Meson-Nucleon Scattering in the Tamm-Dancoff Approximation*

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An attempt is made to calculate meson-nucleon scattering by using charge-symmetric pseudoscalar meson theory with pseudoscalar coupling and without the use of perturbation theory. The Tamm-Dancoff formalism is used, with all states which are not directly coupled to the one-meson, one-nucleon state omitted. In this approximation an inhomogeneous integral equation in three-dimensional momentum space is derived for $g(\mathbf{p})$, the probability amplitude for relative momentum $\mathbf{p}$ between meson and nucleon. This equation is reduced to a separate one-dimensional integral equation for each of the six angular momentum and isotopic spin (T) states, $S_{\frac{1}{2}}$, $P_{\frac{1}{2}}$, and $P_{\frac{3}{2}}$ for $T=\frac{1}{2}, \frac{3}{2}$. The phase shifts for these states are given by the value of $g(p)$ on the energy-shell.

All self-energy and renormalization terms are omitted and the present method is inapplicable to the two states of total angular momentum and T equal to $\frac{1}{2}$. The integral equations are solved by semi-numerical methods for the $S_{\frac{1}{2}}$ and $P_{\frac{3}{2}}$ states with $T=\frac{3}{2}$, obtaining the shape of $g(p)$ and the variation of phase shift with energy and coupling constant. It is shown that only for the $P_{\frac{3}{2}}$, $T=\frac{3}{2}$ (corresponding to an attractive potential) is the phase shift much larger than in Born approximation and depends strongly on the coupling constant and on energy. For the other, repulsive, states the phase shift is less than in Born approximation. For energies of about 150 Mev or more and for a large enough coupling constant, the $P_{\frac{3}{2}}$, $T=\frac{3}{2}$ phase shift is larger than any of the others. This is in rough qualitative (but by no means quantitative) agreement with experiment.

I. INTRODUCTION

EXPERIMENTS on the angular distribution and total cross-section for ordinary and charge-exchange scattering of pions on protons have now been performed at various energies up to 135 Mev and higher.[1] An analysis of these experiments[1] indicates that they are compatible with the assumption of charge symmetry and that the results can be well represented in terms of six phase shifts (for each energy) α_1, α_{11}, α_{13}, the phase shifts for the $S_{\frac{1}{2}}$, $P_{\frac{1}{2}}$ and $P_{\frac{3}{2}}$ states of isotopic spin T equal to $\frac{1}{2}$, and α_3, α_{31}, α_{33}, the corresponding phase shifts for $T=\frac{3}{2}$. At energies approaching 150 Mev the dominating feature of such an analysis is the fact[1,2] that α_{33}, the $P_{\frac{3}{2}}$, $T=\frac{3}{2}$ phase shift, is much larger than any of the other five. On the other hand, if a second-order perturbation method is used on symmetric pseudoscalar meson theory, larger phase shifts are obtained for the S states than for P states, giving results not even in qualitative agreement with experiment. Chew[3] pointed out that, since the coupling constant used is very large, such a perturbation method is equivalent to using lowest-order Born approximation for the scattering from a very strong potential, which is well known to give misleading results. In particular the equivalent potential is attractive only for the $P_{\frac{3}{2}}$, $T=\frac{3}{2}$ state, so one would expect the correct phase shift α_{33} to be much larger than on Born approximation, but the phase shifts for all the other five "repulsive" states to be less than on Born approximation. This is in agreement with the earlier observation by Drell and Henley[4] that at least in the S states an effectively repulsive potential is obtained if the perturbation theory is avoided by a contact transformation, and that the S phase shifts are greatly reduced accordingly.

The present paper reports an attempt started[5] in the fall of 1952 to calculate pion-nucleon scattering phase shifts and using approximate methods but not using expansions in powers of the large coupling constant. The method used is essentially the Tamm-Dancoff treatment[6] of conventional three-dimensional field theory, involving an expansion of state vectors in terms of states corresponding to different numbers of virtual mesons and nucleon pairs present in the field. The treatment is applied to the charge-symmetric pseudoscalar meson theory with pseudoscalar coupling. To obtain relatively simple equations, only the states directly coupled to the principal one-meson, one-nucleon state are considered. This approximation is a drastic one but it goes much beyond perturbation theory.

With this approximation the problem can be reduced to an equation of motion in the form of a single integral equation for $g(\mathbf{p})$, the probability amplitude of a relative momentum $\mathbf{p}$ between pion and proton in the principal one-meson, one-nucleon state. This integral equation, derived in Sec. II, can be greatly simplified by using the

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[2] K. Brueckner, Phys. Rev. **86**, 106 (1952).

[3] G. F. Chew, Phys. Rev. **89**, 591 (1953).

[4] S. Drell and H. A. Henley, Phys. Rev. **88**, 1053 (1952).

[5] Dyson, Schweber, and Visscher, Phys. Rev. **90**, 372 (1953).

[6] I. Tamm, J. Phys. (U.S.S.R.) **9**, 449 (1945); S. Dancoff, Phys. Rev. **78**, 382 (1950).

conservation of isotopic spin, angular momentum, and parity (Secs. III to V), as was already pointed out by Chew.[3] For each value of T, j, and parity, the wave function $g(\mathbf{p})$ is thereby reduced to a function of a single variable, the magnitude of the momentum $\mathbf{p}$. In agreement with the general scattering theory of Lippmann and Schwinger[7] and Goldberger,[8] the function $g(\mathbf{p})$ is separated into an incident wave,

$$\delta(E-E(p)-\omega(p)),$$

and a scattered wave,

$$\mathrm{P}(1/[E-E(p)-\omega(p)])f(p)$$

(P=principal part), and it is shown (Sec. VI) that the scattering phase shift is directly related to the value of $f(p)$ on the energy shell, $f(k)$.

The integral equation for $f(p)$, Eq. (27), is an inhomogeneous one, and the Born approximation corresponds to carrying only the inhomogeneous term. Although the scattering phase shift is given directly by the value of $f(p)$ "on the energy shell," for its correct evaluation the integral equation has to be solved for all values of p (off the energy shell).

No attempt is made in the present paper to relate the Tamm-Dancoff formalism to more fully covariant four-dimensional treatments.[9] As a consequence all self-energy and renormalization terms had to be omitted, and the analysis could not be carried to completion for the two states of total angular momentum and isotopic spin of $\frac{1}{2}$. The integral equations are investigated in detail for two of the four remaining states, the $S_{\frac{1}{2}}$ and $P_{\frac{1}{2}}$ states with $T=\frac{3}{2}$, which experimentally show the largest phase shifts and most interesting energy dependence. Even these simplified equations cannot be solved by analytic means so that numerical or semi-numerical methods of solution had to be applied. Mainly to test the suitability of different numerical methods, the solutions for the two states of interest were carried out by slightly different means. In Secs. IX and X we give the approximations thus obtained for the shape of the "wave function" $g(\mathbf{p})$ and for the variation of phase shift with energy and with coupling constant. The results are discussed in Sec. XI.

II. DERIVATION OF THE EQUATION OF MOTION

An arbitrary state vector can be expanded in terms of the complete set of eigenfunctions of the number operators of the various free-field quanta. These are given by[10]

$$N^N=\int d^3p\sum_{i=1}^{2}a_i^*(\mathbf{p})a_i(\mathbf{p}),$$

$$N^{AN}=\int d^3p\sum_{i=1}^{2}b_i^*(\mathbf{p})b_i(\mathbf{p}),\quad\text{with}\quad b_i(\mathbf{p})=a_i^*(\mathbf{p}),\qquad(1)$$

$$N_i^M=\int d^3k c_i^*(\mathbf{k})c_i(\mathbf{k}),$$

for the nucleons, antinucleons, and mesons, respectively. One readily establishes that these have as their eigenvalues the positive integers and zero, and that their eigenfunctions are obtained by repeated applications of the creation operators of the various field quanta on the vacuum state vector.

