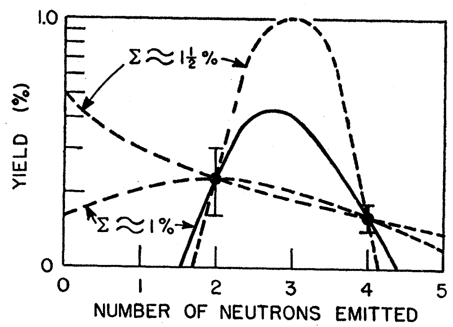

FIG. 5. Possible yield curves of antimony isotopes from reaction of μ^- mesons with iodine.

contradiction to calculations based on the Fermi-gas model which predict that the greatest possible excitation energy is 40 Mev.[1]

The 3.0 percent yield of 110-day Te^{127m} (spin 11/2)[10] relative to 5.2 percent for 9.3-hr Te^{127} (spin 3/2) is plausible when compared to slow-neutron capture by Te^{126} to form these isomers. In the latter case the spin of the target nucleus is 0 and the formation of low-spin states is thus favored. A ratio of 10 to 1 for the cross section of 9.3-hr Te^{127} relative to that of 110-day Te^{127m} is actually found.[13] On the other hand, the target nucleus in the muon reaction is I^{127} with a spin of 5/2 and, furthermore, the neutrino can escape with orbital angular momentum. Hence, it seems reasonable that the yields of the Te^{127} isomers should be more nearly equal.

[13] Seren, Friedlander, and Turkel, Phys. Rev. 72, 888 (1947).

TABLE III. Yields of μ^- reactions from iodine as read off Figs. 4 and 5.

Number of neutrons (n) and protons (p) emitted	Yield, percent
0	8.2
1n	43
2n	34
3n	11
4n	2–3
1p2n	0.23
1p3n	0.4
1p4n	0.16
Rest	<1

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Elastic Scattering of Pions by Nucleons and Pion Production in Nucleon-Nucleon Collisions*

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If the ideas about the pion-nucleon interaction discussed by Chew are taken seriously, it is important to know the accuracy of his calculation of the phase shifts for pion-nucleon scattering resulting from the second-order Tamm-Dancoff approximation. In this paper, exact solutions (found by numerical methods) of the integral equations for the scattering amplitudes resulting from the second-order Tamm-Dancoff approximation are compared with his approximate results.

Aitken, Mahmoud, and Watson have shown that a similar integral equation occurs in a theory (based on the same ideas as those discussed by Chew) of pion production in nucleon collisions. The accuracy of their approximate solution of this integral equation is discussed.

I. INTRODUCTION

As Chew[1] has stated, a theory of the pion-nucleon interaction in which the region of interaction is spread out (or in which a cutoff is introduced in momentum space) makes possible an evaluation of higher-order effects in the scattering. Even when the coupling is weak, the second-order perturbation results may be seriously in error because reactive effects in the scattering may be important.

The scattering amplitudes $(k'|K|k_0)$ satisfy an integral equation of the form[2]

$$(k'|K|k_0)=(k'|V|k_0)+\int\frac{1}{\omega_k-\omega_0}(k'|V|k)(k|K|k_0)k^2dk, \quad (1)$$

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[1] Geoffrey F. Chew, Phys. Rev. 89, 591 (1953).

[2] Throughout this paper, all masses are expressed in units of μ, all momenta in units of μc, and all energies in units of μc^2. μ is the mass of the pion (about 280 electron masses).

where k_0 is the initial momentum of the meson in the center-of-mass system and $\omega_0=(k_0^2+1)^{\frac{1}{2}}$ its initial energy. The phase shifts for pion nucleon scattering are related to $(k'|K|k_0)$ as follows:

$$\frac{1}{\pi\omega_0 k_0}\tan\delta=(k_0|K|k_0). \tag{2}$$

The construction of the potential $(k'|V|k_0)$ from quantum field theory has been discussed by Brueckner and Watson.[3] For pseudoscalar symmetric coupling, the terms in $(k'|V|k)$ proportional to f^2 are[4]

$$(k'|V|k)=\frac{1}{3\pi}f^2\lambda\frac{k'k}{(\omega_{k'}\omega_k)^{\frac{1}{2}}}\frac{1}{\omega_{k'}+\omega_k-\omega_0}, \tag{3}$$

where $\lambda=4$, -2, and -2 for the spin and isotopic spin states $(\frac{3}{2},\frac{3}{2})$, $(\frac{3}{2},\frac{1}{2})$, and $(\frac{1}{2},\frac{3}{2})$, respectively.[5] Also, f^2 is related to g^2 as follows:

$$f^2=(1/4\pi)g^2(\mu/2M)^2. \tag{4}$$

Perturbation theory takes

$$(k_0|K|k_0)=(k_0|V|k_0). \tag{5}$$

That is, Eq. (1) is solved in the Born approximation only. The reactive terms, which from the point of view of perturbation theory are terms of higher order than the second in f^2, are included in the integral in Eq. (1).

Chew showed that correcting the second-order perturbation theory by allowing for the reactive effects removes the most serious discrepancy between perturbation theory and what is supposed about the phase shifts from experimental evidence. That is, the reactive effects make the $(\frac{3}{2},\frac{3}{2})$ phase shift much larger in magnitude than the others occurring in the pion-nucleon scattering.

However, his treatment of Eq. (1) is approximate, and if his suggestions are to be taken seriously, it is important to know the accuracy of his results.

