in a complicated way in the exponential function. A preliminary examination of the variational problem using the two-body potential discussed in Sec. III and using one and two parameter variational functions was not found to yield reasonable binding. The extremely singular potentials of the meson theory render a simple variational calculation rather inadequate. It is further felt that the procedures of the Sec. III are simpler and more accurate.

APPENDIX B. THREE-BODY FORCES

The three-body potential can be obtained by evaluating the matrix elements of the operator[5]

$$V=h^-(E-H_0)^{-1}h^-(E-H_0)^{-1}h^+(E-H_0)^{-1}h^+, \quad \text{(A15)}$$

where the operators h^+, h^- act to create or annihilate mesons respectively. The interaction term h we take to be (neglecting the pair terms of the pseudoscalar theory)

$$h=\sum_{i=1}^{3}\frac{f}{\mu}\boldsymbol{\sigma}_i\cdot\nabla\boldsymbol{\tau}_i\cdot\boldsymbol{\varphi}(\mathbf{r}_i). \quad \text{(A16)}$$

The evaluation of the expression for V is simple and gives

$$V=-\left(\frac{f}{\mu}\right)^4(2\pi)^{-6}\int\frac{d\mathbf{k}d\mathbf{k}'(\omega+\omega')}{\omega^3\omega'^3}\boldsymbol{\sigma}_2\cdot\mathbf{k}\boldsymbol{\sigma}_3\cdot\mathbf{k}'$$
$$\times[\mathbf{k}\cdot\mathbf{k}'\boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3-\boldsymbol{\sigma}_1\cdot\mathbf{k}\times\mathbf{k}'\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\times\boldsymbol{\tau}_3]e^{i(\mathbf{k}\cdot\mathbf{r}_{23}+\mathbf{k}'\cdot\mathbf{r}_{13})}$$
$$+(2\text{ cyclic permutations on }1,2,3). \quad \text{(A17)}$$

In taking the expectation value of the potential in an uncorrelated medium, we need consider only the contributions which arise from averaging over the angles of $\mathbf{r}_{13}$, $\mathbf{r}_{23}$, and $\mathbf{r}_{12}$. This gives

$$V=-\left(\frac{f^2}{4\pi}\right)^2\frac{1}{9}\frac{2}{\pi}\mu(\boldsymbol{\sigma}_2\cdot\boldsymbol{\sigma}_3\boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\times\boldsymbol{\sigma}_3\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\times\boldsymbol{\tau}_3)$$
$$\times\frac{e^{-x}}{x}\left[k_0(x')-\frac{k_1(x')}{x'}\right]\approx$$
$$+(\text{all permutations of }1,2,3), \quad \text{(A18)}$$

where $x=\mu r_{13}$, $x'=\mu r_{23}$.

The expectation value of this three-body potential has been evaluated using the approximate methods of Drell and Huang[2] and found to give only about $\frac{1}{10}$ Mev of repulsion. The effect hence is negligible.

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Some General Relations between the Photoproduction and Scattering of π Mesons

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With a partial wave analysis of the photomeson cross sections are combined the principle of charge independence, the hypothesis of time reversibility, and the unitarity of the scattering matrix. This leads to a natural starting point for the study of the photo cross sections. It also leads to some close relations between the photoproduction and scattering of pions in that the complex phases of the matrix elements for photoproduction are explicitly given in terms of the scattering phase shifts. One consequence of this is that there must be an S-wave contribution to the π^0 photoproduction on whose amplitude a lower limit can be given in terms of the S-wave pion-nucleon scattering. A second, independent lower limit on the S-wave term for the π^0 cross sections can be expressed in terms of the π^-/π^+ ratio. Several other nontrivial conditions are imposed on the cross sections.

I. INTRODUCTION

THE purpose of the present note is to discuss certain general relations between photomeson production and meson-nucleon scattering. These follow from the restrictions imposed by the principle of charge independence and by the usual symmetry conditions on the scattering matrix [for example, its unitarity and detailed reversibility].

These arguments (the results of which are given in Sec. II) are purely formal. They do, however, lead to a number of specific relations to be satisfied by the photo cross sections, including two independent lower limits on the S-wave term for neutral photomeson production. One of these is a function of the S-wave meson-nucleon scattering phase shifts. The S- and P-wave interference terms for the photomeson cross sections are also obtained as explicit functions of the scattering phase shifts.

Some of the formal restrictions given here have recently been obtained also by Ross.[1] His analysis was based on a rather specific model, however, whereas we emphasize their very general nature. The present results have also been obtained independently by Fermi[2]

[1] M. Ross, Phys. Rev. 94, 454 (1954). I am indebted to Dr. Ross for sending me his results prior to publication.
[2] E. Fermi (unpublished). I am indebted to Professor Fermi for informing me of his work.

and by Aizu.[3] We observe finally that our arguments could be more simply made following the S-matrix approach of Nakano and Nishijima.[4] We have chosen a different means of discussion [given in the Appendix] for the sake of physical clarity. Briefly stated, we argue that the multipole matrix elements for photoproduction are essentially *real* quantities (in an appropriate representation). Because of the meson-nucleon interaction in the final state, the phases of the various meson partial waves are shifted relative to each other by the amount of the *scattering* phase shifts. The interference of these waves in the photo cross section is thus dependent on the scattering phase shifts.

Since the photoproduction of π^- mesons from free neutrons cannot be easily done, this reaction would seem to be most easily studied using deuterium. For this reason it is necessary to say something about the role played by the deuteron binding on the cross sections. In Sec. IV we shall show that on the basis of some rather general assumptions the observed π^-/π^+ ratio from deuterium should be the same as from free neutrons and protons as long as the cross sections are being studied near the energetic threshold.

II. THE FORMAL REQUIREMENTS ON THE CROSS SECTIONS

We shall suppose the γ-ray energy to be low enough that only final S and P waves for the meson with respect to the nucleon need be considered.[5,6] We denote the momentum vectors of the photon and meson by $\boldsymbol{\kappa}$ and $\mathbf{q}$, respectively. By $\hat{e}$ we denote the polarization vector of the photon. Then, following the notation of reference 5 we may write the scattering matrix for photoproduction as

$$\begin{aligned} T=A\boldsymbol{\sigma}\cdot\hat{e}-B[-i\boldsymbol{\sigma}\cdot(\boldsymbol{\kappa}\times\hat{e})\times\mathbf{q}+(\boldsymbol{\kappa}\times\hat{e})\cdot\mathbf{q}]\kappa^{-1}q^{-1} \\ -C[i\boldsymbol{\sigma}\cdot(\boldsymbol{\kappa}\times\hat{e})\times\mathbf{q}+2(\boldsymbol{\kappa}\times\hat{e})\cdot\mathbf{q}]\kappa^{-1}q^{-1} \\ +\tfrac{1}{2}E[\boldsymbol{\sigma}\cdot\boldsymbol{\kappa}\hat{e}\cdot\mathbf{q}+\boldsymbol{\sigma}\cdot\hat{e}\boldsymbol{\kappa}\cdot\mathbf{q}]\kappa^{-1}q^{-1}, \end{aligned} \quad (1)$$

where $\boldsymbol{\sigma}$ is the nucleon spin operator. The coefficients A, B, C, and E are complex numbers which are functions of q and represent the strength of the various multipole moments for the interaction of the γ ray. The electric dipole absorption strength to produce an S-state meson is given by A. The B and C terms describe magnetic dipole absorption of the γ ray with emission of the meson into a P state, the total angular momentum being $j=\frac{1}{2}$ and $j=\frac{3}{2}$, respectively. The electric quadrupole absorption to produce a P-state meson in the $j=\frac{3}{2}$ state is represented by the E term. There is some theoretical reason[5] for feeling that the electric quadrupole term can be neglected.

