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Born-Type Rigid Motion in Relativity

G. Salzman and A. H. Taub
Physics Department and Mathematics Department, University of Illinois, Urbana, Illinois
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Rigid motions of collections of particles as defined by Born, Herglotz, and Noether are studied in special and general relativity. An approximate method for solving the field equations of general relativity, the equations of motion of a perfect fluid undergoing a rigid motion, and the boundary condition $p=0$ is described and the first-order solution for the first two sets of equations is obtained. It is shown that in the classical limit the boundary conditions may be satisfied to this order.

1. INTRODUCTION

To generalize the classical concept of rigid motion, Born[1] introduced a definition consistent with special and general relativity. As formulated by Herglotz[2] it states: For each pair of nearby particles of the body the orthogonal interval between the corresponding pair of world lines is constant during the motion. The orthogonal interval is the distance between the two world lines, measured along an infinitesimal hyperplane orthogonal to both lines in the sense of four dimensions, and calculated with the line element of the space time. It is shown in Fig. 1 in a Minkowski-type diagram, with one of the spatial dimensions suppressed.

Herglotz and Noether[3] then proved that in special relativity every Born-type rigid motion belongs to at least one of two classes. The world lines are either the orthogonal trajectories of a continuous one-parameter family of space-like hyperplanes, or segments of path curves of a continuous one-parameter family of Lorentz transformations of the four-dimensional Minkowski space-time R_4 into itself. We label these classes *plane motions* and *group motions*. Some motions belong to both classes. Any time-like curve determines one plane motion, but in order to determine a group motion the curve must be restricted.

2. NOTATION, DEFINITIONS, AND ASSUMPTIONS

A point (or event) is specified by four numbers x^α, the coordinate values of that point in the x^α-coordinate system. Lower case Greek indices have the range 1,2, 3, 4; Latin ones 1, 2, 3. The usual summation convention is used. The motion of each particle is described by parametric equations,

$$x^\alpha=\psi^\alpha(\xi^\nu,\theta), \tag{1}$$

where ξ^ν are parameters which label the particles and θ is any convenient *time-like* parameter, e.g., the time x^4 in the x^α-coordinate system. The points on a world line are obtained from (1) by holding ξ^ν fixed and allowing θ to run through its range of values. For any ξ^ν it is assumed that θ and $\psi^4(\xi^\nu,\theta)$ are monotone increasing functions of each other. Let θ_I be the smallest value that θ assumes; call it the initial value. The parameters ξ^ν are chosen to satisfy the equations

$$\psi^\alpha(\xi^\nu,\theta_I)=\xi^\alpha. \tag{2}$$

The particle located at ξ^α when $\theta=\theta_I$ is labeled by ξ^i, the *initial* values of its spatial coordinates x^i. To each particle ξ^i corresponds the *initial* value ξ^4 of x^4 for that particle, i.e.,

$$\xi^4=\sigma(\xi^i). \tag{3}$$

Substituting (3) into (1), we get

$$x^\alpha=\psi^\alpha(\xi^i,\sigma(\xi^i),\theta)$$

or simply,

$$x^\alpha=x^\alpha(\xi^i,\theta). \tag{4}$$

There is thus a three-parameter family $\{C_{\xi^i}\}$ of world lines for a motion of a system of particles. The *initial* hypersurface from which the world lines start, which we call the σ-hypersurface.

$$x^4=\sigma(x^i), \tag{5}$$

is assumed to be spacelike. All necessary differentiability of the functions $x^\alpha(\xi^i,\theta)$ and $\sigma(\xi^i)$ is assumed. Also, Eqs. (4) are assumed to define a nonsingular transformation between the coordinates x^α and the coordinates ξ^i,θ. The x^α-coordinate system has a metric tensor $g_{\alpha\beta}(x^\nu)$, however, in special relativity the coordinates will, unless otherwise specified, be chosen so as

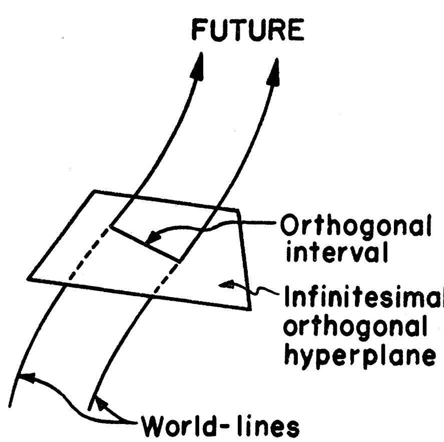

Fig. 1. The orthogonal interval between nearby particles. One of the spatial dimensions is suppressed.

[1] M. Born, Ann. Physik **30**, 1 (1909).
[2] G. Herglotz, Ann. Physik **31**, 393 (1909–1910).
[3] F. Noether, Ann. Physik **31**, 919 (1909–1910).

to have the metric tensor $g_{\alpha\beta}=\eta_{\alpha\beta}$, where

$$\eta_{11}=\eta_{22}=\eta_{33}=-\eta_{44}=1,$$
$$\eta_{\alpha\beta}=0 \quad \text{if} \quad \alpha\neq\beta, \tag{6}$$

i.e., they will be Galilean coordinates. The signature will always be taken as $(+++-)$, i.e., space-like intervals will be positive.

For a family $\{C_{\xi^i}\}$ of world lines let $\mathcal{T}$ be the totality of those points of the space time that lie on world lines. At each point x^α of $\mathcal{T}$, a time like vector is defined by

$$U^\alpha=\partial x^\alpha(\xi^i,\theta)/\partial\theta=x^\alpha(\xi^i,\theta)_{,\theta}. \tag{7}$$

Because of the monotone relationship between $x^4(\xi^i,\theta)$ and θ, U^α points towards the future, i.e., $U^4>0$. Corresponding to this four-velocity vector field $U^\alpha(x^\nu)$ defined over $\mathcal{T}$, and to the choice of signature, is the unit vector field $u^\alpha(x^\nu)$ defined by

$$u^\alpha=(-U_\beta U^\beta)^{-\frac{1}{2}}U^\alpha, \tag{8}$$

where $U_\beta=g_{\alpha\beta}U^\alpha$. It follows from the definition that

$$u_\alpha u^\alpha=-1. \tag{9}$$

$u^\alpha(x^\nu)$ is the normalized four-velocity vector field. $U^\alpha(x^\nu)$ is an unnormalized four-velocity vector field. From Eq. (9) and from the fact that the $g_{\alpha\beta}$'s behave as constants with respect to covariant differentiation, we get

$$u_{\alpha;\beta}u^\alpha=0, \tag{10}$$

where $u_{\alpha;\beta}$ is the covariant derivative of $u_\alpha(x^\nu)$ with respect to x^β.

It is convenient to introduce a proper time s, which is defined by specifying its value on any hypersurface which intersects each of the world lines exactly once and by the equation

$$ds/d\theta=(-U_\beta U^\beta)^{\frac{1}{2}}. \tag{11}$$

It is assumed that a nonsingular transformation exists between the coordinates ξ^i,θ and the coordinates ξ^i,s. Equations (7), (8), and (11) imply that

$$\partial x^\alpha/\partial s=x^\alpha{}_{,s}=u^\alpha.$$

3. DERIVATION OF THE DIFFERENTIAL EQUATIONS OF BORN-TYPE RIGID MOTION

The discussion in this section is the same for flat and nonflat space times, and $g_{\alpha\beta}$ is used for the metric tensor to represent both cases. The infinitesimal displacement vector from the point $x^\alpha(\xi^i,\theta)$ on the world line C_{ξ^i} to the point $x^\alpha(\xi^i+d\xi^i,\theta+d\theta)$ on the nearby world line $C_{\xi^i+d\xi^i}$ is

$$dx^\alpha=\frac{\partial x^\alpha(\xi^i,\theta)}{\partial\xi^i}d\xi^i+x^\alpha(\xi^i,\theta)_{\theta},d\theta \tag{12}$$
$$=x^\alpha{}_{,i}d\xi^i+U^\alpha d\theta.$$

The component of dx^α parallel to $u^\alpha(\xi^i,\theta)$ is $(-dx^\beta u_\beta)u^\alpha$. Therefore the orthogonal displacement from C_{ξ^i} to $C_{\xi^i+d\xi^i}$ at $x^\alpha(\xi^i,\theta)$ is $dx^\alpha-(-dx^\beta u_\beta)u^\alpha$. Its length squared is

$$dl^2=g_{\alpha\beta}[dx^\alpha+dx^\nu u_\nu u^\alpha][dx^\beta+dx^\lambda u_\lambda u^\beta]$$
$$=g_{\alpha\beta}dx^\alpha dx^\beta+u_\nu dx^\nu u_\lambda dx^\lambda=(g_{\alpha\beta}+u_\alpha u_\beta)dx^\alpha dx^\beta. \tag{13}$$

Note the identity

$$(g_{\alpha\beta}+u_\alpha u_\beta)U^\alpha=0. \tag{14}$$

Substituting for dx^α from (12) into (13), and taking account of (14), we get for the square of the orthogonal interval

$$dl^2=(g_{\alpha\beta}+u_\alpha u_\beta)x^\alpha{}_{,i}x^\beta{}_{,j}d\xi^i d\xi^j. \tag{15}$$

If we regard the functions $g_{\nu\lambda}$, u_ν, and $x^\nu{}_{,k}$ in (15) as functions of ξ^k and θ, the statement that the motion is a Born-type rigid motion means that for all points ξ^i,θ in $\mathcal{T}$ and for all infinitesimal $d\xi^k$, dl is independent of θ, or equivalently,

$$[(g_{\alpha\beta}+u_\alpha u_\beta)x^\alpha{}_{,i}x^\beta{}_{,j}]_{,\theta}=0. \tag{16}$$

Theorem: Equation (16) is equivalent to the covariant equation

$$u_{\mu;\nu}+u_{\nu;\mu}+u_{\mu;\lambda}u^\lambda u_\nu+u_{\nu;\lambda}u^\lambda u_\mu=0 \tag{17}$$

in which the functions $g_{\nu\lambda}$ (which enter in the covariant differentiation), and u_ν are regarded as functions of x^α.