Our state vector for the meson-nucleon system, Ψ, can therefore be written as

$$\Psi=\psi^{(0,0;0)}\Phi_0+\frac{1}{(1!)^{\frac{1}{2}}}\sum_r\int d^3p\,\psi^{(1,0;0)}(\mathbf{p},r)a_r^*(\mathbf{p})\Phi_0$$
$$+\frac{1}{(1!)^{\frac{1}{2}}(1!)^{\frac{1}{2}}}\sum_r\sum_i\int d^3p\int d^3k\,\psi^{(1,0;1)}$$
$$\times(\mathbf{p},r;k,i)a_r^*(\mathbf{p})c_i^*(\mathbf{k})\Phi_0+\cdots.\qquad(2)$$

The $\psi^{(m,n;l)}(\mathbf{p}_1s^{(1)},\mathbf{p}_2s^{(2)},\cdots\mathbf{p}_ms^{(m)};\mathbf{q}_1t^{(1)},\mathbf{q}_2t^{(2)},\cdots\mathbf{q}_nt^{(n)};k_1i^{(1)},\mathbf{k}_2i^{(2)},\cdots\mathbf{k}_li^{(l)})$ are then the probability amplitudes for finding m "bare" nucleons with momenta and spins and isotopic spins specified by $\mathbf{p}_1$, $\mathbf{p}_2$, $\cdots\mathbf{p}_m$, $s^{(1)}$, $s^{(2)}$, $\cdots s^{(m)}$; n "bare" antinucleons specified by the $\mathbf{q}_1$, $\cdots\mathbf{q}_n$; $t^{(1)}$, $\cdots t^{(n)}$ variables; and l mesons specified by the $\mathbf{k}$, i variables. This interpretation follows from the fact that the probability amplitude for finding for example a single nucleon of momentum p and spin and isotopic spin specified by s, in an experiment carried out on our system is given by

$$(a_s^*(\mathbf{p})\Phi_0,\Psi)=g^{(1,0;0)}(\mathbf{p},s).\qquad(3)$$

Similarly the one-meson one-nucleon probability amplitude is given by

$$(a_s^*(\mathbf{p})c_i^*(\mathbf{k})\Phi_0,\Psi)=g^{(1,0;1)}(\mathbf{p},s;\mathbf{k},i).\qquad(3a)$$

It should be noticed that $g^{(m,n;l)}(\mathbf{p}_1s^{(1)}\cdots\mathbf{p}_ms^{(m)};\mathbf{q}_1t^{(1)}\cdots\mathbf{q}_nt^{(n)};\mathbf{k}_1i^{(1)}\cdots\mathbf{k}_li^{(l)})$ is an antisymmetric function in the $\mathbf{p}s$ variables and $\mathbf{q}t$ variables, separately, and a symmetric function in the meson variables $\mathbf{k}i$.

The Schrödinger equation,

$$(H_0+H_I)\Psi=E\Psi,\qquad(4)$$

where H_0=non-interacting free-field Hamiltonians and H_I=interaction Hamiltonian,

$$=iG\int d^3x\bar{\psi}(\mathbf{x})\gamma_5\tau_\alpha\phi_\alpha(\mathbf{x})\psi(\mathbf{x}),\qquad(4a)$$

[7] B. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950).
[8] M. L. Goldberger, Phys. Rev. **84**, 929 (1951).
[9] M. Cini, Nuovo cimento **10**, 526 and 614 (1953); Karplus, Kievelson, and Martin, Phys. Rev. **90**, 1072 (1953); S. Deser and P. C. Martin, Phys. Rev. **90**, 1075 (1953); F. J. Dyson, Phys. Rev. **91**, 1543 (1953).
[10] See Appendix for the decomposition of operators.

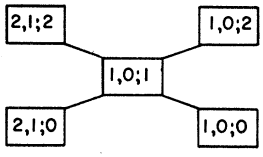


FIG. 1. The states directly coupled to the one-meson one-nucleon state.

can now be transcribed in terms of the amplitudes $g^{(m,n;l)}$. When this is done, an infinite set of coupled integral equations in the amplitudes $g^{(m,n;l)}$ is obtained. As such an infinite set of equations is not readily amenable to calculations when nucleon recoil effects are included, we have solved these equations in the lowest Tamm-Dancoff (hereafter abbreviated as T-D) approximation. The T-D approximation consists in describing the state vector Ψ in terms of only a finite number of amplitudes $g^{(m,n;l)}$ instead of in terms of an infinite set. Within this approximation, however, one then calculates all the amplitudes exactly. The particular approximation we have carried through consists in keeping only those amplitudes which are directly coupled to the one-meson one-nucleon amplitude in the state vector Ψ. This can be represented by Fig. 1. Stated concisely, we have described our state vector Ψ in terms of the amplitudes $g^{(1,0;1)}$, $g^{(1,0;0)}$, $g^{(1,0;2)}$, $g^{(2,1;2)}$, and $g^{(2,1;0)}$ only.

We shall now briefly indicate a method by which the equation of motion for these amplitudes can be obtained from the Schrödinger equation $(H_0+H_I)\Psi=E\Psi$ for the state vector. This equation is to be solved with the usual boundary condition imposed on a scattering state. In the calculations to be described, we have adopted the Goldberger[8] formalism for scattering theory in terms of standing waves. The integral equation for the state vector which incorporates the standing wave foundary condition is then given by

$$\Psi_a{}^{(1)}=\Phi_a+\mathrm{P}(1/[E_a-H_0])H_I\Psi_a{}^{(1)}, \tag{5}$$

where Φ_a is an eigenfunction of H_0 with eigenvalue E_a and corresponds to the incoming plane wave. P denotes the fact that the Cauchy principal value is to be taken for the resonance denominators. In the present problem,

$$\Phi_a=a_s{}^*(\mathbf{p}_0)c_i{}^*(\mathbf{k}_0)\Phi_0, \quad E_a=E(p_0)+\omega(k_0), \tag{6}$$

where $E(p_0)=(M^2+p_0{}^2)^{\frac{1}{2}}$, $\omega(k_0)=(\mu^2+k_0{}^2)^{\frac{1}{2}}$, p_0,s and k_0,i being the variables specifying the nucleon and meson, respectively, in the laboratory system. If we form the matrix element of Eq. (5) with the base vector $a_r{}^*(\mathbf{p})c_n{}^*(\mathbf{k})$, we obtain the equation for $g^{(1,0;1)}(\mathbf{p},r;\mathbf{k},n)$ in the form

$$\begin{aligned}(a_r{}^*(\mathbf{p})c_n{}^*(\mathbf{k})\Phi_0,\Psi_a{}^{(1)})&=g^{(1,0;1)}(\mathbf{p},r;\mathbf{k},n)\\&=(a_r{}^*(\mathbf{p})c_n{}^*(\mathbf{k})\Phi_0,\ a_s{}^*(\mathbf{p}_0)c_i{}^*(\mathbf{k}_0)\Phi_0)\\&\quad+(a_r{}^*(\mathbf{p})c_n{}^*(\mathbf{k})\Phi_0,\ \mathrm{P}(1/[E_a-H_0])H_I\Psi_a{}^{(1)}).\end{aligned} \tag{7}$$

Consider the last term in this expression which contains the interaction energy operator H_I. This operator can be expressed in terms of creation and destruction operators for the various field quanta according to the decompositions given in the Appendix. One of the terms in the decomposition of H_I corresponds to the creation of a meson with the scattering of the nucleon. We consider this term $H_I{}^{(a)}$ in detail. Explicitly, it is given by

$$\begin{aligned}H_I{}^{(a)}=iG\int d^3p_1\int d^3p_2\int d^3k\sum_{i,j}a_i{}^*(\mathbf{p}_1)a_j(\mathbf{p}_2)\\\times c_i{}^*(\mathbf{k})\delta(\mathbf{p}_1-\mathbf{p}_2+\mathbf{k})\bar{u}_i(\mathbf{p}_1)\gamma_5\tau_l u_j(\mathbf{p}_2).\end{aligned} \tag{8}$$

It is obvious that the only nonvanishing contribution to this matrix element comes from the $g^{(1,0;0)}$ amplitude of $\Psi_a{}^{(1)}$. One obtains for this