[3] K. Aizu (unpublished). I am indebted to Professor C. N. Yang for informing me of Aizu's analysis, which was presented at the Japanese Conference on High Energy Physics in the Fall of 1953 (unpublished).

[4] T. Nakano and K. Nishijima, Progr. Theoret. Phys. (Japan) **8**, 53 (1952).

[5] K. Brueckner and K. Watson, Phys. Rev. **86**, 923 (1952).

[6] B. T. Feld, Phys. Rev. **89**, 330 (1953).

The four basic photomeson reactions are:

$$\begin{aligned} &\gamma+p\rightarrow\pi^++n, \\ &\gamma+n\rightarrow\pi^-+p, \\ &\gamma+p\rightarrow\pi^0+p, \\ &\gamma+n\rightarrow\pi^0+n. \end{aligned} \quad (2)$$

We shall use the superscripts "+," "−," or "0" to indicate the first three processes, respectively. The fourth will be designated by $(n\rightarrow\pi^0)$.

For each of these four processes we may expect a different matrix T of the form of Eq. (1). If we make use of the charge independence hypothesis, these T's are not independent, but satisfy the relationships[7]

$$\begin{aligned} T^+&=[\sqrt{2}t_3+(1/\sqrt{2})t_1]-\sqrt{2}S \\ T^-&=[\sqrt{2}t_3+(1/\sqrt{2})t_1]+\sqrt{2}S \\ T^0&=[2t_3-\tfrac{1}{2}t_1]+S \\ T(n\rightarrow\pi^0)&=[2t_3-\tfrac{1}{2}t_1]-S. \end{aligned} \quad (3)$$

Thus the four scattering matrices T depend upon the three quantities t_3, t_1, and S, which themselves are undetermined by the symmetry principle alone. The term S arises from nucleon recoil and leads to final meson-nucleon states of isotopic spin $I=\frac{1}{2}$, only. t_3 and t_1 describe the pure isotopic spin states $I=\frac{3}{2}$ and $I=\frac{1}{2}$, respectively, for the final state containing the meson and nucleon.

Evidently, if we combine Eqs. (1) and (3), the latter equations will represent conditions on the coefficients A, B, C, and E of Eq. (1) on introducing the appropriate notation for the t's and S. Thus let

$$t_1(j,l,\phi), \quad S(j,l,\phi)$$

describe the amplitude for producing a final $I=\frac{1}{2}$ state with angular momentum j and orbital angular momentum l. Similarly, $t_3(j,l,\phi)$ is the amplitude for a final $I=\frac{3}{2}$ state with the same total (and orbital) angular momentum. In these expressions ϕ represents the γ-ray multipole transition and for Eq. (1) may be:

$$\begin{aligned} \phi&=e_1-\text{electric dipole}, \\ &=m_1-\text{magnetic dipole}, \\ &=e_2-\text{electric quadrupole}. \end{aligned}$$

With this notation, Eqs. (3) imply that the A of Eq. (1) may be written as

$$\begin{aligned} A^+&=[\sqrt{2}t_3(\tfrac{1}{2},0,e_1)+(1/\sqrt{2})t_1(\tfrac{1}{2},0,e_1)]-\sqrt{2}S(\tfrac{1}{2},0,e_1), \\ A^0&=[2t_3(\tfrac{1}{2},0,e_1)-\tfrac{1}{2}t_1(\tfrac{1}{2},0,e_1)]+S(\tfrac{1}{2},0,e_1); \end{aligned} \quad (4)$$

A^- is obtained from A^+ and $A(n\rightarrow\pi^0)$ from A^0 by changing the sign of S [see Eqs. (3)]. B, C, and E have the same structure but involve $(j=\frac{1}{2},\ l=1,\ \phi=m_1)$, $(j=\frac{3}{2},\ l=1,\ \phi=m_1)$, and $(j=\frac{3}{2},\ l=1,\ \phi=e_2)$, respectively.

[7] K. M. Watson, Phys. Rev. **85**, 852 (1952).

The t's and S's represent twelve complex functions of q which are completely undetermined by the symmetry principles used thus far. The importance of introducing these quantities is, however, that their complex phases can be calculated explicitly in terms of the phase shift for meson-nucleon scattering in the appropriate final state. This is done in the Appendix and depends upon the assumed unitarity of the scattering matrix $[S=1+2\pi iT]$ and the principle of time reversibility for dynamical systems.

From the results in the Appendix each t and S in Eqs. (4) can be written as

$$e^{i\gamma}N,$$

where γ and N are real and γ is determined by the phase shifts for meson-nucleon scattering. (Of course, all the t's and S's are undetermined to within a common, irrelevant phase factor.) Thus

$$\begin{aligned} t_3(\tfrac{1}{2},0,e_1)&=ie^{i\alpha_3}d_3,\\ t_1(\tfrac{1}{2},0,e_1)&=ie^{i\alpha_1}d_1,\\ S(\tfrac{1}{2},0,e_1)&=ie^{i\alpha_1}\delta D,\end{aligned}\tag{5}$$

where α_3 and α_1 are the phase shifts used by Anderson, Fermi, Martin, and Nagle[8] to describe the S-wave pion scattering in the $I=\frac{3}{2}$ and $I=\frac{1}{2}$ states, respectively. The d_1, d_3, and δD are *real* functions of q, and near the energetic threshold they are constant. These refer to electric dipole absorption of the γ ray. The phase shifts α_1 and α_3 are to be evaluated at the energy of the meson and nucleon (referred to the center-of-mass system) in the final state.

The remaining t's and S's, as obtained in the Appendix, are

$$\begin{aligned} &t_3(\tfrac{3}{2},1,m_1)=e^{i\alpha_{33}}M_3^{\frac{3}{2}}, \quad t_3(\tfrac{1}{2},1,m_1)=e^{i\alpha_{31}}M_3^{\frac{1}{2}}\\ &t_1(\tfrac{3}{2},1,m_1)=e^{i\alpha_{13}}M_1^{\frac{3}{2}}, \quad t_1(\tfrac{1}{2},1,m_1)=e^{i\alpha_{11}}M_1^{\frac{1}{2}}\\ &S(\tfrac{3}{2},1,m_1)=e^{i\alpha_{13}}\delta M_1^{\frac{3}{2}}, \quad S(\tfrac{1}{2},1,m_1)=e^{i\alpha_{11}}\delta M_1^{\frac{1}{2}}\\ &t_3(\tfrac{3}{2},1,e_2)=ie^{i\alpha_{33}}E_3^{\frac{3}{2}}, \quad t_1(\tfrac{3}{2},1,e_2)=ie^{i\alpha_{13}}E_1^{\frac{3}{2}},\\ &S(\tfrac{3}{2},1,e_2)=ie^{i\alpha_{13}}\delta E_1^{\frac{3}{2}}.\end{aligned}\tag{6}$$

These α's are the P-wave phase shifts used by Fermi *et al.*[8] (The first subscript is twice the isotopic spin, the second is twice the total angular momentum.) These terms correspond to magnetic dipole absorption of the γ ray. The M's δM's, E's, and $\delta E_1^{\frac{3}{2}}$ are *real* functions of q, which vary linearly with q near the energetic threshold but are otherwise undetermined (as are d_3, d_1, and δD) by our symmetry arguments (presumably a dynamical theory is required to determine these).