Proof: From (11) it follows, since $U_\beta U^\beta\neq 0$, that Eq. (16) is equivalent to

$$[(g_{\alpha\beta}+u_\alpha u_\beta)x^\alpha{}_{,i}x^\beta{}_{,j}]_{,s}=0. \tag{18}$$

If some dummy indices are changed, Eq. (18) becomes

$$[(g_{\alpha\mu}+u_\alpha u_\mu)u^\mu{}_{,\beta}+(g_{\beta\mu}+u_\beta u_\mu)u^\mu{}_{,\alpha}$$
$$+(g_{\alpha\beta,\mu}+u_{\alpha,\mu}u_\beta+u_\alpha u_{\beta,\mu})u_\mu]x^\alpha{}_{,i}x^\beta{}_{,j}=0.$$

Replacing $u_{\alpha,\mu}$ by $g_{\alpha\lambda,\mu}u^\lambda+g_{\alpha\lambda}u^\lambda{}_{,\mu}$; $g_{\alpha\beta,\gamma}$ by $g_{\mu\beta}\{{}_{\alpha\gamma}{}^\mu\}+g_{\alpha\mu}\{{}_{\beta\gamma}{}^\mu\}$, where $\{{}_{\beta\gamma}{}^\alpha\}$ are the Christoffel symbols of the second kind; changing dummies, collecting terms, using the fact that the covariant derivatives of $g_{\alpha\beta}$ vanish, and taking account of (10) we get

$$D_{\alpha\beta}x^\alpha{}_{,i}x^\beta{}_{,j}=0, \tag{19}$$

where

$$D_{\alpha\beta}=u_{\alpha;\beta}+u_{\beta;\alpha}+u_{\alpha;\lambda}u^\lambda u_\beta+u_{\beta;\lambda}u^\lambda u_\alpha.$$

Note the identities

$$D_{\alpha\beta}u^\alpha=0, \tag{20a}$$
$$D_{\alpha\beta}u^\beta=0. \tag{20b}$$

The Jacobian of the transformation from the coordinates x^α to the coordinates ξ^i,s is the determinant of the matrix $\|J^\alpha{}_\lambda\|$, where $J^\alpha{}_i=x^\alpha{}_{,i}$ and $J^\alpha{}_4=x^\alpha{}_{,s}=u^\alpha$. Equations (19), (20a), and (20b) imply that

$$D_{\alpha\beta}J^\alpha{}_\lambda J^\beta{}_\nu=0. \tag{21}$$

But by assumption $\|J^\alpha{}_\lambda\|$ is a nonsingular matrix. Therefore $D_{\alpha\beta}=0$, i.e., (16) implies (17). The reversibility of the steps in the proof insures that (17) implies (16), and therefore the two equations are equivalent.

There are only six independent equations contained in (17). This follows from the symmetry in μ and ν and from the fact that the equation obtained by multiplying (17) by u^μ is satisfied identically.

Equation (17) may be written in the equivalent form

$$(u_{\alpha;\beta}+u_{\beta;\alpha})(\delta^\alpha{}_\mu+u^\alpha u_\mu)(\delta^\beta{}_\nu+u^\beta u_\nu)=0, \qquad (22)$$

where $\delta^\alpha{}_\mu$ is the Kronecker delta. If we define

$$P^\alpha=(\delta^\alpha{}_\mu+u^\alpha u_\mu)V^\mu,$$

where V^μ is an arbitrary vector, then P^α is an arbitrary vector orthogonal to u^α. (22) may then be written in the equivalent form

$$u_{\alpha;\beta}P^\alpha P^\beta=0. \qquad (23)$$

(22) is also equivalent to

$$(U_{\alpha;\beta}+U_{\beta;\alpha})(\delta^\alpha{}_\mu+u^\alpha u_\mu)(\delta^\beta{}_\nu+u^\beta u_\nu)=0. \qquad (24)$$

This can be seen as follows. Let

$$e^{2\psi}=-U_\alpha U^\alpha \qquad (25)$$

define ψ. Note that $e^\psi\neq 0$. Then

$$u_\alpha=e^{-\psi}U_\alpha. \qquad (26)$$

Differentiation of Eq. (25) gives

$$\psi_{,\beta}=-e^{-\psi}U_{\alpha;\beta}u^\alpha. \qquad (27)$$

Differentiating Eq. (26) and using (26) and (27), we get

$$u_{\alpha;\beta}=e^{-\psi}U_{\mu;\beta}(\delta^\mu{}_\alpha+u^\mu u_\alpha). \qquad (28)$$

Using the relation

$$(\delta^\sigma{}_\mu+u^\sigma u_\mu)(\delta^\mu{}_\alpha+u^\mu u_\alpha)=(\delta^\sigma{}_\alpha+u^\sigma u_\alpha),$$

we obtain from (28)

$$(u_{\alpha;\beta}+u_{\beta;\alpha})(\delta^\alpha{}_\mu+u^\alpha u_\mu)(\delta^\beta{}_\nu+u^\beta u_\nu) =e^{-\psi}(U_{\alpha;\beta}+U_{\beta;\alpha})(\delta^\alpha{}_\mu+u^\alpha u_\mu)(\delta^\beta{}_\nu+u^\beta u_\nu). \qquad (29)$$

Since $e^{-\psi}\neq 0$, either (22) and (24) are each satisfied or they are each not satisfied by any given vector field $U_\alpha(x^\nu)$. They are thus equivalent.

4. THE RESULTS OF HERGLOTZ AND NOETHER

As already mentioned, Herglotz and Noether proved that in special relativity every Born-type rigid motion is either a plane motion or a group motion, or both.

A four-velocity field of a plane motion in special relativity can be characterized as follows. Let C_0 be a world line of the motion. C_0 is given by equations

$$x^\alpha=x^\alpha(\theta). \qquad (30)$$

A four-velocity is determined along C_0, namely,

$$U^\alpha(\theta)=x^\alpha(\theta)_{,\theta}. \qquad (31)$$

The hyperplane which intersects C_0 orthogonally at $x^\alpha(\theta)$ consists of those points x^ν which satisfy

$$U_\nu(\theta)[x^\nu-x^\nu(\theta)]=0. \qquad (32)$$

Equation (32) can be regarded as assigning a value of θ to each point x^ν of $\mathcal{T}$, i.e., it defines a function $\theta(x^\nu)$. The vector field defined by

$$U^\alpha(x^\nu)=U^\alpha[\theta(x^\nu)], \qquad (33)$$

where $\theta(x^\nu)$ is defined by Eq. (32), is a four-velocity vector field for the plane motion.

An important property of plane motions is that the local angular velocity four-vector a^μ vanishes identically, i.e.,

$$a^\mu=\tfrac{1}{2}(-g)^{-\frac{1}{2}}\epsilon^{\mu\alpha\beta\gamma}u_\alpha u_{\beta;\gamma}=0, \qquad (34)$$

at all x^ν in $\mathcal{T}$, where g is the determinant of the matrix $\|g_{\alpha\beta}\|$, and $\epsilon^{\mu\alpha\beta\gamma}$ is the contravariant Levi-Civita tensor density, which equals $+1$ or -1 according as $\mu\alpha\beta\gamma$ is an even or odd permutation of the natural order 1, 2, 3, 4, and zero otherwise. Gödel showed[4] that a^μ represents the local angular velocity of matter relative to the compass of inertia, i.e, relative to the direction of the local proper time.

A four-velocity vector field of a group motion can be written

$$U^\alpha(x^\nu)=F^\alpha{}_\nu x^\nu+A^\alpha, \qquad (35)$$

where $F^\alpha{}_\nu=F^{\alpha\lambda}\eta_{\lambda\nu}$, $F^{\alpha\lambda}=-F^{\lambda\alpha}$, $A^\alpha{}_{,\beta}=0$, and $F^{\alpha\lambda}{}_{,\beta}=0$.