$$\begin{aligned}&iG\int d^3p_1\int d^3p_2\int d^3k_1\int d^3p'\sum_{i,j}(\Phi_0,\ a_r(\mathbf{p})c_n(\mathbf{k})\\&\quad\times\mathrm{P}(1/[E_a-H_0])a_i{}^*(\mathbf{p}_1)a_j(\mathbf{p}_2)c_l{}^*(\mathbf{k}_1)a_s{}^*(\mathbf{p}')\Phi_0)\\&\quad\cdot\delta(\mathbf{p}_1-\mathbf{p}_2+\mathbf{k}_1)\bar{u}_i(\mathbf{p}_1)\gamma_5\tau_l u_j(\mathbf{p}_2)g_s{}^{(1,0;0)}(\mathbf{p}')\\&=iG\int d^3p_1\int d^3p_2\int d^3k_1\int d^3p'\sum_{i,j}(\Phi_0,\ a_r(\mathbf{p})c_n(\mathbf{k})\\&\quad\cdot\mathrm{P}(1/[E_a-H_0])a_i{}^*(\mathbf{p}_1)\delta_{js}\delta(\mathbf{p}'-\mathbf{p}_2)c_l{}^*(\mathbf{k}_1)\Phi_0)\\&\quad\cdot\delta(\mathbf{p}_1-\mathbf{p}_2+\mathbf{k}_1)\bar{u}_i(\mathbf{p}_1)\gamma_5\tau_l u_j(\mathbf{p}_2)g_s{}^{(1,0;0)}(\mathbf{p}')\\&=iG\int d^3p_1\int d^3p_2\int d^3k_1\sum_i(\Phi_0,\ a_r(\mathbf{p})c_n(\mathbf{k})\\&\quad\cdot a_i{}^*(\mathbf{p}_1)c_l{}^*(\mathbf{k}_1)\Phi_0)\cdot\mathrm{P}(1/[E-E(p_1)-\omega(k_1)])\\&\quad\cdot\delta(\mathbf{p}_1-\mathbf{p}_2+\mathbf{k}_1)\bar{u}_i(\mathbf{p}_1)\gamma_5\tau_l u_j(\mathbf{p}_2)g_j{}^{(1,0;0)}(\mathbf{p}_2)\\&=iG\int d^3p_1\int d^3p_2\int d^3k_1\delta_{ri}\delta(\mathbf{p}_1-\mathbf{p})\delta_{nl}\delta(\mathbf{k}_1-\mathbf{k})\\&\quad\cdot\mathrm{P}(1/[E-E(p_1)-\omega(k_1)])\cdot\delta(\mathbf{p}_1-\mathbf{p}_2+\mathbf{k}_1)\\&\quad\times\bar{u}_i(\mathbf{p}_1)\gamma_5\tau_l u_j(\mathbf{p}_2)g_j{}^{(1,0;0)}(\mathbf{p}_2)\\&=iG\mathrm{P}(1/[E-E(p)-\omega(k)])\bar{u}_r(\mathbf{p})\gamma_5\tau_n\\&\quad\times u_j(\mathbf{p}_1+\mathbf{k}_1)g_j{}^{(1,0;0)}(\mathbf{p}_1+\mathbf{k}_1).\end{aligned} \tag{9}$$

Following this procedure, we obtain an equation for $g^{(1,0;1)}$ which involves coupling to the four other amplitudes. Similarly, one derives equations of motion for the amplitudes $g^{(1,0;0)}$, etc., which as a result of our particular approximation couple back only to the $g^{(1,0;1)}$ amplitude.[11] It is therefore possible to substitute the expression for these amplitudes back in the equation for $g^{(1,0;1)}$ and thus obtain an integral equation for $g^{(1,0;1)}(\mathbf{p},\mathbf{k})$ alone. This restates the Schrödinger equation in our approximation. The integral equation for the $g(\mathbf{p},\mathbf{k})$ [hereafter the superscript (1,0,1) will be

[11] If more states were kept, then one would have gotten a set of coupled integral equations, which could not be reduced to a single integral equation except by having the kernel in the form of an infinite sum with the coupling constant as the expansion parameter, which would be against the spirit of the present approach.

dropped, since we shall be concerned with this amplitude only] is

$$[E-E(p)-\omega(k)]g_{u\alpha}(\mathbf{p},\mathbf{k})=-\frac{G^2}{16\pi^3}\int d^3s\frac{M^2}{E(p)E(\mathbf{p}+\mathbf{s})\omega(s)}\bar{u}(\mathbf{p})\gamma_5\left\{\frac{3\Lambda_+(\mathbf{p}+\mathbf{s})}{E-E(\mathbf{p}+\mathbf{s})-\omega(k)-\omega(s)}\right.$$
$$\left.+\frac{3\Lambda_-(-\mathbf{p}-\mathbf{s})}{E-2E(p)-E(\mathbf{p}+\mathbf{s})-\omega(s)-\omega(k)}\right\}\gamma_5 u(\mathbf{p})g_{u\alpha}(\mathbf{p},\mathbf{k})+\frac{G^2}{16\pi^3}\int d^3s\frac{M^2}{E(s)E(\mathbf{s}+\mathbf{k})\omega(k)}$$
$$\times 2\,\mathrm{Sp}[\Lambda_+(\mathbf{s})\gamma_5\Lambda_-(-\mathbf{s}-\mathbf{k})\gamma_5]\cdot\left[\frac{1}{E-E(s)-E(\mathbf{s}+\mathbf{k})-E(p)}+\frac{1}{E-2E(p)-2\omega(k)-E(s)-E(\mathbf{s}+\mathbf{k})}\right]g_{u\alpha}(\mathbf{p},\mathbf{k})$$
$$+\frac{G^2}{16\pi^3}\int d^3s\int d^3t\frac{M^2}{E(s)E(t)\omega(\mathbf{s}+\mathbf{t})}\delta(0)\cdot\frac{6\,\mathrm{Sp}[\Lambda_+(\mathbf{s})\gamma_5\Lambda_-(\mathbf{t})\gamma_5]}{E-E(p)-\omega(k)-E(s)-E(t)-\omega(\mathbf{s}+\mathbf{t})}g_{u\alpha}(\mathbf{p},\mathbf{k})$$
$$-\frac{G^2}{16\pi^3}\int d^3s\left(\frac{M^4}{E(p)E^2(\mathbf{p}+\mathbf{k})E(\mathbf{p}+\mathbf{k}+\mathbf{s})\omega(k)\omega(s)}\right)^{\frac{1}{2}}\cdot\bar{u}(\mathbf{p})\gamma_5\left\{\frac{\tau_\alpha\Lambda_+(\mathbf{p}+\mathbf{k})\tau_\delta}{E-E(\mathbf{p}+\mathbf{k})}\right.$$
$$+\frac{\tau_\alpha\Lambda_-(-\mathbf{p}-\mathbf{k})\tau_\delta}{E-E(p)-E(\mathbf{p}+\mathbf{k})-E(\mathbf{p}+\mathbf{k}+\mathbf{s})-\omega(k)-\omega(s)}+\frac{\tau_\delta\Lambda_+(\mathbf{p}+\mathbf{s})\tau_\alpha}{E-E(\mathbf{p}+\mathbf{s})-\omega(k)-\omega(s)}$$
$$\left.+\frac{\tau_\delta\Lambda_-(-\mathbf{p}-\mathbf{s})\tau_\alpha}{E-E(\mathbf{p}+\mathbf{s})-E(\mathbf{p}+\mathbf{k}+\mathbf{s})-E(p)}\right\}\cdot\gamma_5 w(\mathbf{p}+\mathbf{s}+\mathbf{k})g_{w\delta}(\mathbf{p}+\mathbf{s}+\mathbf{k};-\mathbf{s}).\quad(10)$$

Here

$$\Lambda_+(\mathbf{p})=\frac{p+iM}{2iM}\quad\text{and}\quad\Lambda_-(\mathbf{p})=\frac{p-iM}{2iM}.\quad(11)$$

This equation contains nine terms describable by the following time-ordered Feynman graphs. The last four terms, which connect $g(\mathbf{p},\mathbf{k})$ with g for other momenta by an integral equation, arise from the graphs of Fig. 2

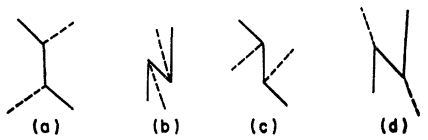


FIG. 2. The graphs responsible for meson-nucleon scattering.

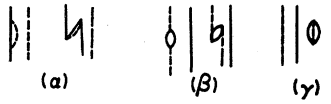
which correspond to Compton scattering. The other five terms arise as a result of the self-energy processes indicated by the diagrams (α), (β), and (γ) of Fig. 3; (α) giving rise to the first two terms, (β) to the next two, and (γ) to the last one. These five terms, however, involve $g(\mathbf{p},\mathbf{k})$ only as a multiplicative factor. Since the $g(\mathbf{p},\mathbf{k})$ only occur in these five terms as a multiplicative factor, the correct treatment would be to transfer them to the left-hand side and obtain an equation of the form

$$[S_E(\mathbf{p})+\Delta_E(\mathbf{k})+\Delta E+1]g(\mathbf{p},\mathbf{k})=\int L(\mathbf{p},\mathbf{k};\mathbf{p}',\mathbf{k}')g(\mathbf{p}',\mathbf{k}')d^3\mathbf{k}'.\quad(12)$$

The first two factors on the left-hand side in a covariant treatment would correspond to using modified propagation functions (S_F' and Δ_F' rather than S_F and Δ_F), thus allowing for self-energy parts in the nucleon and meson lines in intermediary states. The ΔE term arises from graph (γ) and corresponds to a redefinition of the vacuum energy. In the present formalism we are prevented from evaluating these terms correctly since they diverge, and there exists at present no unambiguous method of subtracting all the self-energy parts.