We may now use Eqs. (3), (5), and (6) to write out in detail the coefficients A, B, C, and E of Eq. (1):

$$\begin{aligned} A^+&=i\left\{\sqrt{2}e^{i\alpha_3}d_3+e^{i\alpha_1}\left[\frac{d_1}{\sqrt{2}}-\sqrt{2}\delta D\right]\right\},\\ B^+&=\sqrt{2}e^{i\alpha_{31}}M_3^{\frac{1}{2}}+e^{i\alpha_{11}}\left[\frac{1}{\sqrt{2}}M_1^{\frac{1}{2}}-\sqrt{2}\delta M_1^{\frac{1}{2}}\right],\\ C^+&=\sqrt{2}e^{i\alpha_{33}}M_3^{\frac{3}{2}}+e^{i\alpha_{13}}\left[\frac{1}{\sqrt{2}}M_1^{\frac{3}{2}}-\sqrt{2}\delta M_1^{\frac{3}{2}}\right],\\ E^+&=i\left\{\sqrt{2}e^{i\alpha_{33}}E_3^{\frac{3}{2}}+e^{i\alpha_{13}}\left[\frac{1}{\sqrt{2}}E_1^{\frac{3}{2}}-\sqrt{2}\delta E_1^{\frac{3}{2}}\right]\right\},\end{aligned}\tag{7}$$

and

$$\begin{aligned} A^0&=i\{2e^{i\alpha_3}d_3-e^{i\alpha_1}(\tfrac{1}{2}d_1-\delta D)\},\\ B^0&=2e^{i\alpha_{31}}M_3^{\frac{1}{2}}-e^{i\alpha_{11}}(\tfrac{1}{2}M_1^{\frac{1}{2}}-\delta M_1^{\frac{1}{2}}),\\ C^0&=2e^{i\alpha_{33}}M_3^{\frac{3}{2}}-e^{i\alpha_{13}}(\tfrac{1}{2}M_1^{\frac{3}{2}}-\delta M_1^{\frac{3}{2}}),\\ E^0&=i\{2e^{i\alpha_{33}}E_3^{\frac{3}{2}}-e^{i\alpha_{13}}(\tfrac{1}{2}E_1^{\frac{3}{2}}-\delta E_1^{\frac{3}{2}})\}.\end{aligned}\tag{8}$$

The matrix elements A^-, B^-, C^-, and E^- for the $n\rightarrow\pi^-$ reaction are obtained from those of Eqs. (7) by changing the signs of δD, $\delta M_1^{\frac{1}{2}}$, $\delta M_1^{\frac{3}{2}}$, and $\delta E_1^{\frac{3}{2}}$. To obtain $A(n\rightarrow\pi^0)$, $B(n\rightarrow\pi^0)$, $C(n\rightarrow\pi^0)$, and $E(n\rightarrow\pi^0)$ we again change the sign of δD, $\delta M_1^{\frac{1}{2}}$, $\delta M_1^{\frac{3}{2}}$, and $\delta E_1^{\frac{3}{2}}$ in Eqs. (8).

The differential cross section, as obtained from Eq. (1) is (θ is the angle between $\mathbf{q}$ and $\boldsymbol{\kappa}$)

$$\begin{aligned}\sigma(\theta)=(q/\mu c)\sigma_0\{&|A|^2+|B|^2+\tfrac{1}{2}|C|^2(5-3\cos^2\theta)\\ &+2\,\mathrm{Im}[A^*(C-B)]\cos\theta-\mathrm{Re}[B^*C](3\cos^2\theta-1)\\ &+\mathrm{Re}[A^*E]\cos\theta-\tfrac{1}{2}\,\mathrm{Im}[E^*(B-C)](3\cos^2\theta-1)\\ &+\tfrac{1}{8}|E|^2(1+\cos^2\theta)\}.\end{aligned}\tag{9}$$

Here "Im" means "imaginary part of" and "Re" means "real part of." The factor before the bracket is chosen to make A, B, C, and E dimensionless and of order unity. For this purpose we arbitrarily take

$$\sigma_0=1.1(10)^{-2},\ \mathrm{cm}^2/\mathrm{sterad}.\tag{10}$$

An alternate form for Eq. (9) is

$$\sigma(\theta)=(q/\mu c)\sigma_0\{e_0+e_2'+e_2\cos^2\theta+e_1\cos\theta\},\tag{11}$$

where e_0 is defined to be $|A|^2$ and thus represents the S-state term, which is constant near the energetic threshold. e_2' and e_2 vary as $(q/\mu c)^2$ and e_1 as $(q/\mu c)$ near threshold.

For energies not near threshold it may be desirable to include in σ_0 the kinematical "recoil correction,"

$$\sigma_0\simeq\frac{1}{(1+\kappa/M)^2},\tag{12}$$

to the usual density of states factor.

Making use of the energy dependence, four e's in Eq. (11) can in principle be determined for each of the four basic photo cross sections. This gives a total of 16 coefficients which are expressed in terms of the 12 d's, M's, C's, δD, $\delta E_1^{\frac{3}{2}}$, and δM's.

[8] Anderson, Fermi, Martin, and Nagle, Phys. Rev. **91**, 155 (1953).

As in Eq. (9) we shall continue to denote differential cross sections by $\sigma(\theta)$. Total cross sections will be indicated by σ. We shall use $+$, $-$, 0, and $n\rightarrow\pi^0$ to designate the four reactions of expressions (2).

In Sec. IV we shall give some discussion of the photoproduction of mesons from deuterons, which may give some information concerning these reactions.

III. DISCUSSION OF THE CROSS SECTIONS

Among the implications of the analysis just given two of the most interesting have to do with the S-wave contribution to the π^0 cross sections. It is known that this is much smaller,[9,10] near threshold than is the S-wave term for the π^+ cross sections.[11] Nevertheless, we shall be able to establish two, independent lower limits on the S-wave amplitudes for producing π^0 mesons. One results from the π^-/π^+ ratio,

$$\mathcal{R}(\theta)=\sigma^-(\theta)/\sigma^+(\theta) \quad \text{or} \quad \mathcal{R}=\sigma^-/\sigma^+, \tag{13}$$

near threshold, and the other depends upon the S-wave pion-nucleon scattering amplitudes. The first depends upon the fact that $R(\theta)$ can differ from unity only if one or more of the "nucleon recoil" terms δD, $\delta M_1^{\frac{3}{2}}$, or $\delta M_1^{\frac{1}{2}}$ is different from zero. These quantities are also present, however, in the π^0 amplitudes of Eqs. (8). The second lower limit depends upon the observation that it is *impossible* to make a choice of the "parameters" d_3, d_1, and δD in such a manner that [see Eqs. (7) and (8)]

$$A^+\neq 0 \quad \text{and} \quad A^0=0$$

(unless $\alpha_1=\alpha_3$, which is certainly not the case,[8,12] except at some specific energies).