We now verify that any vector field defined by (33) or by (35) satisfies (24). Differentiation of Eq. (32) gives

$$U_\nu(\theta)_{,\theta}\theta_{,\beta}[x^\nu-x^\nu(\theta)]+U_\nu(\theta)[\delta^\nu{}_\beta-U^\nu(\theta)\theta_{,\beta}]=0;$$

or, solving for $\theta_{,\beta}$,

$$\theta_{,\beta}=-\{U_\nu(\theta)_{,\theta}[x^\nu-x^\nu(\theta)]-U_\nu(\theta)U^\nu(\theta)\}^{-1}U_\beta(\theta). \qquad (36)$$

Differentiation of Eq. (33) and use of (36) gives

$$U_{\alpha,\beta}=U_\alpha(\theta)_{,\theta}\{U_\nu(\theta)_{,\theta}[x^\nu(\theta)-x^\nu] +U_\nu(\theta)U^\nu(\theta)\}^{-1}U_\beta(\theta). \qquad (37)$$

It follows from (14) that the $U_{\alpha,\beta}$ given by (37) satisfies

$$U_{\alpha,\beta}(\delta^\beta{}_\nu+u^\beta u_\nu)=0.$$

From this equation and the fact that $U_{\alpha;\beta}=U_{\alpha,\beta}$ in a Galilean coordinate system it follows that the U_α given by (33) satisfies (24). For a group motion we have, from (35) and from the antisymmetry of $F_{\alpha\beta}$,

$$U_{\alpha,\beta}+U_{\beta,\alpha}=F_{\alpha\beta}+F_{\beta\alpha}=0. \qquad (38)$$

Thus the U_α given by (35) satisfies (24).

For group motions the world lines are given by

$$x^\alpha=L^\alpha{}_\beta(\theta)\xi^\beta+\left(\int_{\theta_I}^{\theta}L^\alpha{}_\beta(\vartheta)d\vartheta\right)A^\beta, \qquad (39)$$

where $\{L^\alpha{}_\beta(\theta)\}$ is a continuous one-parameter family of homogeneous Lorentz transformations.

[4] K. Gödel in *Proceedings of the Sixth International Congress of Mathematicians* (American Mathematical Society, Providence, 1950), Vol. 1, p. 175.

5. GROUP MOTIONS

For an appropriate choice of the parameter θ, the four-velocity vector field $U^\alpha(x^\nu)$ of a group motion satisfies the ten Killing's equations,

$$U_{\alpha;\beta}+U_{\beta;\alpha}=0. \quad (40)$$

The concept of group motion is of course not restricted to flat space time. See, e.g., Eisenhart,[5] page 221 ff for a discussion of groups of motions. It is clear that any solution of (40) is a solution of (24), i.e., any group motion is a Born-type rigid motion.

To the four-velocity vector field $u^\alpha(x^\nu)$ corresponds an unnormalized four-acceleration vector field $\dot{u}^\alpha=u^\alpha{}_{;\beta}u^\beta$.

Theorem: For a group motion,

$$\dot{u}_\alpha=(\log_e(-U_\beta U^\beta)^{\frac{1}{2}})_{,\alpha}, \quad (41)$$

where U_β satisfies (40).

Proof: Let ψ and u_α be defined by (25) and (26). Equation (40) implies that

$$U_{\alpha;\beta}u^\alpha u^\beta=0. \quad (42)$$

Multiplying Eq. (27) by u^β and taking account of (42), we get

$$\psi_{,\beta}u^\beta=0. \quad (43)$$

Differentiation of Eq. (26) gives

$$u_{\alpha;\beta}=-\psi_{,\beta}u_\alpha+e^{-\psi}U_{\alpha;\beta}. \quad (44)$$

Multiplying Eq. (44) by u^β, using (43), (40), and (27), we get

$$\dot{u}_\alpha=\psi_{,\alpha}$$

which, since $\psi=\log_e(-U_\beta U^\beta)^{\frac{1}{2}}$, is (41).

Theorem: A rigid motion is a group motion if and only if the normalized four-velocity vector field u^α satisfies the equation

$$\dot{u}_{\alpha,\beta}-\dot{u}_{\beta,\alpha}=0 \quad (45)$$

at all points x^ν of $\mathcal{T}$.

Proof: The four-acceleration $\dot{u}_\alpha$ of a group motion is the gradient of a scalar, which implies Eq. (45). This proves the *only if* part of the theorem.

Consider a rigid motion which satisfies (45). Equation (45) is the necessary and sufficient condition which insures that a function $\psi(x^\nu)$ is defined by

$$\psi(x^\nu)=\int^{x^\nu}\dot{u}_\alpha dx^\alpha. \quad (46)$$

Let the vector field U_α be defined by $U_\alpha=e^\psi u_\alpha$. Then

$$U_{\alpha;\beta}+U_{\beta;\alpha}=e^\psi(u_{\alpha;\beta}+u_{\beta;\alpha}+u_\alpha\psi_{,\beta}+u_\beta\psi_{,\alpha}).$$

Substituting $\dot{u}_\beta$ for $\psi_{,\beta}$ and taking account of (17), we get (40). This proves the *if* part of the theorem.

[5] L. Eisenhart, *Riemannian Geometry* (Princeton University Press, Princeton, 1949).

The theorem proved by Herglotz and Noether can then be stated in the following manner. In special relativity a normalized four-velocity vector field $u_\mu(x^\nu)$ which satisfies Eq. (17), must also satisfy at least one of the two equations (34) or (45).

The rigid motion equation, (17), may be written

$$u_{\mu;\nu}+u_{\nu;\mu}=-\dot{u}_\mu u_\nu-\dot{u}_\nu u_\mu. \quad (47)$$

From the definition of the local angular velocity four-vector a^μ given in Eq. (34), we obtain

$$\begin{aligned}2(-g)^{\frac{1}{2}}\epsilon_{\mu\nu\sigma\tau}a^\mu&=\delta_{\nu\sigma\tau}{}^{\alpha\beta\gamma}u_\alpha u_{\beta;\gamma}\\&=u_\nu(u_{\sigma;\tau}-u_{\tau;\sigma})+u_\sigma(u_{\tau;\nu}-u_{\nu;\tau})\\&\quad+u_\tau(u_{\nu;\sigma}-u_{\sigma;\nu}). \quad (48)\end{aligned}$$

From (9), (10), and (48), we find

$$2(-g)^{\frac{1}{2}}\epsilon_{\mu\nu\sigma\tau}a^\mu u^\nu=u_{\tau;\sigma}-u_{\sigma;\tau}+u_\sigma\dot{u}_\tau-u_\tau\dot{u}_\sigma,$$

or

$$u_{\mu;\nu}-u_{\nu;\mu}=-\dot{u}_\mu u_\nu+\dot{u}_\nu u_\mu+2(-g)^{\frac{1}{2}}\epsilon_{\sigma\tau\nu\mu}a^\sigma u^\tau. \quad (49)$$

Adding Eqs. (47) and (49) gives us

$$u_{\mu;\nu}=-\dot{u}_\mu u_\nu-(-g)^{\frac{1}{2}}\epsilon_{\mu\nu\sigma\tau}a^\sigma u^\tau. \quad (50)$$

From (10), (34), and (50) it follows that

$$\dot{u}_\mu u^\mu=a_\mu u^\mu=0. \quad (51)$$

From the antisymmetry of $\epsilon_{\mu\nu\sigma\tau}$ in μ and ν, it follows that any $u_\mu(x^\sigma)$ that satisfies (49) also satisfies (47), and therefore (17). Equations (50) and (51) are equivalent to Eq. (17). The Herglotz-Noether theorem can then be stated: In special relativity, if a normalized four-velocity vector field $u_\mu(x^\sigma)$ satisfies Eq. (49), i.e., if it is the field of a Born-type rigid motion, then at least one of the two conditions—

(1) $\dot{u}_\mu$ is the gradient of a scalar and the motion is a *group motion*; or
(2) a^σ vanishes and the motion is a *plane motion*;
must hold throughout $\mathcal{T}$.

The geometrical properties of the world lines of Born type rigid motions in special relativity were examined in detail by Salzman.[6] In particular it was shown that a^μ does not vanish for the group motion which is such that the world lines of the particles undergoing the motion may be identified with the world lines of the particles constituting a rotating rigid disk. This means that there is no three-dimensional manifold orthogonal to this set of world lines. Hence the metric

$$dl^2=(g_{\alpha\beta}+u_\alpha u_\beta)x^\alpha{}_{,i}x^\beta{}_{,j}d\xi^i d\xi^j, \quad (15)$$

which measures the orthogonal distance between the world line of the particle ξ^i and that of the particle $\xi^i+d\xi^i$, is not the metric tensor of a three-space immersed in the Minkowski space time. Nevertheless, the quantity

$$l=\int dl \quad (52)$$

[6] G. Salzman, thesis, University of Illinois, 1953 (unpublished).

can be computed for any curve in the ξ^1, ξ^2, ξ^3 space and it will be independent of θ. We may take a curve such that

$$(\xi^1)^2+(\xi^2)^2=b^2, \quad \xi^3=0,$$

where b is a constant. This curve goes through the particles initially (i.e., for $\theta=\theta_I$) on the circumference of the rigid disk. When $\theta>\theta_I$ this curve is mapped into a locus described by Eqs. (39), where $L^\alpha{}_\beta(\theta)$ has been given by Herglotz. The value of l for this curve computed from Eqs. (15) and (52) has been called the *intrinsic circumference* of the rotating disk. The value of $l\neq 2\pi b$.