The difficulty in renormalizing the vacuum self-energy term lies in the fact that we have diagonalized the total energy in the subspace of complete Hilbert space composed of not more than (2,1;2) particles. This means that the vacuum self energy is taken into account in the (1,0;1) state but not in the other intermediary states, thus causing a coupling between the physical system and the vacuum fluctuations. In this respect our present formalism is therefore wrong. A modified Tamm-Dancoff treatment, developed recently by one of us (F.J.D.)[9] bypasses this difficulty by defining the amplitudes with respect to the true vacuum state vector Ψ_0, rather than with respect to the bare vacuum state vector Φ_0 as done here. In such a treatment no vacuum self-energy term is present, and only the finite quantity $E-E_0$ (E_0=energy eigenvalue of true vacuum state) occurs. Furthermore, this modified form of the Tamm-Dancoff method has the great advantage that the self-energy parts can be unambiguously renormalized and the finite parts retained. In the present work we neglect the contribution due to the self energy terms.[12] The terms corresponding to the Compton

FIG. 3. The graphs involving renormalization corrections.



[12] Some of the renormalization terms have been evaluated by W. Visscher and will soon be published by him

diagrams (a), (b), (c), and (d) of Fig. 2 do not differ markedly in the "new" Tamm-Dancoff method from the ones evaluated in the present paper.

After neglect of the self-energy terms, we are left only with the last four terms in the integral equation (10). We write

$$\sum u(p)g_{u\alpha}(\mathbf{p},\mathbf{k})=g_\alpha'(\mathbf{p},\mathbf{k}),\quad Q_{\alpha\delta}'=\tau_\delta\tau_\alpha,\quad Q_{\alpha\delta}=\tau_\alpha\tau_\delta,$$

and go to the center-of-mass system, setting

$$g_\alpha'(\mathbf{p},\mathbf{k})\rightarrow g_\alpha'(\mathbf{p},\ -\mathbf{p})\equiv g_\alpha'(\mathbf{p}).$$

By utilizing the relations $\gamma_5\Lambda_+(\mathbf{p})\gamma_5=\Lambda_-(\mathbf{p})$; $\gamma_5\Lambda_-(\mathbf{p})\gamma_5=\Lambda_+(\mathbf{p})$, the equation may be written in the center-of-mass system as

$$[E-E(p)-\omega(p)]g'_\alpha(\mathbf{p})=-\frac{G^2}{16\pi^3}\Lambda_+(\mathbf{p})\int d^3s\frac{M^2}{(E(p)\omega(p)E(s)\omega(s))^{\frac{1}{2}}}\Big\{Q_{\alpha\delta}\frac{\Lambda_-(0)}{E-M}+Q_{\alpha\delta}\frac{\Lambda_+(0)}{E-E(p)-\omega(p)-E(s)-\omega(s)-M}$$
$$+Q_{\alpha\delta}'\frac{\Lambda_-(\mathbf{p}+\mathbf{s})}{E-E(\mathbf{p}+\mathbf{s})-\omega(p)-\omega(s)}+Q_{\alpha\delta}'\frac{\Lambda_+(-\mathbf{p}-\mathbf{s})}{E-E(\mathbf{p}+\mathbf{s})-E(p)-E(s)}\Big\}g_\delta'(\mathbf{s}).\quad(13)$$

Hereafter, since we are only going to deal with the amplitudes $g_\alpha'(\mathbf{p})$, we shall drop the prime on the g's, as there will be no confusion.

III. SEPARATION OF STATES OF DIFFERENT ISOTOPIC SPIN

The only operators in (13) involving isotopic spin are clearly Q and Q'. Their properties and eigenvalues may very easily be determined. $Q_{\alpha\delta}=\tau_\alpha\tau_\delta$, $Q_{\alpha\delta}^2=Q_{\alpha\beta}Q_{\beta\delta}=\tau_\alpha\tau_\beta\tau_\beta\tau_\delta=3Q_{\alpha\delta}$. Thus $Q_{\alpha\delta}$ has the eigenvalues 3 or 0. To assign these to isotopic spins, we note that $Q_{\alpha\delta}$ means the annihilation of the meson δ which is originally present, followed by the creation of a new meson, α. Now the annihilation can obviously only take place if $T=\frac{1}{2}$, because only for this isotopic spin do we have a state of the nucleon by itself. Hence $Q_{\frac{1}{2}}=3$ and $Q_{\frac{3}{2}}=0$. The other operator is

$$Q_{\alpha\delta}'=\tau_\delta\tau_\alpha=2-\tau_\alpha\tau_\delta,$$

and therefore has the eigenvalues,

$$Q_{\frac{1}{2}}'=-1,\quad Q_{\frac{3}{2}}'=2.$$

IV. REDUCTION OF WAVE FUNCTION TO LARGE COMPONENTS

Since $g(\mathbf{p})$ refers to a state of positive nucleon energy, it is clear that $\Lambda_-(\mathbf{p})g(\mathbf{p})=0$. We now write the four-component function

$$g(\mathbf{p})=\begin{pmatrix}g_+(\mathbf{p})\\ g_-(\mathbf{p})\end{pmatrix},$$

where $g_+(\mathbf{p})$ and $g_-(\mathbf{p})$ are two-component Pauli spinors. By eliminating the small components $g_-(\mathbf{p})$ in favor of the large components $g_+(\mathbf{p})$ we get

$$g(\mathbf{p})=\left(1+\frac{\gamma_5(\boldsymbol{\sigma}\cdot\mathbf{p})}{E(p)+N}\right)\begin{pmatrix}g_+(\mathbf{p})\\ 0\end{pmatrix}.$$

Since $(\boldsymbol{\sigma}\cdot\mathbf{p})$ is an invariant under rotations in 3-space, it is clear that $g(\mathbf{p})$ has the same total angular momentum as $g_+(\mathbf{p})$. It is convenient to make the reduction in this way, because $g_+(\mathbf{p})$ may be a function with definite orbital angular momentum, thus definite parity, while $g(\mathbf{p})$ clearly may not be.

Equation (13) then becomes

$$[E-E(p)-\omega(p)]g_\alpha(\mathbf{p})=-\frac{G^2}{16\pi^3}\frac{[-\beta\gamma_5(\boldsymbol{\sigma}\cdot\mathbf{p})+\beta E(p)+M]}{2M}\Big[\int d^3s\left(\frac{M^4}{E(p)\omega(p)E(s)\omega(s)}\right)^{\frac{1}{2}}\cdot\Big\{Q_{\alpha\delta}\frac{1-\beta}{2(E-M)}$$
$$+Q_{\alpha\delta}\frac{1+\beta}{2[E-E(p)-\omega(p)-E(s)-\omega(s)-M]}\Big\}+\int d^3s\frac{M^2}{[E^2(\mathbf{p}+\mathbf{s})E(p)\omega(p)E(s)\omega(s)]^{\frac{1}{2}}}$$
$$\cdot\Big\{Q_{\alpha\delta}'\frac{\beta\gamma_5[\boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{s})]-\beta E(\mathbf{p}+\mathbf{s})+M}{2M[E-E(\mathbf{p}+\mathbf{s})-\omega(p)-\omega(s)]}+Q_{\alpha\delta}'\frac{\beta\gamma_5[\sigma\cdot(\mathbf{p}+\mathbf{s})]+\beta E(\mathbf{p}+\mathbf{s})+M}{2M[E-E(\mathbf{p}+\mathbf{s})-E(p)-E(s)]}\Big\}\Big]\left(1+\frac{\gamma_5(\boldsymbol{\sigma}\cdot\mathbf{s})}{E(s)+M}\right)g_\delta^+(\mathbf{s}).\quad(14)$$

To find the equation for $g_+(\mathbf{p})$ in terms of $g_+(\mathbf{s})$, we pick out the terms on the right-hand side which do not mix small and large components (i.e., those with Dirac matrices 1 or γ_5^2, and with any β's or σ's). Remembering that

$\gamma_5^2=1$, and that $\beta=1$ for the "large" component, we get

$$[E-E(p)-\omega(p)]g_{+\alpha}(\mathbf{p})=\frac{G^2}{64\pi^3}\Big[2\int d^3s\frac{1}{[E(p)\omega(p)E(s)\omega(s)]^{\frac{1}{2}}}\cdot Q_{\alpha\delta}\Big[\frac{(\boldsymbol{\sigma}\cdot\mathbf{p})(\boldsymbol{\sigma}\cdot\mathbf{s})}{(E-M)[E(s)+M]}$$
$$-\frac{E(p)+M}{E-E(p)-\omega(p)-E(s)-\omega(s)-M}\Big]+\int d^3s\frac{1}{[E^2(\mathbf{p}+\mathbf{s})E(p)\omega(p)E(s)\omega(s)]^{\frac{1}{2}}}Q_{\alpha\delta}'\Big\{\frac{1}{E-E(\mathbf{p}+\mathbf{s})-\omega(p)-\omega(s)}$$
$$\cdot\Big(\frac{(\boldsymbol{\sigma}\cdot\mathbf{p})(\boldsymbol{\sigma}\cdot\mathbf{s})[E(\mathbf{p}+\mathbf{s})+M]}{E(s)+M}-(\boldsymbol{\sigma}\cdot\mathbf{p})[\boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{s})]+[E(p)+M][E(\mathbf{p}+\mathbf{s})-M]-\frac{[\boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{s})](\boldsymbol{\sigma}\cdot\mathbf{s})[E(p)+M]}{E(s)+M}\Big)$$
$$-\frac{1}{E-E(\mathbf{p}+\mathbf{s})-E(p)-E(s)}\Big((\boldsymbol{\sigma}\cdot\mathbf{p})(\boldsymbol{\sigma}\cdot\mathbf{s})\frac{E(\mathbf{p}+\mathbf{s})-M}{E(s)+M}+(\boldsymbol{\sigma}\cdot\mathbf{p})[\boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{s})]+[E(p)+M][E(\mathbf{p}+\mathbf{s})+M]$$
$$+[\boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{s})](\boldsymbol{\sigma}\cdot\mathbf{s})\frac{E(p)+M}{E(s)+M}\Big)\Big\}\Big]g_{+\delta}(\mathbf{s}). \quad (15)$$