It ought not be claimed that the need for other than moderately accurate solutions (as Chew's are) is really very great, since Eq. (1) with the potential Eq. (3) diverges, so that it is necessary to cut off the integral in Eq. (1) at its upper limit or to use a source function in the integrand (we use a cutoff). Therefore Eq. (1) with Eq. (3) could not represent a final theory of the pion-nucleon scattering even if it were known that terms arising from the fourth- and higher-order Tamm-Dancoff approximations were negligible. However, because Chew's suggestions do seem to lead to results in reasonable agreement with experimental evidence, and because equations like Eq. (1) have not been studied without approximation to see what sort of approximations do give accurate results, it was decided to carry out these calculations.

Aitken, Mahmoud, and Watson[6] found that an integral equation very similar to Eq. (1) occurs in a theory of pion production in nucleon-nucleon collisions based on the same ideas as those discussed by Chew.[1] This equation has also been solved numerically in order to test the accuracy of their approximate solutions.

II. RANGE OF PARAMETERS

Four values of the upper limit for the integral in Eq. (1) are studied. This cutoff should be something like the ratio of the nucleon mass to the pion mass (1836/280=6.557), and this is one of the values used. Chew suggested that lower values might be better, and in fact in his original paper,[1] he used very low values. Other values used here are $K=3.5$, $K=4.25$, and $K=5.0$.

Values of k_0 ranging from 0.5375 to 2.0 are used (pion energy in the laboratory system from 25.48 to 272.5 (Mev). Values of f^2 from 0 to 0.6 are used.

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[3] K. Brueckner and K. Watson, Phys. Rev. **90**, 699 (1953). The existence of an integral equation like Eq. (1) depends on the possibility of constructing this potential. Equation (24) of reference 3 is

$$(k|\Omega|k_0)=\frac{1}{k_0^2}\delta(k-k_0)+\frac{1}{\omega_k-\omega_0}\int(k|V|k'')(k''|\Omega|k_0)k''^2dk''.$$

Since

$$(k'|K|k_0)=\int(k'|V|k)(k|\Omega|k_0)k''^2dk'',$$

we need only multiply Eq. (24), reference 3, by $k^2(k'|V|k)$ and integrate over k to get our Eq. (1).

[4] K. Brueckner and K. Watson, Phys. Rev. **92**, 1023 (1953). Using an expansion of the potential $(k'|V|k)$ in powers of f^2 in Eq. (1) falls into the general category known as the Tamm-Dancoff approximation. Thus in this paper we are dealing with the second order Tamm-Dancoff approximation.

[5] For the $(\frac{1}{2},\frac{1}{2})$ state, the potential has to be renormalized. See G. F. Chew, this issue [Phys. Rev. **95**, 285 (1953)]. This state is not considered in this paper. In working our Eq. (3), only the term

$$H'=(g/2M)\boldsymbol{\sigma}\cdot\nabla\boldsymbol{\tau}\cdot\boldsymbol{\phi}$$

in the nonrelativistic limit (for the nucleon) of the pseudoscalar symmetric coupling is retained. As is well known, it leads only to P-state interactions. The term

$$H'=(g^2/2M)\boldsymbol{\phi}\cdot\boldsymbol{\phi}$$

leads to an S-wave interaction, for which the potential $(k'|V|k)$ is

$$(k'|V|k)=\lambda\frac{1}{(\omega_k\omega_{k'})^{\frac{1}{2}}}.$$

With this expression for $(k'|V|k)$, which separates into a product of two functions, one a function of k' only and the other a function of k only, Eq. (1) may be solved exactly as follows: Let

$$\Lambda(k_0)=\int\frac{k'^2dk'}{\sqrt{\omega_{k'}}}\frac{1}{\omega_{k'}-\omega_0}(k'|K|k_0).$$

This is an as yet undetermined function of k_0. Equation (1) is

$$(k|K|k_0)=\frac{\lambda}{\sqrt{\omega_k\omega_0}}+\frac{\lambda}{\sqrt{\omega_k}}\Lambda.$$

Substituting this into the previous equation gives

$$\Lambda=\frac{\lambda}{\sqrt{\omega_0}}\int\frac{k^2dk}{\omega_k(\omega_k-\omega_0)}+\lambda\Lambda\int\frac{k^2dk}{\omega_k(\omega_k-\omega_0)}.$$

If

$$-M=\int\frac{k^2dk}{\omega_k(\omega_k-\omega_0)},$$

the previous equation is

$$\Lambda=-(\lambda/\sqrt{\omega_0})M-\lambda M\Lambda,$$

which is easily solved for Λ. Substituting this into the expression for $(k|K|k_0)$ above gives

$$(k_0|K|k_0)=\frac{\lambda}{\omega_0}\frac{1}{1+\lambda M}.$$

[6] Aitken, Mahmoud, and Watson, Phys. Rev. **93**, 1349 (1954).

III. METHOD OF SOLUTION

First, the range of integration in the integral in Eq. (1) is broken into two parts: $(0,2k_0)$ and $(2k_0,K)$. This required $2k_0 \leqslant K$, so that at the high energies the low cutoffs could not be used.