It should be pointed out that although it is possible to define other *intrinsic three-dimensional geometries* associated with a given three-parameter family of world lines with the *distance* between the particle ξ^i and $\xi^i+d\xi^i$ given by

$$dl^2=\mathcal{G}_{ij}d\xi^i d\xi^j,$$

the definition given by Eq. (15) has the virtue that dl is independent of θ.

6. THE DYNAMICAL EQUATIONS FOR A COMPRESSIBLE, NONVISCOUS, NON-HEAT CONDUCTING FLUID

Consider an isolated system of particles whose streamlines are the family $\{C_{\xi^i}\}$ of a rigid motion. Let $u^\alpha(x^\nu)$ be the unit four-velocity vector field defined by the streamlines. The system is described by a stress-energy tensor $T^{\alpha\beta}$ from which field quantities with correct relativistic transformation properties are defined. All field quantities, $T^{\alpha\beta}$ included, are understood to be the macroscopic averages of the corresponding microscopic quantities, and it is assumed that $T^{\alpha\beta}=T^{\beta\alpha}$. A method of performing Lorentz-invariant kinetic theory averages is given by Taub.[7] The result is that the macroscopic stress-energy tensor may be decomposed in the same way that Eckart[8] decomposed the corresponding microscopic stress-energy tensor, i.e.,

$$T^{\alpha\beta}=wu^\alpha u^\beta+Q^\alpha u^\beta+Q^\beta u^\alpha+S^{\alpha\beta}, \tag{53}$$

where

$$\begin{aligned} w&=T^{\alpha\beta}u_\alpha u_\beta,\\ Q^\alpha&=-T^{\lambda\beta}(\delta^\alpha{}_\lambda+u^\alpha u_\lambda)u_\beta,\\ S^{\alpha\beta}&=T^{\lambda\nu}(\delta^\alpha{}_\lambda+u^\alpha u_\lambda)(\delta^\beta{}_\nu+u^\beta u_\nu).\end{aligned} \tag{54}$$

These field quantities are interpreted as follows. The invariant w is the proper energy density, i.e., the energy density as determined by someone instantaneously at rest with respect to an element of the fluid, Q^α is the heat flow four vector, and $S^{\alpha\beta}$ is the stress tensor. If ρ is the proper density of matter then ϵ, the internal energy per unit rest mass of the fluid, is defined by

$$w=\rho(c^2+\epsilon). \tag{55}$$

The invariant pressure p is defined by

$$S^\alpha{}_\alpha=3p. \tag{56}$$

In what follows it is assumed that the fluid being discussed is incapable of conducting heat or of maintaining shearing stresses. Thus $Q^\alpha=0$, and in the proper locally Lorentz coordinate system $S^{\alpha\beta}=0$ for $\alpha\neq\beta$, $S^{ii}=p$, and $S^{44}=0$. If we write $S^{\alpha\beta}=pg^{\alpha\beta}$, where $g_{\alpha\beta}$ is the metric tensor of the proper locally Lorentz coordinate system, then this is correct for all values of α and β except $\alpha=\beta=4$, when it gives $-p$ instead of 0. In these coordinates $u^\alpha u^\beta=0$ unless $\alpha=\beta=4$, when it equals 1. Thus

$$S^{\alpha\beta}=p(g^{\alpha\beta}+u^\alpha u^\beta) \tag{57}$$

is correct in the proper locally Lorentz coordinates, and since it is a tensor it is the correct form in any coordinates for the stress tensor of a fluid that can sustain only a pressure. Substituting Eqs. (55) and (57) into (53), and taking account of the assumption $Q^\alpha=0$, we get

$$T^{\alpha\beta}=\rho[c^2+\epsilon+(p/\rho)]u^\alpha u^\beta+pg^{\alpha\beta}. \tag{58}$$

Defining the dimensionless function

$$\mu=1+(1/c^2)[\epsilon+(p/\rho)] \tag{59}$$

enables us to write

$$T^{\alpha\beta}=\mu\rho c^2u^\alpha u^\beta+pg^{\alpha\beta}. \tag{60}$$

The equations of motion of the fluid are

$$T^{\alpha\beta}{}_{;\beta}=0, \tag{61}$$

and

$$(\rho u^\alpha)_{;\alpha}=0. \tag{62}$$

Equation (61) expresses the conservation of momentum and energy. Equation (62) states that no particles are created or annihilated.

Let the invariant absolute temperature τ and the invariant specific entropy S as measured by an observer at rest with respect to an element of the fluid be defined by

$$d\epsilon+pd(1/\rho)=\tau dS. \tag{63}$$

Then the conservation of energy equation, $T^{\alpha\beta}{}_{;\beta}u_\alpha=0$, and the conservation of matter imply conservation of entropy along the streamlines, $S_{,\alpha}u^\alpha=0$, as follows.

By use of (60) and (62), Eq. (61) becomes

$$\rho u^\beta(\mu c^2u^\alpha)_{;\beta}+(pg^{\alpha\beta})_{;\beta}=0.$$

By using the fact that $g^{\alpha\beta}{}_{;\beta}=0$, and taking account of (59), this becomes

$$\rho u^\beta[\epsilon+(p/\rho)]_{,\beta}u^\alpha+\mu\rho c^2\dot{u}^\alpha+p_{,\beta}g^{\alpha\beta}=0.$$

Multiplying by u_α, and noting Eqs. (9) and (10), we get

$$-\rho[\epsilon_{,\beta}+p(1/\rho)_{,\beta}]u^\beta=0,$$

which implies, since $-\rho\tau\neq 0$, that

$$S_{,\beta}u^\beta=0. \tag{64}$$

[7] A. Taub, Phys. Rev. **74**, 328 (1948).
[8] C. Eckart, Phys. Rev. **58**, 919 (1940).

If all parts of the fluid are initially at the same entropy, then $S=constant$ holds throughout $\mathcal{T}$. In this case Eq. (63) determines p and ρ as functions of each other, once $\epsilon(p,\rho)$ is specified.

7. RIGID MOTION IN GENERAL RELATIVITY

The covariant derivatives that enter in Eqs. (61) and (62) are with respect to the coordinates in which the gravitational field is represented by the metric tensor $g_{\alpha\beta}$. We are concerned with problems in which this gravitational field is due to the fluid undergoing the motion described by Eqs. (61) and (62). Thus the $g_{\alpha\beta}$'s are determined by the Einstein field equations,

$$R_{\alpha\beta}-\tfrac{1}{2}g_{\alpha\beta}R=-kT_{\alpha\beta}, \tag{65}$$

in which the $T_{\alpha\beta}$ are the same as those occurring in Eq. (61).

The problem is then to solve the system of equations,

$$\left.\begin{aligned} R_{\alpha\beta}-\tfrac{1}{2}g_{\alpha\beta}R&=-kT_{\alpha\beta}, && (65)\\ u_{\alpha;\beta}+u_{\beta;\alpha}+\dot{u}_\alpha u_\beta+\dot{u}_\beta u_\alpha&=0, && (47)\\ u_\alpha u^\alpha&=-1, && (9)\\ T^{\alpha\beta}{}_{;\beta}&=0, && (61)\\ (\rho u^\alpha)_{;\alpha}&=0. && (62)\end{aligned}\right\}(66)$$

for the sixteen functions $g_{\alpha\beta}$, u_α, ρ, and p, where $T^{\alpha\beta}$ is assumed given by Eqs. (59) and (60) and ϵ is assumed to be a given function of p and ρ. Equation (61) is a consequence of Eq. (65), but is included in the system (66) because explicit use is made of it.

Theorem: A hydrodynamical system which satisfies the Eqs. of (66) and for which

$$\mu\partial\epsilon/\partial p\neq 0 \quad\text{and}\quad S=constant,$$

is performing a group motion.

Proof: Equation (9) implies (10). If Eq. (47) is multiplied by $g^{\alpha\beta}$ and (10) taken into account, then

$$u^\alpha{}_{;\alpha}=0. \tag{67}$$

Equations (62) and (67) imply that

$$\rho_{,\alpha}u^\alpha=0. \tag{68}$$

By using (60) and taking account of (62) and (68), Eq. (61) becomes

$$\rho p_{,\beta}u^\beta u^\alpha\partial\epsilon/\partial p+p_{,\beta}u^\alpha u^\beta+p_{,\beta}g^{\alpha\beta}+\mu\rho^2c^2\dot{u}^\alpha=0. \tag{69}$$

If Eq. (69) is multiplied by u_α, and (9) and (10) taken into account, then

$$-\rho p_{,\beta}u^\beta\partial\epsilon/\partial p=0,$$

or, since $\partial\epsilon/\partial p$ and ρ are nonzero throughout $\mathcal{T}$,

$$p_{,\beta}u^\beta=0 \tag{70}$$

throughout $\mathcal{T}$. Equation (69) then becomes, since $\mu\neq 0$

$$\dot{u}_\alpha=-p_{,\alpha}/(\mu\rho c^2). \tag{71}$$

From (71) we get

$$\dot{u}_{\alpha,\beta}-\dot{u}_{\beta,\alpha}=(\mu+\rho\partial\mu/\partial\rho)(p_{,\alpha}\rho_{,\beta}-p_{,\beta}\rho_{,\alpha})/(\mu\rho c^2).$$

But $S=constant$ implies that ρ and p are functions of each other, which in turn implies that $p_{,\alpha}\rho_{,\beta}-p_{,\beta}\rho_{,\alpha}=0$. Therefore Eq. (45) is satisfied at all x^ν in $\mathcal{T}$, and the motion is a group motion.

