctions $\chi^{(0)}(X,\mu;s)$ can be written

$$\chi^{(0)}(X,\mu;s)\equiv\chi^{(0)}(X,\mu;P,\tau,\epsilon)=\chi^{(0)}(\mu;P,\tau,\epsilon)e^{iP\cdot X}/(2\pi)^{\frac{3}{2}}, \quad (33)$$

where $P\cdot X$ is the inner product of the vectors P and X. The coefficients $\chi^{(0)}(\mu;P,\tau,\epsilon)$ are independent of X. These are the usual "spinor coefficients" of the plane wave.

Equation (31) becomes

$$\langle\rho(X)\rangle_0=-\frac{2e^2R}{(2\pi)^6}\int dX'\phi(X')\times\int\int\frac{dPdP'}{[\omega(P)+\omega(P')]}e^{i(P-P')\cdot(X-X')}\times\sum_{\tau}\sum_{\tau'}\sum_{\mu}\sum_{\mu'}[\bar{\chi}^{(0)}(\mu';P,\tau,-)\chi^{(0)}(\mu;P,\tau,-)\times\chi^{(0)}(\mu';P',\tau',+)\bar{\chi}^{(0)}(\mu;P',\tau',+)]. \quad (34)$$

It can be shown that spinor coefficients satisfy the following identity

$$\sum_{\tau}\bar{\chi}^{(0)}(\mu;P,\tau,\epsilon)\chi^{(0)}(\mu',P,\tau,\epsilon)=\frac{1}{2\epsilon\omega(P)}[\epsilon\omega(P)\delta(\mu',\mu)-\sum_{j=1}^{3}P_j\alpha_j^{\mu}(\mu',\mu)-m\beta^{\mu}(\mu',\mu)]. \quad (35)$$

In Eq. (35), $P_j(j=1,2,3)$ are the Cartesian components of the momentum P. The quantities $\alpha_i^{\mu}(\mu,\mu')$ and $\beta^{\mu}(\mu,\mu')$ are the elements of the usual Dirac matrices.

After substituting Eq. (35) into (34), we get traces of products of the matrices α_i^{μ} and β^{μ}. Using the well-known values of the traces of products of the Dirac operators such as $\mathrm{tr}\alpha_i=0$, $\mathrm{tr}\alpha_i^2=4$, etc., we obtain the result.

$$\langle\rho(X)\rangle_0=\frac{2e^4R}{(2\pi)^{9/2}}\int\int dPdP'\times\frac{[-\omega(P)\omega(P')+P\cdot P'+m^2]}{\omega(P)\omega(P')[\omega(P)+\omega(P')]}\times e^{-i(P'-P)\cdot X}\phi^F(P'-P), \quad (36)$$

where $\phi^F(P)$ is the Fourier transform of $\phi(X)$, the electrostatic potential.

$$\phi^F(P)=\frac{1}{(2\pi)^{\frac{3}{2}}}\int e^{iP\cdot X}\phi(X)dX. \quad (37)$$

It should be noted that the integrand is almost an invariant in four-dimensional space. It is convenient to introduce new variables in the integrand, namely the vector $P_1=P'-P$ and the vector P.

Then we have

$$\langle\rho(X)\rangle_{\infty}=\frac{2eR}{(2\pi)^{9/2}}\int\int dP_1dP\times\frac{[-\omega(P)\omega(P+P_1)+\omega^2(P)+P_1\cdot P]}{\omega(P)\omega(P+P_1)[\omega(P)+\omega(P+P_1)]}\times e^{-iP_1\cdot X}\phi^F(P_1). \quad (36a)$$

The result (36a) can be shown to be the same as the usual one, for if one expands $[-\omega(P)\omega(P+P_1)+\omega(P)+P_1\cdot P]/\{\omega(P+P_1)[\omega(P)+\omega(P+P_1)]\}$ in a series about $P_1=0$, one can evaluate the integral as a series, each nonvanishing term of which, (aside from the first nonvanishing term) corresponds to a term of the series obtained by Schwinger [Eq. (2.47), reference 2] for the time independent external charge. The first term of the series obtained from (36a) is the usual logarithmically-divergent charge renormalization. Hence, $\langle\rho(X)\rangle_\infty$ as obtained above is the usual result for polarization of the vacuum.

If the vector potential $A_i(X)$ is not identically zero, one similarly obtains the usual result for $\langle j_i(X)\rangle_\infty$.

For the expectation value of $\langle N\rangle_\infty$ using, say, the first initial condition (the second initial condition differs only by a factor one-half) one finds in a manner similar to that for $\langle\rho(X)\rangle_\infty$, when the vector potential $A_i\equiv 0$, the following result:

$$\langle N\rangle_\infty=-4e^2\int\int dPdP_1 \times\frac{|\phi(P_1)|^2[-\omega(P)\omega(P+P_1)+\omega^2(P)+P_1\cdot P]}{\omega(P)\omega(P+P_1)[\omega(P+P_1)+\omega(P)]^2}. \quad (38)$$

As in the case of the integral for $\langle\rho(X)\rangle_\infty$, one may evaluate the integral by expanding $[-\omega(P)\omega(P+P_1)+\omega^2(P)+P\cdot P_1]/\{\omega(P+P_1)[\omega(P)+\omega(P+P_1)]^2\}$ as a power series in P_1. In contrast to the case of the integral for $\langle\rho(X)\rangle_\infty$, the first, as well as all other, nonvanishing terms arising after the integrations are finite for suitable electrostatic potentials $\phi(X)$. Hence $\langle N\rangle_\infty$ is finite to the second order. From this result Friedrichs (reference 3) concludes $\langle N\rangle_\infty$ is finite and the canonical transformation exists to all orders. This result is true despite the fact that $\langle\mathfrak{IC}\rangle_\infty$ is infinite.

If the vector potential $A_i(X)$ is not identically zero, one obtains for $\langle N\rangle_\infty$ in addition to the term on the right-hand side of (38) the terms

$$4e^2\sum_{j=1}^{3}\int\int \times\frac{dPdP_1|A_j{}^F(P_1)|^2[+\omega(P)\omega(P+P_1)+\omega^2(P)+P_1\cdot P]}{\omega(P)\omega(P+P_1)[\omega(P)+\omega(P+P_1)]^2},$$

where $A_j{}^F(P_1)$ is the Fourier transform of $A_j(X)$. In obtaining these terms we have used Lorentz condition $\mathrm{div}A_i(X)=0$. Otherwise the derivation is similar to that used to obtain Eq. (38). On expanding $[+\omega(P)\omega(P+P_1)+\omega^2(P)+P_1\cdot P]/\{\omega(P+P_1)[\omega(P)+\omega(P+P_1)]^2\}$ in a power series in P_1 and integrating with respect to P, one obtains a divergent term for any nonzero functions $A_i(P_1)$. Hence, when the vector potential is not zero, $\langle N\rangle_\infty$ is infinite unless one uses a renormalization procedure.

Since the number operator N is invariant under a Lorentz transformation, we can generalize our result and say that for a class of time-dependent potentials $\phi(X,t)$, $A_i(X,t)$ which can be transformed by a Lorentz transformation to a suitable electrostatic potential, the expectation value $\langle N\rangle$ is finite, even though the expectation values of other operators may be infinite.