Returning to Eq. (13) we note that if

$$\delta D=0,$$

then R can differ from unity only when P-wave contributions are important and we must have

$$\mathcal{R}=1+(q/\mu c)^2 \text{ times a constant}$$

near threshold. Such a striking energy dependence should not be difficult to detect. It seems more reasonable, however, to assume that $\delta D\neq 0$, so we suppose

$$\mathcal{R}=1+\epsilon, \tag{14}$$

where ϵ is a constant (near threshold). Let us now define [see Eqs. (7) and (8)]

$$\begin{aligned} D&\equiv\sqrt{2}d_3+(1/\sqrt{2})d_1,\\ D_0&=2d_3-\tfrac{1}{2}d_1. \end{aligned} \tag{15}$$

These are real quantities which are constant near threshold. The smallness of the S-wave term in σ^0 implies that

$$D_0\ll D, \tag{16}$$

(close to threshold) so to a first approximation

$$d_1=4d_3,$$
and
$$d_3=(1/3\sqrt{2})D. \tag{17}$$

Then, in the energy range for which α_1 and α_3 are small

$$\begin{aligned} A^+&=ie^{i\alpha_1}\{D-\sqrt{2}\delta D+\tfrac{1}{3}D[-i(\alpha_1-\alpha_3)\\ &\qquad -\tfrac{1}{2}(\alpha_1-\alpha_3)^2]\},\\ A^0&=ie^{i\alpha_1}\{D_0+\delta D+(\sqrt{2}/3)D[-i(\alpha_1-\alpha_3)\\ &\qquad -\tfrac{1}{2}(\alpha_1-\alpha_3)^2]\}. \end{aligned} \tag{18}$$

The matrix elements of A^- and $A(n\rightarrow\pi^0)$ are obtained by changing the sign of δD. From this we obtain the cross sections very near threshold:

$$\begin{aligned} \sigma^+(\theta)&=(q/\mu c)\sigma_0[D-\sqrt{2}\delta D]^2,\\ \sigma^-(\theta)&=(q/\mu c)\sigma_0[D+\sqrt{2}\delta D]^2,\\ \sigma^0(\theta)&=(q/\mu c)\sigma_0[D_0+\delta D]^2,\\ \sigma(\theta)(n\rightarrow\pi^0)&=(q/\mu c)\sigma_0[D_0-\delta D]^2, \end{aligned} \tag{19}$$

which are independent of θ.

Comparison with Eq. (14) shows that

$$\delta D\simeq(\epsilon/4\sqrt{2})D. \tag{20}$$

Since ϵ is probably appreciably less than unity,[11] Eq. (20) leads us to expect that $\delta D\ll D$, and that perhaps δD and D_0 are of about the same magnitude. Knowledge of δD would also permit us to put a lower limit on the average of the π^0 cross sections [see Eqs. (19)]:

$$\tfrac{1}{2}[\sigma^0+\sigma(n\rightarrow\pi^0)]\geq(q/\mu c)\sigma_0\delta D^2\simeq\sigma^+(\epsilon^2/32)(1+\tfrac{1}{2}\epsilon). \tag{21}$$

(A more specific discussion is given as "Model I" below.)

Returning to the general equations (7) and (8), it is instructive to put a lower limit on $|A^0|^2$ by treating d_1, d_3, and δD as adjustable parameters. We minimize subject to the condition that $|A^+|^2$ is held *constant*. This leads to the *minimum* value,[13]

$$\begin{aligned} |A^0|^2/|A^+|^2\geq\tfrac{1}{2}-3\cos^2(\alpha_1-\alpha_3)\{[1+8\cos^2(\alpha_1-\alpha_3)]^{\frac{1}{2}}\\ +1+2\cos^2(\alpha_1-\alpha_3)\}^{-1}. \end{aligned} \tag{22}$$

Expression (22) increases from a zero lower limit at $\alpha_1-\alpha_3=0$ to

$$|A^0|^2/|A^+|^2\geq\tfrac{1}{2},$$

at $|\alpha_1-\alpha_3|=\pi/2$. The right-hand side of (22) is plotted in Fig. 1. Remembering that (22) represents only a lower limit, we must be prepared to expect that $|A^0|^2$ will not be at all negligible for a considerable range of γ-ray energies. Indeed, when $(\alpha_1-\alpha_3)$ is not small, Eq. (22) implies that $|A^0|^2$ and $|A^+|^2$ are of

[9] A. Silverman and M. Stearns, Phys. Rev. **88**, 1225 (1952).
[10] Goldschmidt-Clermont, Osborne, and Scott, Phys. Rev. **89**, 329 (1953).
[11] G. Bernardini, Phys. Rev. **93**, 930 (1954).
[12] Bodansky, Sachs, and Steinberger, Phys. Rev. **93**, 1367 (1954); J. Tinlot and A. Roberts, Phys. Rev. **90**, 951 (1953).

[13] Equation (22) also represents a lower limit on $\{|A^0|^2+|A(n\rightarrow\pi^0)|^2\}/\{|A^+|^2+|A^-|^2\}$, subject to the condition that the denominator is held constant.

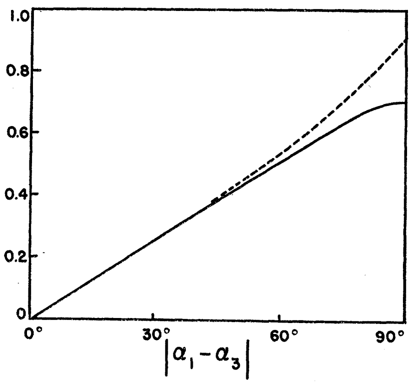

FIG. 1. The solid curve represents the square root of the expression (22). This is a lower limit on the ratio of the π^0 to the π^+ S-wave amplitudes for photoproduction. The dotted curve is the square root of the same ratio obtained from Eq. (23). These curves permit an indication of the S-wave–P-wave interference.

about the same size. This has a simple physical origin—when charge-exchange scattering (which depends upon $\alpha_1-\alpha_3$) is important, a meson which originally was produced with a charge may lose this via charge-exchange scattering before being emitted. Thus, even if there were no mechanism for directly producing π^0 mesons, they would still be emitted as a result of charge-exchange interactions.

Another estimate of $|A^0|$ can be obtained by taking D constant and $D_0=\delta D=0$ for all energies [see Eq. (15)]. This neglects any contribution from expression (21) to the S wave in σ^0, so it also may very well underestimate $|A^0|$. We now obtain

$$|A^+|^2+|A^0|^2=D^2,$$

and

$$|A^0|^2=(4/9)D^2[1-\cos(\alpha_1-\alpha_3)]. \quad (23)$$

Using these equations we again calculate $|A^0|/|A^+|$. This result is also plotted in Fig. 1. We note that for $|\alpha_1-\alpha_3|<60°$ this does not differ very much from that value obtained from the inequality (22).

An upper bound may also easily be obtained for $|A^0|/|A^+|$. This is just the reciprocal of the expression (22). Thus both upper and lower limits for this ratio are determined by the phase shifts α_1 and α_3. Not only does this hold for the electric dipole amplitudes, but all the other multipole transition terms are likewise restricted by the expression (22) and its reciprocal if we replace α_1 and α_3 by the appropriate scattering phase shifts.

As mentioned above it is reasonable to expect D_0 and δD to have comparable magnitudes. Since it is sometimes helpful to try definite models, we shall introduce[14] (to avoid confusion we shall explicitly mention it when using a model hereafter):

$$\text{Model I: } D_0=\delta D. \quad (24)$$

[14] This model was suggested to the author by G. Bernardini.