If $\epsilon(p,\rho)$ is any nonnegative function and if $\partial\epsilon/\partial p\neq 0$, then μ is a positive function and $\mu\partial\epsilon/\partial p\neq 0$. We will restrict ourselves to isentropic motions of fluids that satisfy this condition, and therefore we replace system (66) by

$$\left.\begin{aligned} R_{\alpha\beta}-\tfrac{1}{2}g_{\alpha\beta}R&=-kT_{\alpha\beta}, && (65)\\ U_{\alpha;\beta}+U_{\beta;\alpha}&=0, && (40)\\ u^\alpha&=(-g_{\mu\nu}U^\mu U^\nu)^{-\frac{1}{2}}U^\alpha, && (8)\\ T^{\alpha\beta}{}_{;\beta}&=0, && (61)\\ (\rho u^\alpha)_{;\alpha}&=0. && (62)\end{aligned}\right\}(72)$$

Equations (40) and (8) of system (72) imply (47) and (9), respectively, of system (66), and therefore every solution of (72) is a solution of (66).

8. A METHOD OF INTEGRATING THE SYSTEM OF EQS. (72)

The method we shall pursue is to reduce the system (72) to a set of equations for the $g_{\alpha\beta}$. In this set the U^α will be considered as known since the coordinate system may be chosen so that the U^α may be prescribed arbitrarily. The reduction is accomplished as follows: p and ρ may be expressed as functions of $g_{\alpha\beta}U^\alpha U^\beta$ if one assumes that $\mu\partial\epsilon/\partial p\neq 0$ and that the fluid is performing isentropic motion in accord with Eqs. (72). We examine solutions of (72) for which the U^α chosen is such that the angular velocity vector field given by (34) is nonvanishing. Since the motion is a group motion, the dynamical Eqs. (71) may be written

$$p_{,\alpha}/(\mu\rho c^2)=-[\log_e(-U_\beta U^\beta)^{\frac{1}{2}}]_{,\alpha}. \tag{73}$$

Since the motion is isentropic $p=p(\rho)$, which implies that the integrability conditions of Eq. (73) are satisfied. Therefore

$$\int(\mu\rho c^2)^{-1}dp=-\log_e(-U_\beta U^\beta)^{\frac{1}{2}}, \tag{74}$$

and we can explicitly obtain

$$\rho=\rho(g_{\alpha\beta}U^\alpha U^\beta),\quad p=p(g_{\alpha\beta}U^\alpha U^\beta), \tag{75}$$

where the right-hand members are known functions of $g_{\alpha\beta}U^\alpha U^\beta$. This enables us to eliminate the unknown functions ρ and p from $T_{\alpha\beta}$.

It is known (see, e.g., Eisenhart,[5] page 5) that given a vector field U^α, there exists a coordinate system in which $\bar{U}^\alpha=\delta^\alpha{}_4$. It is therefore no restriction at all to specify U^α, because $g_{\alpha\beta}$ is yet to be determined.

Thus, given a velocity field $U^\alpha(x^\nu)$, Eqs. (8) and (75) enable one to write $T_{\alpha\beta}$ explicitly in terms of the unknown functions $g_{\alpha\beta}$. Equations (65) then determine the $g_{\alpha\beta}$'s. Equations (40) are additional conditions on the first derivatives of the $g_{\alpha\beta}$'s. Equations (8) just define $u^\alpha(x^\nu)$, and there is no question about satisfying them.

That Eq. (62) is automatically satisfied can be seen as follows. Since the motion is isentropic, $\rho=\rho(p)$ and

$$\rho_{,\alpha}=p_{,\alpha}d\rho/dp. \qquad (76)$$

Equation (73) may be written

$$p_{,\alpha}=-\mu\rho c^2\dot{u}_\alpha.$$

It follows from this and from (10) that

$$p_{,\alpha}u^\alpha=0.$$

From this and from (76) we get

$$\rho_{,\alpha}u^\alpha=0,$$

which is Eq. (68). Since the motion is a group motion (67) holds, and (67) and (68) imply (62).

The problem is thus reduced to that of solving the system

$$\left.\begin{aligned} &R_{\alpha\beta}-\tfrac{1}{2}g_{\alpha\beta}R=-kT_{\alpha\beta}(g_{\mu\nu}U^\mu U^\nu), \quad (77)\\ &U_{\alpha;\beta}+U_{\beta;\alpha}=0, \quad (40)\end{aligned}\right\}(78)$$

for the ten unknown $g_{\alpha\beta}$'s, the U^μ's being considered as given.

We will be interested in dynamical solutions which satisfy the condition $p=0$ on the boundary hypersurface of the region $\mathcal{T}$, which hypersurface can be described by equations of the form

$$x^i=h^i(b_1,b_2,ct).$$

The solutions will be required to be such that the functions p and ρ are nonnegative throughout $\mathcal{T}$.

9. THE ROTATING RIGID BODY IN GENERAL RELATIVITY

We assume that in the x^α-coordinates, which for heuristic reasons are called, r, z, ϕ, ct, the world lines are given by

$$\begin{aligned} x^1&=r=r_I,\\ x^2&=z=z_I,\\ x^3&=\phi=\phi_I+(\omega/c)ct,\\ x^4&=ct;\end{aligned} \qquad (79)$$

i.e., the time coordinate ct is the parameter used to describe the motion, and the initial hypersurface from which the world lines start is $ct=0$. r_I, z_I, and ϕ_I are the parameters ξ^i that label the world lines.

The four-velocity vector field defined by Eqs. (79) with this choice of parameter is

$$\|U^\alpha\|=\|0,0,\omega/c,1\|. \qquad (80)$$

Let $g_{\alpha\beta}(x^\nu)$ be the metric tensor of the x^α-coordinate system. Killing's Eqs. (40) can also be written as

$$g_{\alpha\beta,\lambda}U^\lambda+U^\lambda{}_{,\beta}g_{\lambda\alpha}+U^\lambda{}_{,\alpha}g_{\lambda\beta}=0. \qquad (81)$$

Substituting Eq. (80) into Eq. (81) we get, since the second and third terms vanish,

$$g_{\alpha\beta,3}[\omega/c]+g_{\alpha\beta,4}=0.$$

The general solution of this equation is

$$g_{\alpha\beta}(x^\nu)=\chi_{\alpha\beta}(r,\,z,\,\phi-(\omega/c)ct),$$

where $\chi_{\alpha\beta}$ is an arbitrary function of r, z, and $\phi-(\omega/c)ct$ We restrict ourselves to looking for *steady state* motions with *cylindrical* symmetry, i.e., all the field quantities are assumed independent of ct and of ϕ, respectively. Therefore

$$g_{\alpha\beta}=g_{\alpha\beta}(r,z),$$

and this together with (80) insures that Killing's equations are satisfied. The problem is thus reduced to that of finding integrals of Eq. (77), i.e., sets of $g_{\alpha\beta}(r,z)$'s, which have the property that when they are substituted into Eqs. (75), nonnegative functions $\rho(x^\nu)$ and $p(x^\nu)$ are determined which vanish everywhere except in a region $\mathcal{T}$, and p is continuous on the boundary of $\mathcal{T}$. Also, U^α must be timelike throughout $\mathcal{T}$. In order to solve (77) an approximation method is used.

10. AN APPROXIMATION METHOD FOR INTEGRATING EQ. (77)

The method is based on the smallness of the constant k of Eq. (77). Newton's gravitational constant G is related to k by the equation (in our choice of units)

$$k=8\pi G/c^4=[8\pi\times 6.67\times 10^{-8}\text{ dyne cm}^2\text{ gm}^{-2}]/[(3\times 10^{10}\text{ cm sec}^{-1})^4]$$
$$=2.07\times 10^{-48}\text{ g}^{-1}\text{ cm}^{-1}\text{ sec}^2.$$

It is convenient to replace k by $\kappa\mathrm{k}$, with $\kappa=1$ g^{-1} cm^{-1} sec^2, and k the pure number 2.07×10^{-48}, and to assume the metric tensor expanded in a power series in k,

$$g_{\alpha\beta}=\sum_{n=0}^{\infty}\mathrm{k}^n g^{(n)}{}_{\alpha\beta}=\sum_{n=0}^{\infty}\gamma^{(n)}{}_{\alpha\beta}. \qquad (82)$$

Then, substituting this expression for $g_{\alpha\beta}$ into both members of (77), we seek a solution in which each equation obtained by equating the coefficients of like powers of k satisfied. This has the effect of replacing (77) by a sequence of equations of the form

$$\Theta_{\lambda\nu}{}^{\alpha\beta}\gamma^{(n)}{}_{\alpha\beta}=\mathrm{k}^n\phi^{(n)}{}_{\lambda\nu}, \quad n=0,1,2,\cdots, \qquad (83)$$

where $\Theta_{\lambda\nu}{}^{\alpha\beta}$ is a linear differential operator and $\phi^{(n)}{}_{\alpha\beta}$ is a known function which depends only on the $g^{(m)}{}_{\alpha\beta}$'s with $m<n$, and their derivatives. Thus if Eqs. (83) are solved in order, $\phi^{(n)}{}_{\alpha\beta}$ is a known function of x^ν at each step.