V. ELIMINATION OF ANGULAR AND SPIN DEPENDENCE OF THE INTEGRAL EQUATION

Let

$$A=E(p)+\omega(p)+E(s)+\omega(s)+M-E,$$
$$B=E(p)+E(s)-E,\quad C=\omega(p)+\omega(s)-E, \quad (16)$$

and let S_n and R_n be the operators defined by

$$S_ng(\mathbf{s})=\frac{1}{4\pi}\int d\Omega_sP_n(\Theta_s-\Theta_p)g(\mathbf{s})$$

and

$$R_ng(\mathbf{s})=\frac{1}{4\pi}\int d\Omega_s\frac{(\boldsymbol{\sigma}\cdot\mathbf{p})}{|p|}\frac{(\boldsymbol{\sigma}\cdot\mathbf{s})}{|s|}P_n(\Theta_s-\Theta_p)g(\mathbf{s}). \quad (17)$$

S_n clearly has the "eigenvalue" $1/(2n+1)$ for a state g with an orbital angular momentum $l=n$ in the sense that $S_ng(s,\Theta_s)=\delta_{nl}[1/(2l+1)]g(s,\Theta_p)$. The eigenvalues of R_n may be found as follows. Let $\phi(\mathbf{s})=[(\boldsymbol{\sigma}\cdot\mathbf{s})/|s|]g(\mathbf{s})$. Then, as has already been pointed out, $\phi(s)$ has the same total j as $g(\mathbf{s})$ but has the opposite parity. Thus if $g(s)$ has $l=j\pm\frac{1}{2}$, $\phi(s)$ has $l=j\mp\frac{1}{2}$ so that ϕ and g are symmetrically related to each other, and $g(\mathbf{s})=\phi(\mathbf{s})\cdot(\boldsymbol{\sigma}\cdot\mathbf{s})/|s|$, but inserting the definition of ϕ in (17), we may write $R_ng(\mathbf{s})=S_n\phi(\mathbf{s})(\boldsymbol{\sigma}\cdot\mathbf{p})/|p|$. Hence, R_n has eigenvalue $[1/(2n+1)]\delta_{n,l\pm1}$ in the same sense as before.

We define also:

$$[E(\mathbf{p}+\mathbf{s})]^{-1}=\sum_{n=0}^{\infty}X_nP_n(\Theta_s-\Theta_p),$$
$$[B+E(\mathbf{p}+\mathbf{s})]^{-1}=\sum_{n=0}^{\infty}Y_nP_n(\Theta_s-\Theta_p),$$

and

$$[C+E(\mathbf{p}+\mathbf{s})]^{-1}=\sum_{n=0}^{\infty}Z_nP_n(\Theta_s-\Theta_p). \quad (18)$$

Then we can evaluate such integrals as

$$\frac{1}{4\pi}\int d\Omega_s[E(\mathbf{p}+\mathbf{s})]^{-1}(\boldsymbol{\sigma}\cdot\mathbf{p})(\boldsymbol{\sigma}\cdot\mathbf{s})g(\mathbf{s})$$
$$=\sum_{n=0}^{\infty}|p||s|R_nX_ng(|s|),$$

and Eq. (18) becomes

$$[E-E(p)-\omega(p)]g(p)=\frac{G^2}{8\pi^2}\int_0^{\infty}\frac{s^2ds}{[E(p)\omega(p)E(s)\omega(s)]^{\frac{1}{2}}}\Big\{Q\frac{psR_0}{(E-M)(E(s)+M)}+Q\frac{[E(p)+M]S_0}{A}$$
$$+\frac{Q'}{2}\sum_n\Big[(Y_n-Z_n)\Big(S_n[E(p)+M]+\frac{psR_n}{E(s)+M}\Big)+\Big(\frac{1}{C}X_n+\frac{1}{B}X_n-\frac{1}{C}Z_n-\frac{1}{B}Y_n\Big)$$
$$\times\Big(\frac{psR_n[E(p)+E(s)+M]}{E(s)+M}+[E(p)+M][E(p)+E(s)-M]S_n\Big)\Big]\Big\}g(s). \quad (19)$$

The coefficients X_n, Y_n, and Z_n in the expression (18) can be easily determined. For this purpose write

$$E^2(\mathbf{p}+\mathbf{s})\equiv M^2+p^2+s^2+2psx\equiv\bar{E}^2(1+2rx+r^2), \quad (20a)$$

with $x=\cos(\Theta_s-\Theta_p)$. Then, clearly,

$$E(p+s)=\bar{E}(1+r),\quad E(p-s)=\bar{E}(1-r),$$

and

$$\bar{E}=[E(p+s)+E(p-s)]/2.$$

Also, by comparing coefficients in (20a),

$$r=ps/\bar{E}^2,\quad M^2+p^2+s^2=\bar{E}^2(1+r^2). \quad (20b)$$

Introducing further the abbreviations

$$z=E(\mathbf{p}+\mathbf{s})/\bar{E},\quad b=B/\bar{E},\quad\text{and}\quad c=C/\bar{E}, \quad (20c)$$

we get

$$x=(z^2-1-r^2)/2r;$$

and the coefficients become

$$X_n=(n+\tfrac{1}{2})(r\bar{E})^{-1}\int_{1-r}^{1+r}dzP_n\left(\frac{z^2-1-r^2}{2r}\right),$$

$$Y_n=(n+\tfrac{1}{2})(r\bar{E})^{-1}\int_{1-r}^{1+r}dz\left(\frac{z}{b+z}\right)P_n\left(\frac{z^2-1-r^2}{2r}\right),$$

and

$$Z_n=(n+\tfrac{1}{2})(r\bar{E})^{-1}\int_{1-r}^{1+r}dz\left(\frac{z}{c+z}\right)P_n\left(\frac{z^2-1-r^2}{2r}\right).$$

It is convenient to define the integrals

$$K_n(C)=\frac{1}{2ps}\int_{\bar{E}-R}^{\bar{E}+R}\frac{1}{C+Z}P_n\left(\frac{Z^2-\bar{E}^2-R^2}{2R}\right)dZ,$$

$$R=r\bar{E}, \quad \text{and} \quad Z=z\bar{E}, \tag{22}$$

and similarly $K_n(B)$. Then $X_n-Y_n=(2n+1)BK_n(B)$, and $X_n-Z_n=(2n+1)CK_n(C)$. Considering states with a definite l and j, we have seen above that the sum over n in (19) reduces to a single term, and S_n and R_n have the eigenvalues $\delta_{nl}/(2n+1)$ and $\delta_{n,l\pm1}/(2n+1)$, respectively, for $j=l\pm\frac{1}{2}$. Then (19) becomes

$$[E-E(p)-\omega(p)]g(p)=\frac{G^2}{8\pi^2}\int_0^\infty\frac{s^2ds}{[E(p)\omega(p)E(s)\omega(s)]^{\frac{1}{2}}}\cdot\Big\{Q\left(\frac{E(p)+M}{A}\right)_{S_l^j}+Q\left(\frac{1}{(E-M)}\frac{ps}{[E(s)+M]}\right)_{P_l^j}$$

$$+\tfrac{1}{2}Q'[E(p)+M][(A-2M)K_l(C)+(E-M)K_l(B)]+\tfrac{1}{2}Q'\frac{ps}{E(s)+M}[AK_{l\pm1}(C)+(E+M)K_{l\pm1}(B)]\Big\}g(s). \tag{23}$$

Thus finally we have an integral equation in one variable, the absolute momentum of the meson, which may be solved numerically. For scattering problems, $g(p)$ has a singularity because the factor $E-E(p)-\omega(p)$ becomes zero at a certain value of p, *viz.*, the momentum of the incident meson. Hence the integral equation has a singularity at this point, and its solution can be written in the form

$$g(p)=c\delta(E-E(p)-\omega(p))+\frac{1}{E-E(p)-\omega(p)}\times\int_0^\infty\frac{s^2ds}{[E(p)\omega(p)E(s)\omega(s)]^{\frac{1}{2}}}L(p,s)g(s), \tag{24}$$

where

$$L(p,s)=\text{expression in curly bracket in (23).} \tag{24a}$$

We may now define a nonsingular amplitude f by setting

$$g(p)=\delta(E-E(p)-\omega(p))+\mathrm{P}\frac{1}{E-E(p)-\omega(p)}f(p), \tag{25}$$

where we have set the normalization constant c in the previous expression equal to unity without any loss of generality since the normalization of $g(p)$ has not been fixed. In (25) we have chosen to take the principal value of the singular factor $[E-E(p)-\omega(p)]^{-1}$, since the use of purely real wave functions will simplify the numerical work. This means that $g(p)$ is required to be a standing wave (Lippmann and Schwinger,[7] Goldberger[8]). Next we shall find the relation of $f(p)$ to the scattering phase shift.