The justification for this model is that one might expect a much smaller electric dipole moment for the $(n\rightarrow\pi^0)$ process than for the $(p\rightarrow\pi^0)$ process. As we expect D_0 and δD to be comparable in magnitude we may hope that Eq. (24) is in any case qualitatively correct.

Using Eq. (24) and Eqs. (19) and (20) we would conclude that

$$\sigma^0=\sigma^+(\epsilon^2/8)(1+\tfrac{1}{2}\epsilon), \quad (25)$$

as long as only S waves need be considered and $(\alpha_1-\alpha_3)\simeq 0$. Thus the value of σ^0 could be obtained from (σ^-/σ^+) and σ^+.

The S wave for π^0 production will probably be most easily detected through its interference with the P-wave term. The interference term may be readily calculated from Eqs. (7), (8), and (9). These expressions are particularly simple if we were to assume:

Model II: The predominant P-wave term is[5] $M_3^{3/2}$. Then

$$\begin{aligned}\mathrm{Im}[A^{+*}(C^+-B^+)]=&-\sqrt{2}\{\cos(\alpha_3-\alpha_{33})\sqrt{2}d_3\\&+\cos(\alpha_1-\alpha_{33})[(d_1/\sqrt{2})-\sqrt{2}\delta D]\}M_3^{3/2},\\ \mathrm{Im}[A^{0*}(C^0-B^0)]=&-2\{\cos(\alpha_3-\alpha_{33})2d_3\\&-\cos(\alpha_1-\alpha_{33})(\tfrac{1}{2}d_1-\delta D)\}M_3^{3/2}.\end{aligned} \quad (26)$$

[For the corresponding case with electric quadrupole radiation, $M_3^{3/2}$ should be replaced by $(M_3^{3/2}-\tfrac{1}{2}E_3^{3/2})$—Eqs. (7), (8), and (9).] If the phase shift α_{33} goes through 90°, we may expect a change in the sign of these interference terms. Since the experimental $\sigma^+(\theta)$ shows constructive interference in the backward direction near threshold, we conclude that $\mathrm{Im}[A^{+*}(C^+-B^+)]$ is negative in this energy range. If Model I is valid, $\mathrm{Im}[A^{0*}(C^0-B^0)]$ is also negative. Indeed, if we combine Models I and II, we can give an explicit expression for the $p\rightarrow\pi^0$ cross section:

$$\begin{aligned}\sigma^0(\theta)=&(q/\mu c)\sigma_0\{e_0+\tfrac{1}{2}(q/\mu c)^2\\&\times[0.14][5-3\cos^2\theta]+e_1\cos\theta\},\end{aligned} \quad (27)$$

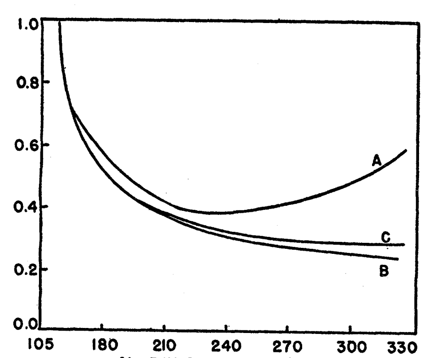

FIG. 2. The square root of the ratio of S- to P-wave contributions to the total π^0 cross section. The expression (28) is that plotted. Curves (A), (B), and (C) refer, respectively, to the value of expression (28) for the Fermi-Metropolis, the Glicksman, and the Martin scattering phase shifts, respectively, (see reference 15). These curves are essentially the ratio of S- to P-wave amplitudes.

where

$$e_0 \simeq \tfrac{1}{8}\epsilon^2(1+\tfrac{1}{2}\epsilon)+[4/9-\tfrac{1}{3}\epsilon][1-\cos(\alpha_1-\alpha_3)],$$
$$e_1 \simeq -(q/\mu c)[0.37]\{(\epsilon/\sqrt{2})\cos(\alpha_3-\alpha_{33}) - \tfrac{2}{3}\sqrt{2}[\cos(\alpha_1-\alpha_{33})-\cos(\alpha_3-\alpha_{33})]\}.$$

Equation (27) represents a combination of our two lower limits on $|A^0|$. That is, the approximate equality of expressions (22) and (23) (see Fig. 1) suggests that we may reasonably take D, D_0, and δD constant, since the dependence on $|\alpha_1-\alpha_3|$ seems to be so important. These three parameters are then fixed by σ^+ and the π^-/π^+ ratio, $R=1+\epsilon$, at threshold if we use Model I [which is not unlikely to be at least qualitatively correct]. When $|\alpha_1-\alpha_3|$ is not small, expression (22) determines the value of e_0 [actually, Eq. (23), to which e_0 essentially reduces when the ϵ terms are negligible] and is not dependent on our use of Model I. We shall discuss below the sensitivity of Eq. (27) to our use of Model II.

The numerical coefficients in Eq. (27) were obtained using Eq. (25) and the experimental[11] value of σ^+ for the S-wave amplitude and the σ^0 cross sections[9,10] at somewhat higher energies for the P-wave amplitude.

For an indication of the order of magnitude of the various terms in Eq. (27), we have plotted in Fig. 2 the quantity

$$[e_0\{0.28(q/\mu c)^2\}^{-1}]^{\frac{1}{2}}, \tag{28}$$

which is the ratio of S- to P-wave amplitudes, for the Fermi-Metropolis, Glicksman, and Martin[15] scattering phase shifts. The S-wave term is evidently not negligible.

In Fig. 3 we have plotted the angular asymmetry ratio (i.e., the ratio of the number of mesons produced in the forward hemisphere to those in the backward hemisphere), as deduced from Eq. (27) for the above three sets of scattering phase shifts.

We should now like to argue that, with the possible exception of the $[5-3\cos^2\theta]$ angular dependence, Eq. (27) may very well give us a reasonable description of the cross section $\sigma^0(\theta)$. The value of e_0 is independent of our assumption that only $M_3^{\frac{3}{2}}$ leads to P-wave production and so are Eq. (28) and the resulting curves of Fig. (2) [to the extent that the total cross section σ^0 is known experimentally].

On the other hand, the value of e_1 is dependent on our assumption that only $M_3^{\frac{3}{2}}$ is of importance—however, not as much so as might appear. Had we included the other possible multipole terms, e_1 would be replaced by a sum of terms of the form given in Eq. (27), one for each multipole and meson partial wave state. Each of these terms would be obtained by replacing α_{33} in Eq. (27) by the appropriate phase shift and modifying the coefficient [0.37] to correspond to the actual multipole strength.

At low energies, for which the phase shifts are small,

[15] E. Fermi and N. Metropolis (unpublished). R. L. Martin, Phys. Rev. **94**, 765 (1954). M. Glickmsan, reported by H. Bethe at the 1954 Rochester Conference on High Energy Physics (to be included in the proceedings of the conference).