More explicitly, (77) may be written in the equivalent form

$$R_{\alpha\beta}=-\kappa\mathrm{k}\mathfrak{T}_{\alpha\beta}(g_{\lambda\nu}U^\lambda U^\nu), \qquad (84)$$

where

$$\mathfrak{T}_{\alpha\beta}=T_{\alpha\beta}-\tfrac{1}{2}g_{\alpha\beta}[T_{\lambda\nu}g^{\lambda\nu}].$$

If it is assumed that $\gamma^{(0)}{}_{\alpha\beta}=\eta_{\alpha\beta}$, then Eqs. (83) are

$$\tfrac{1}{2}\left[\eta^{\alpha\beta}\frac{\partial^2}{\partial x^\mu\partial x^\nu}+\delta^\alpha{}_\mu\delta^\beta{}_\nu\Box-\eta^{\alpha\sigma}\delta^\beta{}_\mu\frac{\partial^2}{\partial x^\sigma\partial x^\mu}-\eta^{\beta\sigma}\delta^\alpha{}_\nu\frac{\partial^2}{\partial x^\sigma\partial x^\mu}\right]\gamma^{(n)}{}_{\alpha\beta}=\mathrm{k}^n\phi^{(n)}{}_{\mu\nu},\quad(85)$$

where

$$\phi^{(n)}{}_{\alpha\beta}=-\mathfrak{T}^{(n-1)}{}_{\alpha\beta}-\tfrac{1}{2}W^{(n)}{}_{\alpha\beta},$$

$\mathfrak{T}^{(n)}{}_{\alpha\beta}$ is defined by the equation

$$\mathfrak{T}_{\alpha\beta}=\sum_{n=0}^{\infty}\mathrm{k}^n\mathfrak{T}^{(n)}{}_{\alpha\beta},\quad \Box=\eta^{\alpha\beta}\frac{\partial^2}{\partial x^\alpha\partial x^\beta},$$

and $W^{(n)}{}_{\alpha\beta}$ is defined by the equation

$$R_{\alpha\beta}=\sum_{n=1}^{\infty}\mathrm{k}^n[\Theta_{\alpha\beta}{}^{\nu\lambda}g^{(n)}{}_{\nu\lambda}+\tfrac{1}{2}W^{(n)}{}_{\alpha\beta}].$$

In this method the boundary condition $p=0$ is to be satisfied at the last step in the approximation.

11. USE OF THE APPROXIMATION METHOD IN THE CASE OF THE ROTATING RIGID BODY

Here the problem of Sec. 9 is examined in the first approximation, and it is shown that there exist in the classical limit of the theory rigidly rotating dynamical systems held together by their own gravitational fields.

We assume that

$$\gamma^{(0)}{}_{11}=\gamma^{(0)}{}_{22}=-\gamma^{(0)}{}_{44}=1,\quad \gamma^{(0)}{}_{33}=r^2,\quad \gamma^{(0)}{}_{\alpha\beta}=0\ \text{ if }\ \alpha\neq\beta.$$

This choice satisfies the zeroth order Eq. of (83) and Killing's equations, and enables us to get a formal expression for $\mathfrak{T}^{(0)}{}_{\alpha\beta}$.

The next step is to integrate the first-order field equations. To do this it is convenient to make a coordinate transformation which transforms $\gamma^{(0)}{}_{\alpha\beta}$ into $\eta_{\alpha\beta}$, the Minkowski metric tensor.

$$x^{*1}=x=r\cos\phi=x^1\cos x^3,$$
$$x^{*2}=y=r\sin\phi=x^1\sin x^3,$$
$$x^{*3}=z=z=x^2,$$
$$x^{*4}=ct=ct=x^4.$$

In this coordinate system the first order equation of (83) may be written

$$R^{*(1)}{}_{\alpha\beta}=-\kappa\mathrm{k}\mathfrak{T}^{*(0)}{}_{\alpha\beta},\quad(86)$$

where

$$R^{*(1)}{}_{\alpha\beta}=\Theta_{\alpha\beta}{}^{\mu\nu}\gamma^{*(1)}{}_{\mu\nu}=\tfrac{1}{2}[\Box\gamma^{*(1)}{}_{\alpha\beta}+\gamma^{*(1)}{}_{,\alpha\beta}-\gamma^{*(1)\nu}{}_{\alpha,\nu\beta}-\gamma^{*(1)\nu}{}_{\beta,\nu\alpha}],$$

$$\gamma^{*(1)}{}_{\alpha\beta}=(\partial x^\nu/\partial x^{*\alpha})(\partial x^\lambda/\partial x^{*\beta})\gamma^{(1)}{}_{\nu\lambda},$$

$$\gamma^{*(1)\nu}{}_\alpha=\eta^{\nu\beta}\gamma^{*(1)}{}_{\beta\alpha},$$

$$\gamma^{*(1)}=\eta^{\alpha\beta}\gamma^{*(1)}{}_{\alpha\beta}.$$

The homogeneous equation $R^{*(1)}{}_{\alpha\beta}=0$ has the solution $\gamma^{*(1)}{}_{\alpha\beta}=f^*{}_{\alpha,\beta}+f^*{}_{\beta,\alpha}$, where $f^*{}_\alpha$ are arbitrary functions. Therefore

$$\gamma^{*(1)}{}_{\alpha\beta}(x^{*\nu})=\kappa\mathrm{k}(2\pi)^{-1}\int [\![\mathfrak{T}^{*(0)}{}_{\alpha\beta}(x^{*\prime\nu})]\!]\times[\eta_{ij}(x^{*\prime i}-x^{*i})(x^{*\prime j}-x^{*j})]^{-\frac{1}{2}}dV^{*\prime}+\kappa\mathrm{k}(f^*{}_{\alpha,\beta}+f^*{}_{\beta,\alpha})\quad(87)$$

is a solution of Eq. (86) which has four arbitrary functions in accordance with the freedom of choice of a coordinate system. The three-dimensional volume element $dV^{*\prime}=dx^{*\prime}dy^{*\prime}dz^{*\prime}$. The expression $[\![\mathfrak{T}^{*(0)}{}_{\alpha\beta}(x^{*\prime\nu})]\!]$ is to be evaluated at the retarded time. The spatial distance between the field point and the variable integration point, evaluated in zeroth order, is

$$[\eta_{ij}(x^{*\prime i}-x^{*i})(x^{*\prime j}-x^{*j})]^{\frac{1}{2}}.$$

It can be verified that (87) is a solution of Eq. (86) for any choice of the region of integration.

The region over which the integration in (87) is carried out is related to the boundary condition $p^{(1)}=p([\gamma^{(0)}{}_{\alpha\beta}+\gamma^{(1)}{}_{\alpha\beta}]U^\alpha U^\beta)=0$ as follows: For any prescribed region of integration $p^{(1)}$ will be determined as a function of the coordinates, as will be shown below, the explicit functional form depending on the choice of the region of integration. The boundary described by the equation $p^{(1)}=0$ will then be determined by the region of integration and will vary as this region is varied. Thus, if the region of integration is given, the boundary $p^{(1)}=0$ is determined. It may not be a real boundary for some choices of the region of integration, in the sense that there may be no points in space time whose coordinates satisfy the equation of the boundary. Presumably, if one is given a real boundary $p^{(1)}=0$, one should be able to determine whether a region of integration exists for which the process described above leads to this boundary. This question will not be discussed further here.

In the subsequent discussion we shall assume that the region of integration in Eq. (87) is the interior of a sphere of radius $a^{(0)}$ and shall show that real boundaries exist for this case.

Because we are considering steady state motion the expression $[\![\mathfrak{T}^{*(0)}{}_{\alpha\beta}(x^{*\prime\nu})]\!]$ in Eq. (87) can be evaluated at any time. Transforming back to the x^α-coordinate system, we get

$$\gamma^{(1)}{}_{\alpha\beta}=(\partial x^{*\nu}/\partial x^\alpha)(\partial x^{*\lambda}/\partial x^\beta)\kappa\mathrm{k}(2\pi)^{-1}\times\int\mathfrak{T}^{*(0)}{}_{\nu\lambda}(x^{*\prime\sigma})[\eta_{ij}(x^{*\prime i}-x^{*i})(x^{*\prime j}-x^{*j})]^{-\frac{1}{2}}dV^{*\prime}+\kappa\mathrm{k}(f_{\alpha;\beta}+f_{\beta;\alpha}),\quad(88)$$

where

$$f_{\alpha;\beta}=(\partial x^{*\nu}/\partial x^\alpha)(\partial x^{*\lambda}/\partial x^\beta)f^*{}_{\nu;\lambda}=(\partial x^{*\nu}/\partial x^\alpha)(\partial x^{*\lambda}/\partial x^\beta)[f^*{}_{\nu,\lambda}+O(\mathrm{k})],$$

and of course the terms of order k with respect to $f^*_{\nu,\lambda}$ are ignored.