Equation (1) is then replaced by a system of twenty-five linear equations as follows [replacing $(k'|K|k_0)$ by $K(k')$]:

$$K(k_i)=(k_i|V|k_0)+\sum_{j=0}^{n} w_jG(k_i,k_j)K(k_j)\tfrac{1}{3}\Delta_1 +\sum_{j=n}^{24} w_jG(k_i,k_j)K(k_j)\tfrac{1}{3}\Delta_2. \quad (6)$$

In this equation,

$$\Delta_1=2k_0/n,\quad \Delta_2=(K-2k_0)/(24-n); \quad (7)$$

n is chosen to make Δ_1 and Δ_2 as nearly equal as possible. Also

$$\begin{aligned} k_i&=i\Delta_1, & i&\leqslant n;\\ k_i&=2k_0+(i-n)\Delta_2, & n&\leqslant i\leqslant 24. \end{aligned} \quad (8)$$

The w_j are weight factors corresponding to the method of numerical integration (we use Simpson's rule: $w_0=1$, $w_1=4$, $w_2=2$, $\cdots$, $w_n=1$, $w_{n+1}=4$, $\cdots$, $w_{24}=1$). The kernel $G(k_i,k_j)$ is

$$G(k_i,k_j)=\frac{1}{\omega(k_j)-\omega(k_0)}(k_i|V|k_j). \quad (9)$$

It would be a straightforward matter to solve Eq. (6) (it takes only one minute to solve a system of twenty-five linear equation on the Los Alamos MANIAC), but a singularity occurs in the kernel Eq. (9) for $j=\tfrac{1}{2}n$.

The following method was used to avoid this singularity. $K(k_0)$ through $K(k_n)$ are replaced by new unknowns:

$$Z(k_j)=K(k_j)+K(k_{n-j}),$$
$$Y(k_j)=\frac{1}{k_j-k(\tfrac{1}{2}n)}[K(k_j)-K(k_{n-j})],\quad j\leqslant\tfrac{1}{2}n. \quad (10)$$

This requires n to be an even integer.

The first sum in Eq. (6) is replaced by

$$\begin{aligned}\sum_{j=0}^{n} w_jG(k_ik_j)K(k_j) &=\tfrac{1}{2}\sum_{j=0}^{n/2} w_j[G(k_i,k_j)+G(k_i,k_{n-i})]Z(k_j)\tfrac{1}{3}\Delta_1\\ &\quad+\tfrac{1}{2}\sum_{j=0}^{n/2} w_j[G(k_i,k_j)-G(k_i,k_{n-i})]\\ &\quad\times[k_j-k(\tfrac{1}{2}n)]Y(k_j)\tfrac{1}{3}\Delta_1. \end{aligned}\quad (11)$$

If new kernels are defined as

$$\begin{aligned} G^+(k_i,k_j)&=\tfrac{1}{2}[G(k_i,k_j)+G(k_i,k_{n-i})],\\ G^-(k_i,k_j)&=\tfrac{1}{2}[k_j-k(n/2)][G(k_i,k_j)-G(k_i,k_{n-i})], \end{aligned}\quad (12)$$

Eq. (6) becomes

$$\begin{aligned} &\tfrac{1}{2}Z(k_i)+\tfrac{1}{2}[k_i-k(\tfrac{1}{2}n)]Y(k_i)\\ &=(k_i|V|k_0)+\sum_{j=0}^{n/2} w_jG^+(k_i,k_j)Z(k_j)\tfrac{1}{3}\Delta_1\\ &\quad+\sum_{j=0}^{n/2} w_jG^-(k_i,k_j)Y(k_j)\tfrac{1}{3}\Delta_1\\ &\quad+\tfrac{1}{2}G[k_i,k(\tfrac{1}{2}n)]\{Z[k(\tfrac{1}{2}n)]\\ &\quad+k(\tfrac{1}{2}n)Y[k(\tfrac{1}{2}n)]\}\tfrac{1}{3}\Delta_2\\ &\quad+\sum_{j=n+1}^{24} G(k_i,k_j)K(k_j)\tfrac{1}{3}\Delta_2,\quad i\leqslant n; \end{aligned}\quad (13)$$

$$K(k_i)=\text{same thing},\qquad i>n.$$

Since K is a regular function, so are Z and Y. It can be seen that G^+ and G^- are regular, so that the system Eq. (6) with the singular kernel Eq. (9) has been replaced by the system Eq. (13) in which everything is finite.

However, we have replaced the $n+1$ unknowns $K(k_i)$ by $n+2$ unknowns $Z(k_i)$ and $Y(k_i)$. Therefore we need an additional requirement. For this we have chosen to require that $Y(k_i)$ be continuous. We actually required that

$$Y(\tfrac{1}{2}n)=Y[\tfrac{1}{2}n-1]. \quad (14)$$

IV. RESULTS OF THE PHASE-SHIFT CALCULATION

According to Eq. (3), the result of the calculation depends on the product $f^2\lambda$, and the result also depends on the cutoff K.

Table I presents the phase shifts as a function of $f^2\lambda$, the cutoff K, and the energy $E_{\rm lab}$ in the laboratory system.

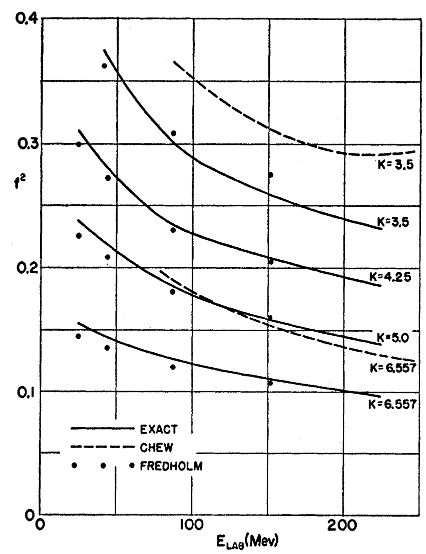

FIG. 1. The value of f^2 for which $\delta_{33}=90°$ at the energy shown on the abscissa.