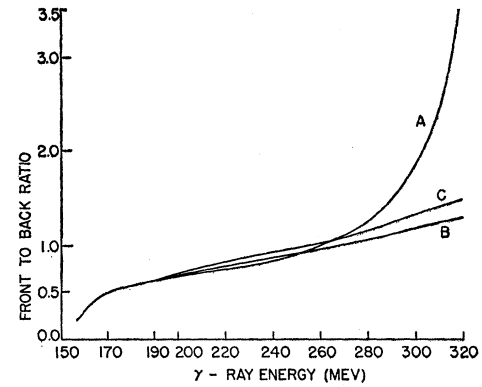

FIG. 3. The ratio of the number of π^0 mesons produced in the forward hemisphere to that in the backward hemisphere (in the center-of-mass system) is given as a function of the γ-ray energy in the laboratory system. The curves (A), (B), and (C) refer to the same sets of phase shifts as the corresponding curves of Fig. 2. This ratio was obtained from Eq. (27).

these terms will add together to give an expression of the form

$$e_1 = -(q/\mu c)[0.37](\epsilon/\sqrt{2}),$$

except for an uncertainty in the actual value of the coefficient [0.37]. e_1 will deviate from this value only when some of the phase shifts become large. To the extent that α_{33} is the important P-wave phase shift we can expect the $M_3^{\frac{3}{2}}$ term to determine then the behavior of e_1, at least qualitatively [the coefficient (0.37) may of course be modified somewhat]. The degree to which this argument is valid depends, naturally, on the actual values of the phase shifts.

We may now summarize our arguments by stating that we feel Eq. (27) to be a reasonable estimate for the actual π^0 cross section as long as higher partial waves are not important. It seems likely that the term $|A^0|$ is not much greater than its lower limit since it has not shown up experimentally. Figures 1 and 2 indicate that $|A^0|$ is large enough to be detected rather easily via its interference with the P wave. Figure 3 implies that this interference is smaller between 220 and 280 Mev than might have been thought from Fig. 2. This results from the special form of e_1 in Eq. (27). The absence of any reported asymmetry about 90° in the experimental cross sections obtained to date is probably not inconsistent with Eq. (27), but may very well prove to be incompatible with some of the suggested sets of phase shifts (see Fig. 3).

The π^+ cross sections can evidently be analyzed in the same manner. Since this has been done by Ross,[1] we shall not repeat the arguments. We do emphasize again, however, that the photo cross sections are sensitive as to the choice of scattering phase shifts and may help in making a choice between alternate sets of these.

A third model may prove useful in the analysis of the cross sections. This model would imply that we replace Eqs. (5) and (6) by [at least for positive phase shifts]:

Model III:

$$t_1(\tfrac{1}{2},0,\phi)=ie^{i\alpha_1}\frac{\sin\alpha_1}{q}d_1',$$
$$t_3(\tfrac{3}{2},1,\phi)=e^{i\alpha_{33}}\frac{\sin\alpha_{33}}{q^2}M_3'^{\frac{3}{2}},\tag{29}$$

etc., where d_1', $M_3'^{\frac{3}{2}}$, etc., are constant, real numbers. The arguments for Model III have been discussed in some detail previously.[16] Equations (29) evidently give the correct energy dependence near threshold. Also, if α_{33} describes simple resonant scattering, then $t_3(\frac{3}{2},1,\phi)$ has the form expected from the theory of resonance reactions.[16] It is not easy to justify Model III on general grounds over an extended energy range. However, it may provide a useful hypothesis for studying the cross sections in certain energy ranges.[17]

We may obviously apply our general framework for analyzing the cross sections to other models. For instance, one might try using one of the approximations to meson theory to deduce certain additional relations among our parameters.

IV. THE PRODUCTION OF CHARGED MESONS FROM DEUTERONS

It seems that the study of the (π^-/π^+) ratio, $\mathcal{R}(\theta)$ can be most easily done with a deuterium target. To interpret such results it is necessary to study the effect of the deuteron binding on $\mathcal{R}$. We shall conclude that near threshold the observed $\mathcal{R}(\theta)$ from deuterium should be the same as that from free protons and neutrons if certain general conditions are met. These conditions are subject to an experimental test.

We shall assume: (1) The electromagnetic interaction which produces a meson is the same for a nucleon bound in a deuteron as for a free nucleon. This is essentially the "impulse approximation."[18] Once produced, the meson may undergo a complicated interaction with the two nucleons, however.

(2) The energy range studied can be taken low enough that only mesons produced into S states [with respect to the nucleon from which they were produced] need be considered. We further assume that direct π^0 production with exchange scattering to produce a charged meson is negligible [partly because of the smallness of the S-wave term in π^0 production and partly because of the estimated smallness of such an exchange process at low energies]. We finally assume that the S-wave matrix element is essentially constant for the energy range involved.

(3) Coulomb forces are neglected, which means that the meson energy must be at least a few Mev.

We then write the transition matrix for producing a meson from a free nucleon [either nucleon "1" or nucleon "2"] as

$$T^{\pm}=D^{\pm}f(q)\{\tau^{(1)\pm}\boldsymbol{\sigma}_1\cdot\hat{e}\exp[i(\boldsymbol{\kappa}-\mathbf{q})\cdot\mathbf{z}_1]+\tau^{(2)\pm}\boldsymbol{\sigma}_2\cdot\hat{e}\exp[i(\boldsymbol{\kappa}-\mathbf{q})\cdot\mathbf{z}_2]\}.\tag{30}$$

The superscripts "+" or "−" refer, as usual, to the charge of the meson produced, the τ's are the appropriate isotopic spin operators, and $\mathbf{z}_1$ and $\mathbf{z}_2$ are the coordinates of the two nucleons. We have used assumption (2) above to express the dependence of T on charge by the real, dimensionless constants D^+ and D^-, which are of order unity.

Thus the ratio $\mathcal{R}$ of Eq. (13) is in this notation

$$\mathcal{R}(\theta)=\mathcal{R}=(D^-/D^+)^2.\tag{31}$$

We now define a modified operator, T_0, by

$$T_0=T^{\pm}/D^{\pm}.\tag{32}$$

With an appropriate choice of phases, we can write

$$T_0=f(q)\{\tau_x^{(1)}\boldsymbol{\sigma}_1\cdot\hat{e}\exp[i(\boldsymbol{\kappa}-\mathbf{q})\cdot z_1]+\tau_x^{(2)}\boldsymbol{\sigma}_2\cdot\hat{e}\exp[i(\boldsymbol{\kappa}-\mathbf{q})\cdot z_2]\},\tag{33}$$

which describes the production of mesons of either charge. We shall now consider the production of pions from deuterons using T_0 as the basis mechanism for their production. We suppose that all other interactions involved are charge independent [actually, we need require only charge symmetry].

Let $T_0(d)$ be the resulting transition matrix for producing charged mesons in deuterium. $T_0(d)$ is [see Eq. (33)] invariant with respect to the charge-symmetry operator R_x which represents a rotation through an angle of 180° about the x axis in charge space:

$$T_0(d)=R_x^{-1}T_0(d)R_x.$$

But R_x interchanges neutrons and protons and positive and negative mesons.[7,19] Thus,

$$T_0^+(d)=T_0^-(d),\tag{34}$$

to within a phase factor.

Since the electromagnetic interaction is weak, $T_0(d)$ is linear in T_0. Therefore, the actual matrix elements in deuterium, $T(d)$, can be obtained from $T_0(d)$ by multiplying by the appropriate constants $D^{\pm}$ [see Eq. (32)][20]:

$$T^+(d)=D^+T_0^+(d),\quad T^-(d)=D^-T_0^-(d).\tag{35}$$

This, in turn, implies that the observed π^-/π^+ ratio

[16] K. Watson, Phys. Rev. **88**, 1163 (1952).

[17] G. Bernardini (private communication), has suggested that on the basis of this model the π^-/π^+ ratio should reflect any marked energy dependence in the phase shift α_1.

[18] G. Chew and M. Goldberger, Phys. Rev. **87**, 778 (1952).