The combination $(\gamma^{(0)}{}_{\alpha\beta}+\gamma^{(1)}{}_{\alpha\beta})U^\alpha U^\beta$ is required, rather than the individual $\gamma^{(1)}{}_{\alpha\beta}$'s. From (80) and the $\gamma^{(0)}{}_{\alpha\beta}$'s assumed,

$$(\gamma^{(0)}{}_{\alpha\beta}+\gamma^{(1)}{}_{\alpha\beta})U^\alpha U^\beta=(\omega r/c)^2-1+\Gamma, \qquad (89)$$

where

$$\Gamma=\gamma^{(1)}{}_{44}+2(\omega/c)\gamma^{(1)}{}_{34}+(\omega/c)^2\gamma^{(1)}{}_{33}.$$

Using (88), straightforward calculation leads to

$$\Gamma(x^\nu)=\frac{\text{K}\text{k}}{2\pi}\int\frac{\Omega(\mathbf{r},\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dV'+2\text{K}\text{k}(\omega/c)rf_1, \qquad (90)$$

where

$$\Omega(\mathbf{r},\mathbf{r}')=\frac{\mu^{(0)}\rho^{(0)}c^2\{1-[(\omega/c)^2rr'\cos(\phi-\phi')]\}^2}{1-(\omega r'/c)^2}-(\tfrac{1}{2}\mu^{(0)}\rho^{(0)}c^2-p^{(0)})(1-(\omega r/c)^2),$$

$\mathbf{r}$ is the ordinary radial three vector, i.e., no time component, from the origin ($r=z=0$) to the point r,z,ϕ, and

$$|\mathbf{r}-\mathbf{r}'|=[r^2+r'^2+(z-z')^2-2rr'\cos(\phi-\phi')]^{\frac{1}{2}}.$$

The function f_1 may be taken to be zero, for if it is not we may go to another coordinate system related to the r,z,ϕ,ct coordinates as follows:

$$\bar{r}=\bar{r}(r,z),\quad \bar{z}=\bar{z}(r,z),\quad \bar{\phi}=\phi,\quad c\bar{t}=ct.$$

In this coordinate system the $\bar{U}^\sigma$ are still given by the right-hand sides of Eq. (80). Hence Eq. (90) will hold in the barred coordinate system. However we may choose the functions $\bar{r}(r,z)$ and $\bar{z}(r,z)$ so that

$$\bar{f}_1=f_1\partial r/\partial\bar{r}+f_2\partial z/\partial\bar{r}=0.$$

Therefore (90) becomes

$$\Gamma(x^\nu)=\frac{\text{K}\text{k}}{2\pi}\int\frac{\Omega(\mathbf{r},\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dV'. \qquad (91)$$

12. THE CLASSICAL LIMIT

Before specializing $\epsilon(p,\rho)$ or restricting ourselves to isentropic flow, we can determine the *classical limit* of $c^2\Gamma$. Note that

$$u^{(0)}=1+(1/c^2)[\epsilon^{(0)}+(p^{(0)}/\rho^{(0)})],$$
$$\tfrac{1}{2}\mu^{(0)}\rho^{(0)}c^2-p^{(0)}=\tfrac{1}{2}\rho^{(0)}c^2\{1+(1/c^2)[\epsilon^{(0)}-(p^{(0)}/\rho^{(0)})]\},$$
$$\text{K}\text{k}=8\pi G/c^4.$$

Therefore, we get from (91)

$$\text{Limit}_{cl}\,c^2\Gamma=\underset{c\to\infty}{\text{Limit}}\,c^2\Gamma=2G\int\frac{\rho^{(0)}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dV'. \qquad (92)$$

Equation (92) is useful in examining the classical limit ($c\to\infty$) of the first order approximation, i.e., the approximation to order k^1. The right-hand member of (92) is (-2) times the Newtonian gravitational potential of a distribution $\rho^{(0)}(\mathbf{r})$ of matter.

Under the assumption that the motion is isentropic and that $\epsilon(p,\rho)=p/[\rho(\gamma-1)]$, where γ is the ratio of the specific heats of the fluid and is assumed to be a constant, Eq. (63) gives

$$p=\kappa\rho^\gamma, \qquad (93)$$

where κ is the constant of integration. Substituting this into (74), we find for Eqs. (75):

$$\left.\begin{aligned}\rho&=\{[(\gamma-1)/\kappa\gamma][A(-g_{\alpha\beta}U^\alpha U^\beta)^{-\frac{1}{2}}-c^2]\}^{1/(\gamma-1)},\\ p&=\kappa\{[(\gamma-1)/\kappa\gamma][A(-g_{\alpha\beta}U^\alpha U^\beta)^{-\frac{1}{2}}-c^2]\}^{\gamma/(\gamma-1)},\end{aligned}\right\} \qquad (94)$$

where A is the constant of integration. Substituting from Eq. (89) for $g_{\alpha\beta}U^\alpha U^\beta$ into the above equation for p, we find the first order approximation to p, namely,

$$p^{(1)}=\kappa\{[(\gamma-1)/\kappa\gamma][A^{(1)}(1-(\omega r/c)^2-\Gamma)^{-\frac{1}{2}}-c^2]\}^{\gamma/(\gamma-1)}.$$

Let $[\chi]_0$ denote the value of a function χ at the origin ($r=z=0$). Then from the last equation,

$$A^{(1)}=\{[\kappa\gamma/(\gamma-1)][[p^{(1)}]_0/\kappa]^{(\gamma-1)/\gamma}+c^2\}(1-[\Gamma]_0)^{\frac{1}{2}},$$

and

$$p^{(1)}=\kappa\{[c^2(\gamma-1)/\kappa\gamma][\Lambda(1-[\Gamma]_0)^{\frac{1}{2}}\times(1-(\omega r/c)^2-\Gamma)^{-\frac{1}{2}}-1]\}^{\gamma/(\gamma-1)}, \qquad (95)$$

where

$$\Lambda=\frac{\kappa\gamma([p^{(1)}]_0/\kappa)^{(\gamma-1)/\gamma}}{(\gamma-1)c^2}+1.$$

If the square roots in Eq. (95) are expanded in powers of k (remembering that Γ and $[\Gamma]_0$ are of order k^1), terms of order k^2 or greater discarded and the classical limit taken, we find with the help of (92) that

$$\text{Limit}_{cl}\,p^{(1)}=\kappa\left\{\frac{\gamma-1}{\kappa\gamma}\left[\tfrac{1}{2}\omega^2r^2+B+G\int\frac{\rho^{(0)}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dV'\right]\right\}^{\gamma/(\gamma-1)}. \qquad (96)$$

where

$$B=\frac{\kappa\gamma}{\gamma-1}\left(\frac{[p^{(1)}]_0}{\kappa}\right)^{(\gamma-1)/\gamma}-G\int\frac{\rho^{(0)}(\mathbf{r}')}{|\mathbf{r}'|}dV'.$$

Classically, a rigidly rotating body of gas which obeys the adiabatic law, Eq. (93), has a pressure p_{cl} given by

$$p_{cl}=\kappa\left\{\frac{\gamma-1}{\kappa\gamma}\left[\tfrac{1}{2}\omega^2r^2+B_{cl}+G\int\frac{\rho_{cl}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dV'\right]\right\}^{\gamma/(\gamma-1)}, \qquad (97)$$

where

$$B_{cl}=\frac{\kappa\gamma}{\gamma-1}\left(\frac{[p_{cl}]_0}{\kappa}\right)^{(\gamma-1)/\gamma}-G\int\frac{\rho_{cl}(\mathbf{r}')}{|\mathbf{r}'|}dV'.$$

This is equivalent to Eq. (236.2) of Jean's *Astronomy and Cosmogony*.[9]

Consider the classical problem of a rigidly rotating body of gas which obeys the adiabatic law. If we attempted a perturbation approach in which the gravitational interaction was ignored in zeroth order, then we would have, from Eq. (97),

$$p_{cl}{}^{(0)}=\kappa\left\{\frac{\gamma-1}{\kappa\gamma}\left[\tfrac{1}{2}\omega^2r^2+\frac{\kappa\gamma}{\gamma-1}\left(\frac{[p_{cl}{}^{(0)}]_0}{\kappa}\right)^{(\gamma-1)/\gamma}\right]\right\}^{\gamma/(\gamma-1)}.$$

If the zeroth-order density $\rho_{cl}{}^{(0)}$ corresponding to this $p_{cl}{}^{(0)}$ was used to calculate a first approximate gravitational field and this field was used to calculate the first order pressure, we would get, from (97),

$$p_{cl}{}^{(1)}=\text{Limit}_{cl}p^{(1)}.$$

Thus the classical limit of the first order approximation gives the same density and pressure distribution as does the scheme described in this paragraph.

13. AN EXAMPLE

From (89) and (94) we see that

$$p^{(0)}=\kappa\{[c^2(\gamma-1)/\kappa\gamma][\Lambda(1-(\omega r/c)^2)^{-\frac{1}{2}}-1]\}^{\gamma/(\gamma-1)}.$$

This shows that $p^{(0)}$ is a monotone increasing function of r from 0 to c/ω, and that $p^{(0)}\to\infty$ as $r\to c/\omega$. There are thus no self-contained rotating rigid bodies in the zeroth approximation, i.e., an external cylindrical wall would be needed to keep the fluid moving in circular orbits rather than flowing outward. This is to be expected because in this approximation the particles are interacting only through the pressure.