VI. RELATION OF $f(p)$ TO THE SCATTERING PHASE SHIFT

The asymptotic coordinate space wave function corresponding to $g(p)$ is determined wholly by the behavior of $\delta(E-E(p)-\omega(p))+\mathrm{P}(1/[E-E(p)-\omega(p)])$ near its singularity. If we designate the momentum of the incident meson by k, and assume that $f(p)$ is well-behaved near $p=k$, we may write

$$g(p)=\delta(p-k)+\mathrm{P}(1/[k-p])f(k)+R(p),$$

where $R(p)$ is a nonsingular function. A typical wave of angular momentum l behaves for large r as

$$g(\mathbf{r})\sim[j_l(kr)-\tan\delta\eta_l(kr)]Y_l^m(\mathbf{r})$$

(definition of the phase shift). Asymptotically, one finds that the singular part of the wave function is proportional to

$$g(\mathbf{r})=[\sin(kr-(l+\tfrac{1}{2})\pi/2)+\tan\delta_l\cos(kr-(l+\tfrac{1}{2})\pi/2)].$$

Expanding the radial function in terms of spherical Bessel functions $j_l(pr)$ whose asymptotic behavior is $\sin(pr-(l+\frac{1}{2})\pi/2)$, we find

$$g(p)=\delta(p-k)-(1/\pi)\tan\delta(1/[k-p])+R(p),$$

where R is a function of p which is regular at k. Therefore the identification may be made that

$$f(k)=-(1/\pi)\tan\delta, \tag{26}$$

and Eq. (24) becomes

$$f(p)=\frac{G^2}{8\pi^2}\int_0^\infty\frac{s^2}{[E(p)\omega(p)E(s)\omega(s)]^{\frac{1}{2}}}L(p,s)g(s)ds, \tag{27}$$

where $L(p,s)$ is given by the expression within the curly brackets of (23) and $g(s)$ stands for

$$g(s)=\delta(E-E(s)-\omega(s))+\mathrm{P}(1/[E-E(s)-\omega(s)])f(s). \tag{27a}$$

VII. DISCUSSION OF THE INTEGRAL EQUATION

The integral equation we have obtained after omitting all self-energy terms above for the meson-nucleon scat-

tering problem is an inhomogeneous one, and the terms therein arise as follows. The first and second terms in $L(p,s)$ arise from graphs (a) and (b), of Fig. 2, graph (a) contributing only to the $p_{\frac{1}{2}}$ state and (b) to the $S_{\frac{1}{2}}$ state respectively, and only for isotopic spin $\frac{1}{2}$. The next two terms arise from graphs (c) and (d), (c) giving the $K(C)$ terms and (d) giving the $K(B)$ terms.

The kernel L in the integral equation is real for $E<M+\mu$, where E is the actual energy of the system in the center-of-mass system. For energies greater than $M+\mu$, L becomes complex over the region of momenta for which the production of an additional real meson is possible. Thus for p (or s) between 0 and p_m, where p_m satisfies $\omega(p_m)+E(p_m)=E-\mu$, there is a pair of singularities in $\ln[(\bar{E}+C+R)/(\bar{E}+C-R)]$ at $[R/(C+\bar{E})]=\pm 1$, between which the logarithm is imaginary. Keeping the real and imaginary parts, one could calculate the total cross section for the production of an additional meson. This singularity in the kernel also affects the scattering problem. For simplicity, the principal part of the kernel alone was retained. If the energy is only slightly above threshold, $E-M-\mu\ll\mu$, the effect of the imaginary part of the kernel is small. Further, if we consider the cross section for meson production near threshold as small of first order, then the error made in the scattering calculation by retaining only the principal part is small of second order.

We now have the task of solving a one-dimensional integral equation with a real, nonsymmetric kernel $L(p,s)$ for the wave function $f(p)$, for each of the eigenstates of orbital and total angular momentum and isotopic spin. The quantity which can be compared directly with experiment is then the phase shift for each state δ, which is given in terms of the wave function "on the energy shell" $f(k)$ by the relation (26).

We focus our attention now on the S and P states. The functions K_l required for these states are

$$K_0(C)=\frac{1}{2ps}\ln\frac{\bar{E}+C+R}{\bar{E}+C-R},$$

$$K_1(C)=-\frac{1}{2ps}(M^2+p^2+s^2-C^2)K_0(C)+\frac{1}{2ps}\left(1-\frac{C}{\bar{E}}\right),$$

and

$$K_2(C)=-\frac{3}{4ps}(M^2+p^2+s^2-C^2)K_1(C)+\frac{C}{4\bar{E}^3}-\tfrac{1}{2}K_0(C); \tag{28}$$

and similarly for $K_l(B)$.

It will be convenient to compare our results with the equivalent Born approximation results. The lowest-order Born approximation can be obtained from Eq. (27) by retaining only the δ-function term in the expression for $f(s)$ in the integral on the right-hand side. The general expression for $f_B(p)$ is

$$f_B(p)=\frac{G^2}{8\pi^2}\left(\frac{E(k)\omega(k)}{E(p)\omega(p)}\right)^{\frac{1}{2}}\frac{k}{E}L(p,k);$$
$$E=E(k)+\omega(k). \tag{28a}$$

For the S and the P states, the expression for $L(p,k)$ involves K_0, K_1, and K_2 functions, for which the general expressions have been given above. If we now consider $(E-M)/M$ and μ/M small compared with unity, then the expansion parameters $R/(\bar{E}+C)$ or $R/(\bar{E}+B)$ involved in the logarithms in K_0, K_1, and K_2 are small,[13] hence the logarithms can be expanded in terms of the parameter, and the first terms retained. It may also be noticed[14] that $A-2M\sim\bar{E}+C$. Under these approximations the following simpler expressions for the first-order Born approximation are obtained.

$S_{\frac{1}{2}}$: $L(p,k)$

$$=Q\frac{E(p)+M}{M}+\tfrac{1}{2}Q'\frac{E(p)+M}{\bar{E}}\left(1+\frac{E-M}{\bar{E}+B}\right),$$

$P_{\frac{1}{2}}$: $(pk)^{-1}L(p,k)$

$$=Q\frac{1}{(E-M)(E(k)+M)}$$
$$+\tfrac{1}{2}Q'\left[-\frac{E(p)+M}{\bar{E}^3}\left(\frac{2\bar{E}+C}{3(\bar{E}+C)}+\frac{E-M}{2(\bar{E}+B)}\right)\right.$$
$$\left.+\frac{1}{\bar{E}[E(k)+M]}\left(1+\frac{2M}{\bar{E}+C}+\frac{E+M}{\bar{E}+B}\right)\right], \tag{29}$$

and

$P_{\frac{3}{2}}$: $(pk)^{-1}L(p,k)$

$$=-\tfrac{1}{2}Q'\frac{E(p)+M}{\bar{E}^3}\left(\frac{2\bar{E}+C}{3(\bar{E}+C)}+\frac{E-M}{2(\bar{E}+B)}\right).$$

The quantity $\bar{E}$ which was previously a function of p and s is now a function of p and k. At low energies, i.e., $p<M$ it is found that the $K(B)$ terms arising from time ordered graph (d) are much smaller than the $K(C)$ terms arising from graph (c). The latter are particularly large as the denominators $\bar{E}+C$ are small. The Born approximation phase shifts are plotted against energy in Fig. 4. All these states are seen to imply a repulsive potential except the $P_{\frac{3}{2}}$, $T=\frac{3}{2}$ state. The use of Born approximation is based on the assumption that the effect of $f(s)$ in the integral equation is negligible compared to that of the delta function. But physically reasonable values of the coupling constant $G^2/4\pi$ are larger than unity rather than being small and the

[13] For $\bar{E}+C$ this is not quite obvious, since this quantity is small. If also $p\ll M$, then

$$\bar{E}+C\approx E(k)+\omega(k)+\omega(p)-E=\omega(p) \tag{a}$$

because of (28a). On the other hand,

$$R=r\bar{E}=pk/\bar{E}\sim pk/M, \tag{b}$$

and, since $p<\omega(p)$, we have $[R/(\bar{E}+C)]<k/M\ll 1$. If p is larger,

$$\bar{E}+C\approx E(p)+\omega(p)+\omega(k)-E\approx\omega(p)+E(p)-M, \tag{c}$$

which makes the result hold *a fortiori*.