TABLE I. Phase shifts in degrees as a function of the coupling constant $f^2\lambda$, the cutoff K, and the energy E_{lab}.

$\frac{1}{4}f^2$ \ K	3.5	4.25	5.0	6.577
		$E_{\text{lab}}=25.48$ Mev		
−0.6	3.316	3.191	3.104	3.066
−0.45	2.760	2.640	2.558	2.504
−0.3	2.089	1.996	1.930	1.870
−0.225	1.695	1.624	1.564	1.509
−0.15	1.236	1.187	1.154	1.105
−0.075	0.6837	0.6673	0.6509	0.6290
+0.075	−0.9079	−0.9517	−1.006	−1.181
+0.15	−2.225	−2.580	−3.256	−32.87
+0.225	−4.416	−6.643	−26.14	+1.466
+0.3	−9.112	−52.60	6.319	0.08861
+0.45	37.88	5.562	1.433	−1.356
+0.6	9.218	2.482	−0.03172	−2.645
		$E_{\text{lab}}=43.96$ Mev		
−0.6	6.678	6.413	6.288	6.102
−0.45	5.615	5.365	5.224	5.114
−0.3	4.314	4.117	3.991	3.858
−0.225	3.535	3.370	3.268	3.141
−0.15	2.606	2.503	2.424	2.330
−0.075	1.469	1.430	1.398	1.343
+0.075	−2.046	−2.163	−2.306	−2.763
+0.15	−5.232	−6.272	−8.316	43.29
+0.225	−11.23	−19.51	71.03	6.280
+0.3	−27.16	51.48	10.32	0.3145
+0.45	33.06	9.473	2.937	−2.598
+0.6	14.68	4.832	0.1999	−5.294
		$E_{\text{lab}}=88.26$ Mev		
−0.6	16.23	15.44	15.08	14.89
−0.45	13.86	13.17	12.72	12.45
−0.3	10.94	10.36	9.933	9.590
−0.225	9.122	8.626	8.266	7.920
−0.15	6.878	6.543	6.292	5.985
−0.075	3.993	3.866	3.740	3.585
+0.075	−6.124	−6.613	−7.253	−9.412
+0.15	−17.14	−22.39	−35.58	27.05
+0.225	−40.97	−80.93	38.54	
+0.3	−88.02	40.48	17.39	14.98
+0.45	38.72	17.91	7.100	−4.962
+0.6	25.46	10.96	1.893	−11.07

$\frac{1}{4}f^2$ \ K	3.5	4.25	5.0	6.577
		$E_{\text{lab}}=25.48$ Mev		
		$E_{\text{lab}}=151.7$ Mev		
−0.6	28.54	26.96	26.43	26.25
−0.45	24.79	23.19	22.53	21.95
−0.3	19.96	18.54	17.81	17.18
−0.225	16.85	15.67	14.98	14.27
−0.15	12.94	12.12	11.59	10.96
−0.075	7.689	7.358	7.093	6.740
+0.075	−12.68	−14.19	−16.00	−23.10
+0.15	−35.75	−50.76	−79.12	28.72
+0.225	−72.28	70.51	39.94	10.62
+0.3	75.67	43.44	24.23	3.828
+0.45	47.31	25.70	12.42	−5.587
+0.6	36.19	17.93	5.521	−13.93
		$E_{\text{lab}}=223.5$ Mev		
−0.6	43.10	41.3	39.97	39.78
−0.45	37.57	35.66	34.12	33.20
−0.3	30.60	28.79	26.67	25.96
−0.225	26.22	24.56	22.35	21.72
−0.15	20.49	19.27	18.03	16.83
−0.075	12.46	11.93	11.30	10.57
+0.075	−24.67	−23.63	−28.38	−45.46
+0.15	−54.59	−68.75	79.41	31.56
+0.225	−86.94	72.37	40.16	14.92
+0.3	+72.98	53.69	26.96	7.101
+0.45		37.36	14.79	−3.810
+0.6		28.86	7.393	−13.00
		$E_{\text{lab}}=272.5$ Mev		
−0.6			49.73	49.39
−0.45			41.97	41.06
−0.3			33.44	31.96
−0.225			28.44	26.77
−0.15			22.44	20.81
−0.075			14.23	13.20
+0.075			−36.67	−59.49
+0.15			76.52	34.11
+0.225			47.25	17.84
+0.3			35.01	9.570
+0.45			22.58	−1.927
+0.6			14.68	−11.21

In Fig. 1, the coupling constant f^2 required to make the $(\frac{3}{2},\frac{3}{2})$ phase shift 90° at the energy shown on the abscissa is plotted as a function of that energy for the four cutoffs considered. This graph is useful in choosing an f^2 which will give a fit to the experimental data for a prescribed cutoff.

The off diagonal scattering amplitudes $(k|K|k_0)$, $k\neq k_0$, are of interest in some calculations, for example, the photomeson production and the nucleon Compton effect. Two examples of these are shown in Fig. 2.[7]

An accurate approximation to our scattering results exists, as will be discussed now.

V. APPROXIMATIONS TO THE EXACT SCATTERING RESULTS

Suppose we have an integral equation

$$y(x)=f(x)+\lambda\int K(x,x')y(x')dx'. \tag{15}$$

[7] The author will be glad to supply tables of these to workers interested in special values of the cutoff and coupling constants.