The rest of the calculation is concerned with showing that in the classical limit there are self-contained dynamical solutions in the first approximation. We restrict ourselves to the case

$$\epsilon=p/[\rho(\gamma-1)],\quad p=\kappa\rho^\gamma,\quad\text{and}\quad \gamma=\tfrac{3}{2},$$

and to the classical limit.

The boundary of a self-contained dynamical solution in the first order approximation is defined by $p^{(1)}=0$. From Eq. (95) it follows that $p^{(1)}=0$ if and only if

$$\left(\frac{\omega r}{c}\right)^2+\Gamma-[\Gamma]_0+(1-[\Gamma]_0)\times\left[\frac{6\kappa}{c^2}\left(\frac{[p^{(1)}]_0}{\kappa}\right)^{\frac{1}{3}}+\left(\frac{3\kappa}{c^2}\right)^2\left(\frac{[p^{(1)}]_0}{\kappa}\right)^{\frac{2}{3}}\right]=0.$$

If this equation is multiplied through by c^2 and the classical limit taken, one obtains for the equation of the boundary in this case:

$$\omega^2r^2+\text{Limit}_{cl}c^2(\Gamma-[\Gamma]_0)+6\kappa([p^{(1)}]_0/\kappa)^{\frac{1}{3}}=0.\quad(98)$$

[9] J. Jeans, *Astronomy and Cosmogony* (Cambridge University Press, London, 1929), p. 259.

It can be shown[7] that the velocity of sound in a fluid of the type under consideration is

$$v=c\left(\frac{\rho}{\mu}\frac{d\mu}{d\rho}\right)^{\frac{1}{2}},$$

where μ must be expressed as a function of ρ alone. We then find that, in the classical limit,

$$\kappa=2v^2/(3\rho^{\frac{1}{2}})$$

which enables us to replace κ in Eq. (98).

If the integral of Eq. (91) is evaluated on the assumption of a spherical zero-order boundary of radius $a^{(0)}$, this result substituted into Eq. (98), the resulting equation brought into dimensionless form by dividing through by $2\omega^2(a^{(0)})^2$, the dimensionless constants

$$\alpha=2([v^{(1)}]_0/\omega a^{(0)})^2,$$
$$\alpha^{(0)}=2([v^{(0)}]_0/\omega a^{(0)})^2,$$
$$\beta=2\pi G[\rho^{(1)}]_0/\omega^2,$$

introduced, the replacements

$$\mathrm{r}=r/a^{(0)},\quad \mathrm{z}=z/a^{(0)}$$

made, and the equation multiplied through by α^2, then the equation of the boundary is

$$f(\mathrm{r}^2,\mathrm{z}^2)=0,\quad(99)$$

where

$$\begin{aligned}f(\mathrm{r}^2,\mathrm{z}^2)=\alpha^3&-\frac{\beta}{3}\left\{\frac{2}{5\cdot7}+\alpha^{(0)}[\alpha^{(0)}+\tfrac{2}{5}]\right\}\mathrm{z}^2\\&+\frac{\beta}{5\cdot7}\left[\frac{47}{9}-4\alpha^{(0)}\right]\mathrm{z}^4-\frac{2\beta}{3\cdot7\cdot9\cdot11}\mathrm{z}^6\\&-\frac{\beta(2+9\alpha^{(0)})}{3\cdot5\cdot7}\mathrm{r}^2\mathrm{z}^2+\frac{5\beta}{7\cdot9\cdot11}\mathrm{r}^2\mathrm{z}^4-\frac{5\beta}{7\cdot11\cdot12}\mathrm{r}^4\mathrm{z}^2\\&+\left\{\frac{\alpha^2}{2}+\frac{\beta}{3}\left[\frac{1}{5\cdot7}+\alpha^{(0)}(\tfrac{1}{5}-\alpha^{(0)})\right]\right\}\\&+\frac{\beta(1-48\alpha^{(0)})}{5\cdot7\cdot12}\mathrm{r}^4-\frac{113\beta}{7\cdot9\cdot11\cdot12}\mathrm{r}^6.\end{aligned}\quad(100)$$

The function $f(\mathrm{r}^2,\mathrm{z}^2)$ is related to the pressure in the classical limit of the first order approximation as follows: Substituting (92) into (96), and setting $\gamma=\frac{3}{2}$, we get

$$\text{Limit}_{cl}p^{(1)}=\kappa\{[1/(6\kappa)][(\omega r)^2+\text{Limit}_{cl}c^2(\Gamma-[\Gamma]_0)+6\kappa([p^{(1)}]_0/\kappa)^{\frac{1}{3}}]\}^3.$$

The expression in the second square bracket is

$$(2\omega^2(a^{(0)})^2/\alpha^2)f(\mathrm{r}^2,\mathrm{z}^2),$$

as can be seen by tracing the steps from Eq. (98) to Eq. (99). Therefore,

$$\text{Limit}_{cl}p^{(1)}=\kappa\{[1/(3\kappa)](\omega a^{(0)}/\alpha)^2f(\mathrm{r}^2,\mathrm{z}^2)\}^3.\quad(101)$$

Thus Limit$_{cl}p^{(1)}$ and $f(r^2,z^2)$ are monotone increasing functions of each other and surfaces $f=constant$ surfaces are surfaces of uniform pressure.

The choice $\alpha^{(0)}=\alpha=2$ and $\beta=100$ gives a self-contained solution. Figure 2 shows the trace of the boundary on the positive part ($z>0$) of the $r-z$ half plane. The fluid mass is in the shape—roughly speaking—of an ellipsoid of revolution about the minor axis. It differs from an ellipsoid to the extent that the equation of the boundary is

$$z=\pm 0.269(0.0684-r^2-0.967r^4)^{\frac{1}{2}},$$

whereas the boundary of an ellipsoid with the same axes is

$$z=\pm 0.277(0.0643-r^2)^{\frac{1}{2}}.$$

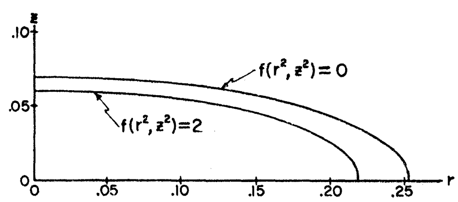

FIG. 2. A self-contained dynamical system in the first approximation.

It has been found that there are other choices of the parameters $\alpha^{(0)}$, α, and β which give non-self-contained solutions in this approximation.

Thus, the classical limit of the first order approximation in general relativity is a theory in which there exist self-contained dynamical systems which perform Born-type rigid motion.

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Comparison of the Cut-Off Meson Theory with Experiment*

GEOFFREY F. CHEW
Department of Physics, University of Illinois, Urbana, Illinois
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The cut-off theory for the interaction of pions with nonrelativistic nucleons is tested against experiments involving a single nucleon, with and without the presence of an electromagnetic field. It is concluded that most of the existing information about the P-wave pion-nucleon interaction can be understood with a renormalized coupling constant, $f^2=0.058$ and a cut-off energy, $\omega_{\max}=5.6\ \mu$. No light is shed on the S-wave pion-nucleon interaction.

I. INTRODUCTION

THE purpose of this paper is to compare with existing experimental data the so-called cut-off form of the Yukawa theory for the interaction of pions with nucleons. Although this form is not Lorentz-invariant[1] and is appropriate only when the nucleon velocity is small compared to the velocity of light, the meson velocity is unrestricted, so the theory can be applied to a very wide range of experiments. These include pion-nucleon scattering, photo-pion production, nucleon-nucleon scattering, and the ground-state properties of the deuteron, as well as the anomalous electromagnetic properties of nucleons (e.g., magnetic moments). It will be shown here that a large amount of the existing experimental information can be correlated by the meson theory with only two arbitrary parameters: a coupling constant and an energy cutoff.

The theory can most easily be characterized by writing down the interaction energy which it postulates between the pion field and a single fixed nucleon (at the origin of the coordinate system):

$$H_{\text{int}}=(4\pi)^{\frac{1}{2}}(f/\mu)\int d\mathbf{r}\rho(r)\sum_{\lambda=1}^{3}\tau_\lambda\boldsymbol{\sigma}\cdot\delta\phi_\lambda(\mathbf{r}). \qquad (1)$$

Here f is the dimensionless unrationalized coupling constant ($\hbar=c=1$), μ is the pion mass, $\rho(r)$ the "source" function, normalized so that $\int\rho(r)d\mathbf{r}=1$, $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ are the Pauli spin and isotopic spin operators for the nucleon, and the ϕ_λ are the three real components of the pion field. The form (1) is often referred to as "gradient" coupling, but we prefer to call it simply "linear" coupling, since it is the only form compatible with the conservation of angular momentum, parity, and isotopic spin which at the same time is linear in the pion field and does not involve antinucleons. The effective nonrelativistic linear interactions of any field theory (including the γ_5 theory) must reduce to the form (1).

Although (1) has been written for an infinitely heavy nucleon, it is not hard to make the interaction Galilean invariant, that is, to include effects of order v/c, where v is the nucleon velocity. This has been done for some of the calculations discussed below, where it was felt that the accuracy of both experiment and calculation

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[1] For a general discussion of the cut-off theory and more references, see W. Pauli, *Meson Theory of Nuclear Forces* (Interscience Publishers, Inc., New York—London, 1946), p. 12.