[14] $A-2M=E(k)+\omega(k)-E+E(p)+\omega(p)-M$, which, by (28a) is equal to expression (c) in the preceding footnote.

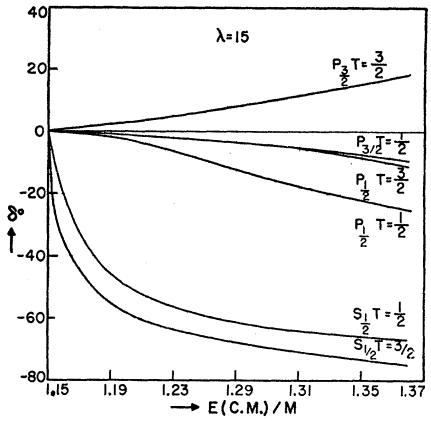


FIG. 4. Born approximation phase shifts as a function of c.m. energy $\lambda=15$.

effect of $f(s)$ is in fact quite large. The higher-order Born approximation could be obtained from Eq. (27) by using as initial trial wave function on the right-hand side the delta function instead of the whole $g(s)$, and then iterating the first-order Born approximation thus obtained. But since the $f(s)$ terms are so important this initial trial wave function is very poor, and the iteration procedure would converge very slowly, if at all. Because of the large G, we expect the first-order Born approximation results to be very poor, and direct solution of the integral equation is clearly necessary

VIII. GENERAL PROPERTIES OF THE INTEGRAL EQUATION

The kernel of our integral equation is a complicated function of the energy E as well as of the variables p and s. Therefore any accurate numerical solution of the equation will have to be found separately for each energy of interest. Before discussing such numerical solutions of our actual equation, however, some qualitative statements can be made on the general problem. Equation (27) may be rewritten by inserting (27a) and (28a) as

$$f(p)=f_B(p)-\frac{G^2}{8\pi^2}\int_0^\infty ds\frac{s^2}{[E(p)\omega(p)E(s)\omega(s)]^{\frac{1}{2}}}\times L(p,s)\frac{1}{E(s)+\omega(s)-E}f(s). \quad (30)$$

We shall now show that the determining feature of this equation is the *sign* of the integral on the right-hand side. If this is the same as the sign of $f(p)$, then the integral will tend to enhance $f(p)$ and the amplitude will be larger than the Born approximation amplitude f; if the integral has a sign opposite to that of $f(p)$, the reverse will be the case. Now $E(s)+\omega(s)-E$ is positive for most values of s; hence $L(p,s)$ must be *negative* in order to enhance f above the Born approximation value. From (29) it is seen that this will be the case, e.g., for the $P_{\frac{3}{2}}$ state if Q' is positive, which is true for $T=\frac{3}{2}$. Thus this state is apt to have a much higher phase shift than the Born approximation indicates, which offers hope that the observed large phase shift can be explained.

The situation for negative $L(p,s)$ is similar to that found in elementary problems with an attractive potential; in fact, the wave equation for the momentum wave function of the hydrogen atom has the form

$$[E-E(p)]g(\mathbf{p})=\int V(s-\mathbf{p})g(\mathbf{s})d^3s, \quad (31)$$

where $V(\mathbf{k})$ is the Fourier transform of the potential energy. The similarity with (23) is obvious, and it is also clear that $L(p,s)$, except for a factor, corresponds to $V(\mathbf{s}-\mathbf{p})$; hence negative L corresponds to a negative (attractive) potential. It is well known that in an attractive potential the scattering can be much larger than the Born approximation. In particular, we can have a *resonance* in which $(f/f_B)=\infty$. The same phenomenon can occur with our integral equation. If $L(p,s)$ is kept fixed and G is permitted to increase, a value of G will finally be reached at which the homogeneous equation, i.e., (30) without the term f_B, will have a solution. For this critical value of G, then, $f(k)=\infty$, and therefore [Eq. (26)] the phase shift δ is 90° as it should be for a resonance. (Incidentally, if L is negative, then also f_B is negative; therefore, as long as G is below its critical value, f is negative, and the phase shift is positive, as it should be for an attractive potential. For G greater than its critical value, f will have a sign opposite to f_B, hence positive, and δ is greater than 90°.)

If $L(p,s)$ is positive, we have the same situation as for a respulsive potential; f is less than the Born approximation, and a resonance cannot occur. An example is clearly the S state of $T=\frac{3}{2}$ for which (29) gives a positive $L(p,s)$. Thus the phase shift of this state will be less than the Born approximation which again gives hope for achieving agreement with experiment.

The $P_{\frac{3}{2}}$ state of $T=\frac{1}{2}$ is obviously also of repulsive character since $Q'=-1$ in this case. For the $P_{\frac{1}{2}}$ state we should consider primarily the terms with $\bar{E}+C$ in the denominator because this quantity is small compared with M, $\bar{E}$, etc. From (29) we find that

$$L(p,k)=pk\left\{Q\frac{1}{(E-M)[E(k)+M]}+\frac{Q'}{2}\left[-\frac{E(p)+M}{\bar{E}^3}\times\left(\frac{2\bar{E}+C}{3(\bar{E}+C)}\right)+\frac{1}{\bar{E}[E(k)+M]}\left(\frac{2M}{\bar{E}+C}\right)\right]\right\}.$$

This is clearly positive for $T=\frac{3}{2}$, $Q'=2$ so that this state is also repulsive. Both $P_{\frac{3}{2}}$, $T=\frac{1}{2}$ and $P_{\frac{1}{2}}$, $T=\frac{3}{2}$ are therefore expected to have small phase shifts in agreement with experiment.

This leaves the two states with $T=\frac{1}{2}$, $j=\frac{1}{2}$ (i.e. S and $P_{\frac{1}{2}}$). These cannot be treated by our present formalism. This is because for these states, and these

alone, it is possible for the meson to become absorbed by the nucleon in the intermediate state [graphs (a) and (b) of Fig. 2] (for all other quantum numbers this intermediate state is forbidden by conservation laws). Now the Tamm-Dancoff integral equation implies that the processes illustrated by the graphs can occur repeatedly, so that we are really considering graphs of the type shown in Fig. 5. These clearly contain self-energy graphs; hence they give the usual divergent results which must be removed by renormalization. But until now renormalization has only been carried out by perturbation theory, and this is not applicable in our strong-coupling problem. Therefore the two states in question of $T=\frac{1}{2}$ and $j=\frac{1}{2}$ are beyond the scope of this paper, and their treatment must be postponed to the future.

To get a better feeling for the working of the integral equation, we shall now discuss the solution of similar integral equations, but with kernels of a simplified form for which exact analytic solutions can be obtained. These cases illustrate the general relation between actual and Born-approximation wave functions as a function of the coupling constant and sign of the effective potential, but in a much simpler form than in our equation. One of these soluble cases is that in which the kernel has a maximum value when p and s are approximately equal to each other. The equation

$$f(p)=f_B(p)+\frac{G^2}{8\pi^2}\int_0^\infty dsL'(p,s)f(s)$$

may then be modified by taking $f(s)$ outside the integral sign, which gives

$$f(p)=f_B(p)/[1+(G^2/8\pi^2)I_1(p)], \qquad (32)$$
$$I_1(p)=-\int_0^\infty dsL'(p,s).$$

The second case is that in which $L(p,s)$ can be written as a product $L_1(p)L_2(s)$. Then $f_B(p)=L_1(p)\times$constant independent of p. If we let $a(p)=f(p)/L_1(p)$, then $a_B(p)=a_B=$a constant independent of p. Thus our equation is

$$a(p)=a_B+\frac{G^2}{8\pi^2}\int_0^\infty dsL_1(s)L_2(s)a(s).$$

The right-hand side is independent of p; therefore $a(p)$ is a constant and is equal to

$$a_B\Big/\left(1+\frac{G^2}{8\pi^2}I_2\right),$$

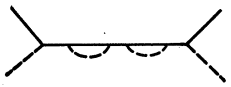
where

$$I_2=-\int_0^\infty dsL_1(s)L_2(s).$$

Finally,

$$f(p)=f_B(p)\Big/\left(1+\frac{G^2}{8\pi^2}I_2\right). \qquad (32a)$$

FIG. 5. Modified propagator correction for the $j=\frac{1}{2}$, $T=\frac{1}{2}$ state.