The integral Eq. (1) is of this form. For Eq. (1) there exists a value of λ for which $y=\infty$. [More precisely, for a given cutoff K there exists a value of $f^2\lambda$ for which $(k|K|k_0)=\infty$, or for which the phase shift is 90°. We may find this value of $f^2\lambda$ from Fig. 1 if we remember that $\lambda=4$ for the $(\frac{3}{2},\frac{3}{2})$ state.] Let us call this value of λ, λ_∞.

The iteration solution of Eq. (15),

$$y(x)=f(x)+\lambda\int K(x,x')f(x')dx +\lambda^2\int K(x,x')K(x',x'')f(x'')dx''+\lambda^3\cdots, \tag{16}$$

could not converge for $\lambda>\lambda_\infty$.

There is another (perhaps well-known) expression for $y(x)$ which is related to the Fredholm theory of integral equations; namely,

$$y(x)=\frac{f(x)+\lambda\int K(x,x')f(x')dx'-\lambda f(x)\int K(x',x')dx'+\cdots}{1-\lambda\int K(x,x)dx+\cdots}, \tag{17}$$

which is formally identical with Eq. (16) provided $\lambda<\lambda_\infty$. However, Eq. (17) can still represent $y(x)$ for $\lambda>\lambda_\infty$, and probably represents it for all $\lambda<\infty$.

The point we want to make is that retaining terms to the first order in λ in the numerator and denominator in Eq. (17) provides an accurate approximation to the solutions of Eq. (1) which we have found by numerical integration.

We may write Eq. (17) as follows:

$$y(x)=\frac{f(x)\left[1+\frac{1}{f(x)}\lambda\int K(x,x')f(x')dx'-\lambda\int K(x',x')dx'\right]}{1-\lambda\int K(x',x)dx'}=\frac{f(x)(1+\lambda\Delta_I(x)-\lambda\Delta_F)}{1-\lambda\Delta_F}. \tag{18}$$

$\Delta_I(x)$ is so labeled because it results from iterating the solution in the numerator, and Δ_F is so labeled because Eq. (17) is related to the Fredholm theory of integral equations.

The value of y at only one value of $x=x_0$ [or the value of $(k|K|k_0)$ at only $k=k_0$, according to Eq. (2)] is needed to find the phase shifts. Chew's approximation for the phase shift, Eq. (1′) of reference 1, results if one takes

$$y(x_0)=f(x_0)/[1-\lambda\Delta_I(x_0)]. \tag{19}$$

He has not given an expression for the off diagonal elements.

Table II presents Δ_F and $\Delta_I(x_0)$, for which we have the following expressions:

$$\frac{2}{\lambda}\frac{1}{f^2}\Delta_I(x_0)=\frac{2}{3\pi}\omega_0\int_1^{\omega_k}d\omega\frac{k^3}{\omega_k^2(\omega_k-\omega_0)}, \tag{20}$$

and

$$\frac{2}{\lambda}\frac{1}{f^2}\Delta_F=\frac{2}{3\pi}\int_1^{\omega_k}d\omega\frac{k^3}{(2\omega_k-\omega_0)(\omega_k-\omega_0)}. \tag{21}$$

The zeros of the denominators of Eqs. (18) and (19) should reproduce Fig. 1. It is seen that Eq. (18) is very accurate in this respect, while Eq. (19) fails badly, especially for larger cutoffs.

Figure 3 compares the extent to which the two approximations reproduce the exact results for the $(\frac{3}{2},\frac{3}{2})$ phase shift as a function of energy for various coupling constants and cutoffs.

To calculate the off-diagonal scattering amplitudes, it is necessary to know $\Delta_I(k)$ for $k\neq k_0$. This quantity is tabulated in Table III for the same cutoffs and energies used in the exact calculations.

Approximate values of $(k|K|k_0)$ calculated from Eq. (18) are compared to our exact results for several

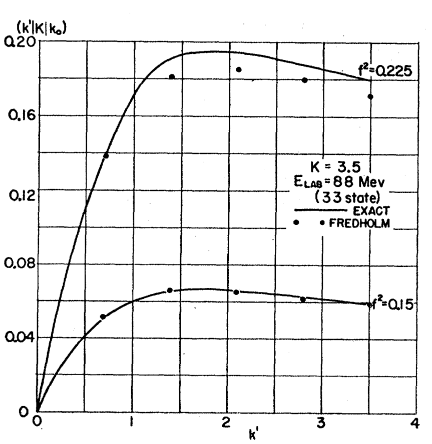

FIG. 2. $(k'|K|k_0)$ as a function of k' according to the approximations described in the text.

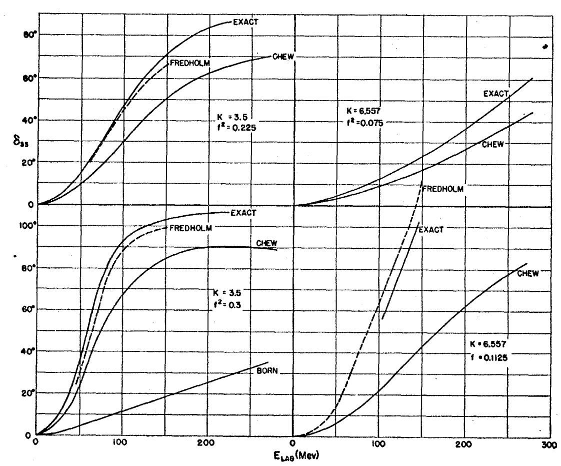

FIG. 3. δ_{33} as a function of energy for several cutoffs and coupling constants according to the approximations described in the text.