[19] N. M. Kroll and L. L. Foldy, Phys. Rev. **88**, 1177 (1952).

[20] There is an approximation involved at this point. That is, it is assumed that double charge exchange scattering is negligible. This would permit a π^- meson to become a π^+ meson after two charge exchanges, for instance. Estimates indicate that this type of process gives only about a 1 percent contribution near threshold.

from deuterium should be

$$\frac{\sigma_{\gamma+d\to\pi^-}(\theta)}{\sigma_{\gamma+d\to\pi^+}(\theta)}=\left(\frac{D^-}{D^+}\right)^2=\mathcal{R}, \tag{36}$$

which is what we set out to demonstrate.

The model implied by assumptions (1) and (2) permits detailed calculations of the cross sections $\sigma_{\gamma+d\to\pi^-}(\theta)$ and $\sigma_{\gamma+d\to\pi^+}(\theta)$ The formal method has been developed previously.[21] This involves a calculation of the multiple scattering of the meson before it "leaves" the nucleons. Estimates of the importance of this multiple scattering indicate that it may give corrections of only about 10 percent to the calculation of Chew and Lewis[22] and Lax and Feshbach,[23] who neglected it. It is thus reasonable to use the calculation of these authors to check the validity of the model. A simple consequence of this model is that $\sigma_{\gamma+d\to\pi^-}(\theta)/\sigma_{\gamma+d\to\pi^+}(\theta)$ should be *independent* of angle near threshold, as is implied by Eq. (36).

The author is indebted to Professor G. Bernardini for several discussions of his experimental program, which provided much of the stimulus for this investigation. He is indebted to Professor R. G. Sachs for several stimulating discussions of the problems treated. He is also indebted to Professor M. Gell-Mann for pointing out a different derivation than that given here as well as some additional consequences of the arguments.†

APPENDIX. DISCUSSION OF THE PHASES

To derive Eqs. (5), (6), and (12), we shall suppose that photomeson production can be developed within the general framework of quantum mechanics. We suppose the system to be described by a Hamiltonian $H+V$, where V represents the interaction of the electromagnetic field with the pertinent particles and H represents the remainder of the Hamiltonian. Evidently, photomeson production may be described as a transition between two eigenstates of H, say "a" and "b," which represent, respectively, the physical states containing a nucleon and a γ ray and a nucleon and a meson. Let the respective eigenvectors be Ψ_a and Ψ_b, which are expected to be extremely complex at small distances (possibly containing virtual heavy mesons, V particles, etc.), but upon which physical requirements impose known asymptotic forms at large distances.

We then have for the transition matrix for radiative absorption,

$$T_{ab}=(\Psi_a,V\Psi_b), \tag{A-1}$$

since V is a (weak) electromagnetic interaction. No approximation is implied concerning the states Ψ_a and Ψ_b. Since Ψ_a can be factored into a "nucleon wave function" and a "photon wave function," we can absorb the photon wave function in the definition of V in Eq. (A-1). This will permit us to consider Ψ_a as describing only a (physical) nucleon, which will simplify our subsequent discussion.

The transition matrix for photoproduction, T_{ba}, is[24]

$$T_{ba}=(\Psi_b^{(-)},V\Psi_a), \tag{A-2}$$

where $\Psi_b^{(-)}$ is related to Ψ_b by the Wigner[25] time reversal operator K:

$$i^{-2M}\Psi_b^{(-)}=K\Psi_{(-b)}. \tag{A-3}$$

Here M is the azimuthal component of the nucleon spin wave function $\mathcal{E}^M$ when referred to the direction $\mathbf{q}$ of the meson as the axis of quantization. The state "$(-b)$" is obtained from "b" by reversing the direction of all momenta and angular momenta. The choice (A-3) of phases is particularly convenient, since it implies that when Ψ_b is a plane wave,

$$\Psi_b=e^{i\mathbf{q}\cdot\mathbf{x}}\mathcal{E}^M,$$

we have

$$\Psi_b^{(-)}=\Psi_b$$

(using the customary representation for K).

We shall calculate T_{ba} [Eq. (A-2)] by breaking the state Ψ_b into partial wave eigenstates of the angular momentum l and the total angular momentum j of the meson-nucleon system. Equations (A-1) and (A-2) evidently remain valid whether Ψ_b (and $\Psi_b^{(-)}$) refer to incident plane waves or to separate eigenstates of j and l. As a matter of fact, for pure eigenstates of (j,l) which have no accidental degeneracies (which seems forbidden by physical requirements) the state Ψ_b is *unique*. This means that $\Psi_b^{(-)}$, which is also a solution of the Schrödinger equation with the same eigenvalues can differ from Ψ_b only by a constant phase factor, which can evidently be calculated from just the asymptotic form of Ψ_b. (That we can calculate this phase from just the asymptotic form of Ψ_b is perhaps the most crucial point in our analysis.)

To calculate these phases, we suppose Ψ_b to represent a pure isotopic spin state and write its explicit form for an incident plane wave (when the meson and nucleon are far apart)

$$\Psi_b(r\to\infty)=f_0^{\frac{1}{2}}(qr)\,\mathcal{E}^M+3f_1^{\frac{1}{2}}(qr)\Lambda_{\frac{1}{2}}^M +3f_1^{\frac{3}{2}}(qr)\Lambda_{\frac{3}{2}}^M+\cdots. \tag{A-4}$$

Here $\mathbf{r}$ is the relative coordinate of the meson and nucleon and $f_l^j(qr)$ is the radial wave function for scattering in the state (j,l). The Λ_j^M's are eigenstates of j and may be written

$$\begin{aligned}\Lambda_{\frac{1}{2}}^M&=\tfrac{1}{3}(\boldsymbol{\sigma}\cdot\mathbf{r}/r)(\boldsymbol{\sigma}\cdot\mathbf{q}/q)\,\mathcal{E}^M,\\ \Lambda_{\frac{3}{2}}^M&=(\mathbf{q}/q)\cdot[\mathbf{r}/r-\tfrac{1}{3}(\boldsymbol{\sigma}\cdot\mathbf{r}/r)\boldsymbol{\sigma}]\,\mathcal{E}^M.\end{aligned} \tag{A-5}$$

[21] K. Watson, Phys. Rev. **89**, 575 (1953).
[22] G. Chew and H. Lewis, Phys. Rev. **84**, 779 (1951).
[23] M. Lax and H. Feshbach, Phys. Rev. **88**, 509 (1952).
† In particular, I should like to thank Professor Gell-Mann for noting a numerical error in one of the coefficients of Eq. (9) in the original manuscript [which was also in reference 5].

[24] The state $\Psi_b^{(-)}$ contains *incoming* scattered waves for the meson, whereas Ψ_b has outgoing scattered waves. The necessity for the use of $\Psi_b^{(-)}$ in Eq. (A-2) was demonstrated in reference 16 and also by Gell-Mann and Goldberger, Phys. Rev. **91**, 398 (1953).
[25] E. P. Wigner, Göttinger Nachr. **31**, 546 (1953).

We are neglecting states with $l>1$. The f_l^j's in Eq. (A-4) are

$$f_l^j = i^l \exp(i\delta_{jl}) \frac{\sin[qr-\frac{1}{2}\pi l+\delta_{jl}]}{qr}, \tag{A-6}$$

where δ_{jl} is the appropriate scattering phase shift.