TABLE II. $2\Delta_I(k_0)/f^2\lambda$ and $2\Delta_F/f^2\lambda$ [see Eqs. (20) and (21)] as a function of the energy E_{lab} and the cutoff K.

	$K=3.5$		$K=4.25$		$K=5.0$		$K=6.557$	
E_{lab} (Mev)	$2\Delta_F/f^2\lambda$	$2\Delta_I/f^2\lambda$	$2\Delta_F/f^2\lambda$	$2\Delta_I/f^2\lambda$	$2\Delta_F/f^2\lambda$	$2\Delta_I/f^2\lambda$	$2\Delta_F/f^2\lambda$	$2\Delta_I/f^2\lambda$
25.48		0.9134	1.698	1.135	2.212	1.351	3.464	1.789
43.96	1.373	1.058	1.850	1.306	2.383	1.547	3.623	2.031
88.26	1.630	1.359	2.171	1.683	2.760	1.991	4.151	2.600
151.7	1.807	1.600	2.442	2.033	3.109	2.435	4.637	3.209
223.5		1.704		2.292		2.817		3.799
272.5		1.666		2.379		2.996		4.127

cutoffs and coupling constants in Fig. 3. Of course Eq. (18) fails quite badly near the values of f^2 and K for which either the exact or approximate K is infinite. Even in this case, the approximation Eq. (18) may be looked upon as accurate in the sense that it reproduces the exact results for a very slightly different value of f^2.

VI. PION PRODUCTION IN NUCLEON-NUCLEON COLLISIONS

Aitken, Mahmoud, and Watson[6] have discussed the three reactions

$$\begin{aligned} p+p&\to\pi^++d,\\ p+p&\to\pi^++n+p,\\ n+p&\to\pi^++n+n. \end{aligned} \tag{22}$$

In the initial state for these processes, one of the nucleons has a momentum $\mathbf{p}$ and the other a momentum $-\mathbf{p}$ (center-of-mass system). The connection between the total energy available in the center-of-mass system and p is

$$E_a=2[(p^2+M^2)^{\frac{1}{2}}-M]. \tag{23}$$

The threshold for pion production occurs at $E_a=1$, or 294 Mev in the laboratory system. The threshold for production of two pions occurs at $E_a=2$, or at 613 Mev in the laboratory system. In this calculation we cover the range of energies for which only a single meson may be produced.

In the intermediate state, the nucleon of momentum $\mathbf{p}$ emits a pion of momentum $\mathbf{k}_0$. Now the reactions may proceed in two ways, which are coherent with each other, so that the final expression for the cross sections should contain the sum of two terms squared. In the first way, the nucleons scatter to their final state momenta. In the second way, the pion emitted by the first nucleon is scattered by the second nucleon, and its momentum in the final state is $\mathbf{k}'$. Since we are going to assume that a pion and nucleon interact only in a $(\frac{3}{2},\frac{3}{2})$ state, the second way cannot occur for the third reaction, since the total charge $(T_z=\frac{1}{2})$ is not sufficient to lead to a $T=\frac{3}{2}$ state (the neutron cannot emit a π^+ particle, and the proton cannot emit a π^- particle). This means that the first two reactions should have larger cross sections than the third (this is essentially the point of the paper of Aitken, Mahmoud, and Watson).

For either way, for the first reaction, a final state interaction[8] between the nucleons leads to the formation of a deuteron. Conservation of energy and momentum demands for this reaction that

$$[k'^2+(2M)^2]^{\frac{1}{2}}-2M+(k'^2+1)^{\frac{1}{2}}=E_a-E_d, \tag{24}$$

where E_d is the binding energy of the deuteron.

This integral equation which describes the pion scattering in the intermediate state is (in the notation of reference 6)

$$(k'|t_1(\tfrac{3}{2},\tfrac{3}{2})|k_0)=\frac{k'k_0}{(\omega_{k'}\omega_0)^{\frac{1}{2}}}\frac{1}{E_a-\omega_{k'}-\omega_0}+\lambda\int\frac{k'k}{(\omega_{k'}\omega_k)^{\frac{1}{2}}}\frac{k^2dk}{E_a-\omega_{k'}-\omega_k}\frac{1}{E_a-\omega_k}\times(k|t_1(\tfrac{3}{2},\tfrac{3}{2})|k_0), \tag{25}$$

TABLE III. $2\Delta_I(k)/f^2\lambda$ as a function of the momentum k, the energy E_{lab}, and the cutoff K.

k \ E_{lab} (Mev)	25.48	43.96	88.26	151.7	223.5	272.5
			$K=3.5$			
0.0	0.8622	0.9671	1.150	1.202	0.9397	0.4454
0.7	0.9433	1.055	1.261	1.357	1.233	0.9617
1.4	1.083	1.205	1.442	1.592	1.568	1.428
2.1	1.202	1.331	1.588	1.769	1.794	1.697
2.8	1.293	1.426	1.696	1.895	1.945	1.870
3.5	1.363	1.499	1.777	1.987	2.052	1.988
			$K=4.25$			
0.00	1.064	1.181	1.400	1.507	1.335	1.198
0.85	1.218	1.347	1.602	1.776	1.776	1.636
1.70	1.456	1.599	1.897	2.139	2.254	2.233
2.55	1.645	1.796	2.120	2.399	2.568	2.591
3.40	1.786	1.943	2.281	2.582	2.780	2.824
4.25	1.892	2.053	2.401	2.715	2.929	2.986
			$K=5.0$			
0	1.260	1.387	1.634	1.780		
1	1.516	1.660	1.958	2.195	2.292	2.240
2	1.875	2.037	2.388	2.708	2.932	2.996
3	2.147	2.319	2.701	3.063	3.346	3.456
4	2.346	2.525	2.925	3.311	3.624	3.757
5	2.496	2.679	3.090	3.491	3.823	3.969
			$K=6.557$			
0.000	1.655	1.797	2.085	2.289	2.261	1.563
1.311	2.220	2.391	2.767	3.116	3.369	3.447
2.623	2.897	3.094	3.544	3.999	4.401	4.599
3.934	3.379	3.591	4.080	4.587	5.057	5.303
5.245	3.726	3.947	4.458	4.995	5.501	5.773
6.557	3.987	4.213	4.738	5.293	5.821	6.108

[8] K. M. Watson, Phys. Rev. 88, 1163 (1952).

TABLE IV. Δ [see Eq. (29)] calculated by various methods as a function of the coupling constant f^2 and the energy E_{lab} for $K=6.557$.

f^2	$E_{\text{lab}}=320$ Mev Δ_{exact}	Δ_{Fred}	$E_{\text{lab}}=350$ Mev Δ_{exact}	Δ_{Fred}	$E_{\text{lab}}=370$ Mev Δ_{exact}	Δ_{Fred}	Δ_{AMW}
0.0375	1.10	1.10	1.05	1.07	1.03	1.06	1.37
0.075	1.55	1.58	1.55	1.63	1.57	1.66	2.15
0.1125	2.71	3.05	3.05	3.68	3.40	4.28	5.04
0.15	13.8	−299	−216	−10.0	−16.1	−6.145	−14.5
0.1875	−4.16	−2.71	−2.84	−1.99	−2.31	−1.691	−2.98
0.225	−1.87	−1.30	−1.39	−1.06	−1.22	−0.95	−1.66
	$E_{\text{lab}}=410$ Mev		$E_{\text{lab}}=500$ Mev		$E_{\text{lab}}=600$ Mev		
0.0375	0.99	1.04	0.93	1.00	0.89	0.96	1.37
0.075	1.63	1.75	1.82	1.99	2.27	2.33	2.15
0.1125	5.17	7.25	36.6	−58.3	−4.28	−5.06	5.04
0.15	−5.34	−3.46	−2.00	−1.79	−1.09	−1.18	−14.5
0.1875	−1.66	−1.31	−0.96	−0.89	−0.87	−0.67	−2.98
0.225	−0.97	−0.78	−0.63	−0.58	−9.43	−0.46	−1.66

where $\lambda=4f^2/3\pi$. This is Eq. (33) of reference 6. The only difference between it and our Eq. (1) is that the sign of K and t are opposite, and the connection between E_a and k_0 is different. For pion-nucleon scattering, $E_a=\omega_0$, whereas for pion production in nucleon-nucleon collisions there is no strict connection between the two.

Aitken, Mahmoud, and Watson wrote the solution of Eq. (25) in the form

$$(k'|t_1(\tfrac{3}{2},\tfrac{3}{2})|k_0)=\frac{\lambda k'k_0}{(\omega_{k'}\omega_0)^{\frac{1}{2}}}\frac{1}{E_a-\omega_{k'}-\omega_0}\Delta. \quad (26)$$

That is, they took the inhomogeneous term (or Born's approximation) and multiplied it by a numerical factor Δ which is supposed to allow for the reactive effects. Then they made further approximations based on the following observations about the magnitudes of the quantities k' and k_0 which are especially appropriate near threshold ($E_a=1$).

Since both nucleons are brought nearly to rest in these reactions, the first nucleon must emit a pion of momentum nearly equal to $\mathbf{p}$, coming nearly to rest in so doing. Thus,

$$k_0\approx p. \quad (27)$$

From Eq. (23), we see that $p\approx M^{\frac{1}{2}}$ for $E_a\approx 1$. Assuming $M\gg 1$, we have then $p\approx\omega_p$, or $k_0\approx\omega_0\approx M^{\frac{1}{2}}$. The second nucleon scatters this pion into a final momentum state $\mathbf{k}'$, the second nucleon having a momentum $-\mathbf{k}'$ in the final state. Energy has to be conserved in the over-all process, so that

$$(k'^2+M^2)^{\frac{1}{2}}-M+(k'^2+1)^{\frac{1}{2}}=E_a. \quad (28)$$

Of course, for the first reaction, k' is strictly connected with E_a through Eq. (24). Near threshold ($E_a=1$)k' will be small, so that $\omega_{k'}\approx 1\approx E_a$.

With these assumptions, Eq. (26) becomes

$$(k'|t_1(\tfrac{3}{2},\tfrac{3}{2})|k_0=-\lambda k'M^{-\frac{1}{2}}\Delta. \quad (29)$$