Quite evidently, we have [see Eq. (A-3)]

$$Kf_l^j = (-1)^l \exp(-2i\delta_{jl}) f_l^j, \tag{A-7}$$

which along with Eqs. (A-4) and (A-5) permits us to write down $\Psi_b^{(-)}$ in terms of Ψ_b for a pure (j,l) state.

Let us first consider only S waves, from which we can derive Eqs. (5). Then $\Psi_b = f_0^{\frac{1}{2}}(qr)\,\mathcal{E}^M$ and

$$\Psi_b^{(-)} = \exp(-2i\delta_{\frac{1}{2}0})\Psi_b. \tag{A-8}$$

Thus Eq. (A-2) becomes

$$\begin{aligned} T_{ba} &= \exp(2i\delta_{\frac{1}{2}0})(\Psi_b, V\Psi_a) \\ &= \exp(2i\delta_{\frac{1}{2}0})(\Psi_a, V\Psi_b)^* \\ &= \exp(2i\delta_{\frac{1}{2}0})(T_{ab})^*, \end{aligned} \tag{A-9}$$

since V is Hermitean.

Now, still restricting ourselves to S waves, let us write T in operator form as in Eq. (1):

$$T = \mathfrak{D}\boldsymbol{\sigma}\cdot\hat{e}. \tag{A-10}$$

Here $\mathfrak{D}$ is essentially the A of Eq. (1) except that it has a creation (or absorption) operator for one meson as a factor and it now refers to a pure isotopic spin state for the meson-nucleon system.

The postulated time-reversal invariance for our system implies[16] that

$$KTK^{-1} = T^\dagger. \tag{A-11}$$

Applied to Eq. (A-10) this gives

$$K\mathfrak{D}K^{-1} = \mathfrak{D}^\dagger,$$

or

$$\mathfrak{D} = K^{-1}\mathfrak{D}^\dagger K \tag{A-12}$$

(since $K\boldsymbol{\sigma}K^{-1} = -\boldsymbol{\sigma}$ and $K\hat{e}K^{-1} = -\hat{e}$).

We designate the eigenvectors of the occupation number operators by ω_0 and ω_1 where ω_0 describes the "existence of one nucleon" and ω_1 describes the existence of one nucleon and one meson. If we choose

$$\omega_0 = K\omega_0,$$

it follows that (if ω results from applying the creation operator to ω_0)

$$K\omega_1 = -\omega_1 \tag{A-13}$$

because of the pseudoscalar nature of the pion field.[26]

Referring to Eq. (A-10), we have

$$\begin{aligned} T_{ba} &= (\omega_1, \mathfrak{D}\omega_0)(\mathcal{E}^M, \boldsymbol{\sigma}\cdot\hat{e}\,\mathcal{E}^{M_a}), \\ T_{ab} &= (\omega_0, \mathfrak{D}\omega_1)(\mathcal{E}^{M_a}, \boldsymbol{\sigma}\cdot\hat{e}\,\mathcal{E}^M), \end{aligned} \tag{A-14}$$

where the M and M_a are nucleon spin components. The quantity $(\omega_1, \mathfrak{D}\omega_0)$ is a complex number which determines the amplitude of the transition—it is one of the quantities $t_1(\frac{1}{2},0,e_1)$, $t_3(\frac{1}{2},0,e_1)$ or $S(\frac{1}{2},0,e_1)$.

Combining Eqs. (A-9) and (A-14), we have

$$(\omega_1, \mathfrak{D}\omega_0) = \exp(2i\delta_{\frac{1}{2}0})(\omega_0, \mathfrak{D}\omega_1)^*. \tag{A-15}$$

Making use of Eqs. (A-12) and (A-13), we have

$$\begin{aligned} (\omega_1, \mathfrak{D}\omega_0) &= (\omega_1, K^{-1}\mathfrak{D}^\dagger K\omega_0) \\ &= (K\omega_1, \mathfrak{D}^\dagger K\omega_0)^* \\ &= -(\omega_1, \mathfrak{D}^\dagger\omega_0)^* \\ &= -(\omega_0, \mathfrak{D}\omega_1). \end{aligned} \tag{A-16}$$

Comparison with Eq. (A-15) shows that

$$(\omega_0, \mathfrak{D}\omega_1) = -\exp(2i\delta_{\frac{1}{2}0})(\omega_0, \mathfrak{D}\omega_1)^*,$$

or

$$-(\omega_1, \mathfrak{D}\omega_0) = (\omega_0, \mathfrak{D}\omega_1) = i\exp(i\delta_{\frac{1}{2}0}) \text{ times a real number.} \tag{A-17}$$

Since this holds for either of the pure isotopic spin states for the meson nucleon system, Eqs. (5) follow immediately on identifying $\delta_{\frac{1}{2}0}$ with the appropriate α.

Only a slight modification is required to obtain the relations (6) and (12). Referring to Eq. (A-8), let us take, for instance,

$$\Psi_b = f_1^{\frac{1}{2}}\Lambda_{\frac{1}{2}}^M; \tag{A-18}$$

then

$$\Psi_b^{(-)} = \exp(-2i\delta_{\frac{1}{2}1})\Psi_b.$$

This relation is formally the same as (A-8), so (A-9) holds also for the $j=\frac{1}{2}$, $l=1$ state. Equation (A-10) is modified in that $\boldsymbol{\sigma}\cdot\hat{e}$ is replaced by the appropriate operator from Eq. (1):

$$\begin{aligned} T = {} & \mathfrak{D}_a[i\boldsymbol{\sigma}\cdot(\boldsymbol{\kappa}\times\hat{e})\times\mathbf{q}/q + (\boldsymbol{\kappa}\times\hat{e})\cdot\mathbf{q}/q]\kappa^{-1} \\ & + \mathfrak{D}_p[-i\boldsymbol{\sigma}\cdot(\boldsymbol{\kappa}\times\hat{e})\times\mathbf{q}/q + (\boldsymbol{\kappa}\times\hat{e})\cdot\mathbf{q}/q]\kappa^{-1}. \end{aligned} \tag{A-19}$$

Here $\mathfrak{D}_a$ absorbs and $\mathfrak{D}_p$ produces the meson. Since the coefficients of $\mathfrak{D}_a$ and $\mathfrak{D}_p$ above change sign under time reversal, Eqs. (A-12) are modified to read

$$\begin{aligned} \mathfrak{D}_a &= -K^{-1}\mathfrak{D}_p^\dagger K, \\ \mathfrak{D}_p &= -K^{-1}\mathfrak{D}_a^\dagger K. \end{aligned} \tag{A-20}$$

The minus sign in these equations changes the sign of Eq. (A-16) so we now have

$$(\omega_1, \mathfrak{D}_p\omega_0) = (\omega_0, \mathfrak{D}_a\omega_1).$$

This modifies Eq. (A-17) in that the factor of i does not appear and Eqs. (6) follow immediately. Equations (12) are obtained in just the same manner as was Eq. (A-17).

We mention again, that all the terms in T are undetermined, of course, to within a *common* irrelevant phase factor. This has not appeared above because we have always made a specific choice of phases.

[26] For instance, if $\boldsymbol{\sigma}\cdot\nabla\phi$ is to remain invariant then $K\phi K^{-1} = -\phi$ since $K\boldsymbol{\sigma}K^{-1} = -\boldsymbol{\sigma}$. R. G. Sachs, Phys. Rev. **87**, 1100 (1952) has explicitly constructed K for this case.