Aitken, Mahmoud, and Watson then found the following approximate value of Δ, namely:

$$\Delta=(1-4f^2M^{\frac{1}{2}}K_c/3\pi)^{-1}, \quad (30)$$

where K_c is the momentum-space cutoff used in the calculation. With Eqs. (29) and (30) they evaluated the cross sections for the three reactions Eq. (22). The results are given in their Eqs. (46), which we reproduce for convenience,

$$\sigma[p+p\to\pi^++d]=\sigma_0\left[\frac{\nu/M}{1-\nu r_0}\right]q_{\text{deut}}^3[1+(4/9)M^{-\frac{1}{2}}\Delta]^2,$$

$$\sigma[p+p\to\pi^++n+p]=\frac{3\sqrt{2}}{8}\sigma_0T_0^2M^{-\frac{1}{2}}[1+(4/9)M^{-\frac{1}{2}}\Delta]^2, \quad (31)$$

$$\sigma[n+p\to\pi^++n+n]=\frac{3\sqrt{2}}{16}\sigma_0T_0^2M^{-\frac{1}{2}},$$

where $\sigma_0=2\pi[4f^2]^3[M^{5/2}]$. The value of q_{deut} appearing in the first of these being the value of k' resulting from our Eq. (24). $\nu=(ME_d)^{\frac{1}{2}}$ and r_0 is the ^3S effective range for low-energy $n-p$ scattering. As we expected, Eqs. (31) contain the sum of two terms squared for the first two reactions, the first term representing a nucleon-nucleon scattering in the intermediate state, and the

TABLE V. Quantities useful in calculating R [see the equation following Eq. (32)].

E_{lab} (Mev)	320	350	370	410	500	600
E_a	1.082	1.179	1.244	1.372	1.656	1.966
p	2.718	2.843	2.923	3.077	3.398	3.722
q	0.3835	0.5764	0.6793	0.8561	1.184	1.494
q_{deut}	0.3558	0.5701	0.6818	0.8724	1.227	1.565
Δ_F	3.344	3.565	3.705	3.970	4.493	4.961
$\Delta_I(q)$	2.822	3.085	3.254	3.575	4.228	4.780
T_0	0.07621	0.1658	0.2249	0.3416	0.5972	0.8973

second (which contains Δ) representing a pion-nucleon scattering in the intermediate state. As we further expected, the possibility of a pion-nucleon scattering is absent for the third reaction (there is no term containing Δ). Their work leading to Eq. (31) involved integration over the momentum k', since they allowed for the fact that Eq. (28) does not really hold for the second reactions, but a spectrum of values of k' is possible, the upper limit of the spectrum of pion kinetic energies being given by

$$T_0+1+2\{[\tfrac{1}{4}(T_0^2+2T_0)+M^2]^{\frac{1}{2}}-M\}=E_a. \quad (32)$$

In a sense our calculation provides a value of their Δ. Our exact solutions of Eq. (25) will not be of the form of Eq. (29) with Eq. (30). However, in order to avoid repeating the work of Aitken, Mahmoud, and Watson, we have evaluated Δ using Eq. (29) and our exact solutions for k' given by Eq. (28) and k_0 given by Eq. (27).

The values of Δ found in this way are shown in Table IV, and compared with the values found in a similar way from the Fredholm approximation and the values found from the approximation of Aitken, Mahmoud, and Watson. A table of quantities useful in calculating the Fredholm approximation and the cross sections is given in Table V.

With these values of Δ, we may calculate

$$R=\frac{\sigma[p+p\rightarrow\pi^++d]+\sigma[p+p\rightarrow\pi^++n+p]}{[n+p\rightarrow\pi^++n+n]}$$

as a function of energy. As mentioned before, it was the point of the work of Aitken, Mahmoud, and Watson

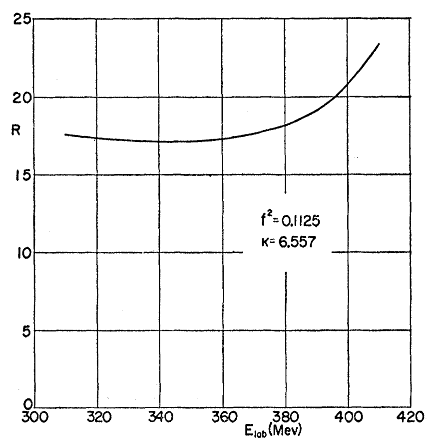

FIG. 4. The ratio R as a function of energy.

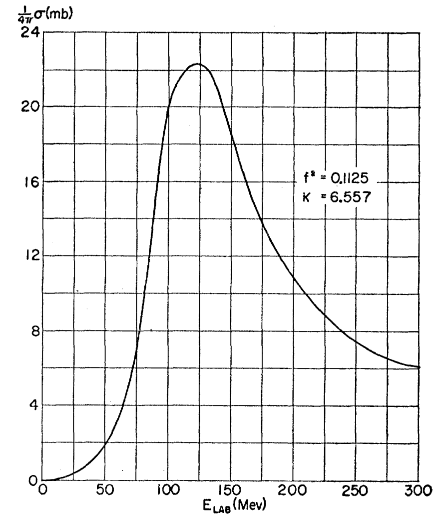

FIG. 5. The π^+-p total cross section as a function of energy.

that this should be much greater than one. The results of this calculation are shown in Fig. 4 for $f^2=0.1125$.

VII. CONCLUDING REMARKS

Whether or not Chew's ideas correspond in any way to reality cannot be decided until the fourth-order Tamm-Dancoff corrections have been calculated. It will be necessary also to calculate the effects of nucleon recoil and the effects of other terms in the nonrelativistic limit of the pseudoscalar symmetric coupling.

However, this calculation has several encouraging features. The first is that according to Fig. 1, less coupling constant is required than would have been supposed from Chew's approximate calculation to make the $(\frac{3}{2},\frac{3}{2})$ phase shift 90° at a given energy. Secondly, with Chew's approximation, it is difficult to make the π^+-p total cross section decrease sufficiently rapidly beyond its peak to fit the experiments.

In Fig. 5, a graph of the π^+-p total cross section is shown which indicates this difficulty may not be present with the exact solutions.

VIII. ACKNOWLEDGMENTS

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