Although our equation is more complicated than the two simple cases mentioned above, yet we may make some qualitative statements about the nature of solutions for attractive and repulsive states. In the case of an attractive state, $f_B(p)$ and I_1 (or I_2) will be negative, and as $(G^2/8\pi^2)\cdot I_1$ (or I_2) reaches the value -1, a resonance occurs. This relation only crudely represents the actual case for the $P_{\frac{3}{2}}$, $T=\frac{3}{2}$ state. Here $(G^2/8\pi^2)\cdot I_1$ (or I_2) is negative and of the order of -1; the ratio $f(p)/f_B(p)$ is very large and is a very sensitive function of energy and coupling constant $G^2/4\pi$. However, we know that in the actual case $f(p)$ cannot be expressed in one of the simpler forms above, and therefore a resonance might not occur.

Another simple property of the kernel of our equation is that the important range of integration over s is of the order of M rather than μ. The integral is practically constant over values of $E-M-\mu\lesssim\mu$. Thus for states other than the sensitive $P_{\frac{3}{2}}$, $T=\frac{3}{2}$ state, the ratio of $f(p)$ to $f_B(p)$ is a slowly varying function in this energy range. Qualitatively, therefore, we can say that the forces represented by this theory have a range of order $1/M$. In an attempt to determine whether forces of longer range may be present, Mitra and Dyson[15] have considered the role of meson-meson interaction, so far with inconclusive results.

In the following section we discuss the solutions obtained for the $S_{\frac{1}{2}}$ and $P_{\frac{3}{2}}$ states for $T=\frac{3}{2}$ for laboratory energies up to 270 Mev by means of straightforward, largely numerical methods.

IX. NUMERICAL SOLUTION FOR THE S STATE WITH $T=3/2$

Two calculations were performed for the S state, one at zero energy and another for an energy of about 270 Mev (laboratory system). These correspond to values of k, the momentum in the center-of-mass system, of zero and $0.3M$, respectively. For the zero-energy case a seminumerical method was used for the solution of the one-dimensional integral equation, Eq. (30). This method only gives a fairly crude approximation, but is designed such that (1) the very lengthy kernel has to be evaluated only for a few values each of its two parameters p and s, (2) the integrals can be evaluated analytically, and (3) that solutions can be found easily for different values of the coupling constant $\lambda=G^2/4\pi$. The method proceeds as follows.

For a fixed value of p, the kernel $L(p,s)$ is evaluated for a few values of s and constants $a'(p)$, $b'(p)$, and $c'(p)$ are chosen such that the simple analytic function

$$a'(p)+[b'(p)/E(s)]+[c'(p)s/E^2(s)] \qquad (33)$$

gives a reasonable approximation to the kernel $L(p,s)$.

[15] A. N. Mitra and F. J. Dyson, Phys. Rev. **90**, 372 (1953).

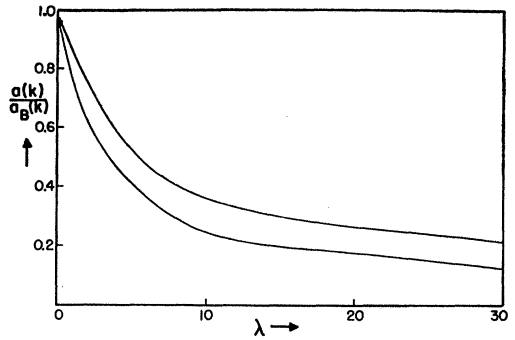


FIG. 6. Ratio of actual wave function to the Born approximation wave function on the energy shell as a function of the coupling constant λ. $(k/M)=0.3$ for upper curve and 0 for lower curve.

This fitting was done for a few values of (p/M) (0, 0.5, 1, and 2); the coefficients a', b', and c' are of course functions of p.

It is convenient to rewrite the integral equation in terms of a "wave function" $a(p)$, instead of $g(p)$, where $a(p)$ is defined by

$$a(p)=g(p)[E(p)\omega(p)]^{\frac{1}{2}}. \tag{34}$$

A form of iteration procedure is then adopted as follows. We choose an initial trial wave function $a_{\rm tr}(p)$ of the form

$$a_{\rm tr}(p)=a+[b/E(p)]+[cp/E^2(p)], \tag{35}$$

where the three constants a, b, and c are left unspecified for the moment. We substitute this function $a_{\rm tr}(s)$ for $a(s)$ in the integral on the right-hand side of the integral equation and replace the kernel $L(p,s)$ by the approximate analytic expression, Eq. (33). For each of the values of p considered the integrals can now be carried out analytically, and an iterated wave function $a_{\rm it}(p)$ obtained. This function $a_{\rm it}(p)$ for each value of p considered consists of the sum of four terms, a known constant and known multiples of λa, λb, and λc. The three constants a, b, and c are then chosen by requiring the initial and iterated functions $a_{\rm tr}(p)$ and $a_{\rm it}(p)$ to agree for three values of p (these values were then adjusted very slightly to give the best overall fit between $a_{\rm tr}(p)$ and $a_{\rm it}(p)$ at all the values of p used.)

For a value of λ of 10 (the value used by M. Lévy in equivalent calculations on the neutron-proton system), for instance, the constants chosen were

$$a=-0.003, \quad b=0.271, \quad c=-0.084. \tag{36}$$

The initial and iterated functions were fairly similar and the result for the function on the energy shell, with the estimated error of this method of solution, is

$$a_{\rm actual}(0)=(0.24\pm0.01)a_{\rm Born}(0), \tag{37}$$

where

$$a_{\rm Born}(0)=1.075.$$

As expected for a "repulsive" state, Eq. (30) shows that the actual phase shift is very much smaller than its Born approximation value. As discussed above such a solution is easily obtained for a number of different values for the coupling constant λ and the ratio of actual phase shift to its Born approximation is plotted against λ in Fig. 6. It will be seen that this curve is similar to that represented by Eq. (31), the equivalent ratio for the simpler forms of the equation, although the actual curve has a slightly higher curvature than Eq. (31).

Similar calculations were carried out at a higher energy namely that corresponding to $k=0.3M$. An estimate of the contribution of the singularity due to the double meson production was made in this case by integrating the rapidly varying logarithm across the singularity. The contribution due to this singularity was found to be less than 10 percent of the value of the integral of the kernel found disregarding this singularity. Having found that this singularity did not contribute much to the scattering at least near the threshold, no effort was made to include this contribution accurately.

At this higher energy $(k=0.3M)$ essentially the same method was used as for the zero energy case, except that some of the integrations were carried out numerically. The iteration process described above was carried for values of (p/M) of 0, 0.3, 1, 2, and 4. For a coupling constant $\lambda=10$, the values of the constants in the trial wave function were found by requiring that the equations at the above-mentioned values of p be satisfied in the least-squares sense and one gets

$$a=-0.053, \quad b=0.454, \quad c=0.094. \tag{38}$$

The agreement between the initial and the iterated wave functions was not quite as close in this case as for zero energy, but was sufficiently good for the accuracy required. These two functions are plotted against p in Fig. 7, together with the Born approximation for this function. The value obtained for the wave function

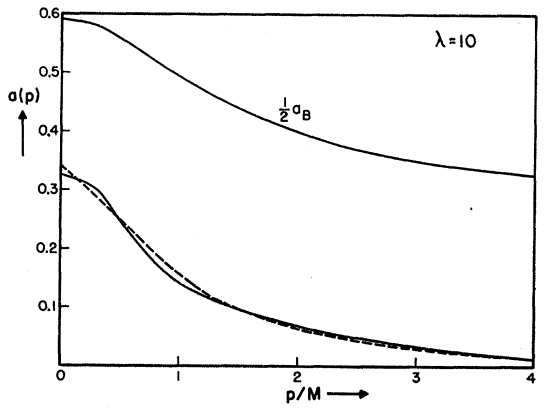


FIG. 7. Actual wave function together with the Born approximation wave function for the $S_{\frac{1}{2}}$, $T=\frac{3}{2}$ state, as a function of the momentum.

on the energy shell was (for $\lambda=10$)

$$a_{\text{actual}}(0.3)=(0.365\pm0.020)a_{\text{Born}}(0.3),$$
$$a_{\text{Born}}(0.3)=1.172. \quad (39)$$

As will be seen from Fig. 6, the ratios of phase shift to their Born approximation for all the values of λ considered were not very different for the zero-energy case and for the 270-Mev case ($k=0.3M$). Since this ratio is such a slowly varying function of energy,[16] a simple interpolation procedure was used for intermediate energies. In Fig. 8 we plot the estimated phase shift as a function of energy together with their Born approximation for $\lambda=15$, (the order of magnitude of λ which will be shown in the next section to give best agreement with experiment for the $P_{\frac{3}{2}}$, $T=\frac{3}{2}$ state.)

X. THE $P_{\frac{3}{2}}$ INTEGRAL EQUATION

The $P_{\frac{3}{2}}$, $T=\frac{3}{2}$ integral equation was solved by expressing the solution $f(p)$ in a manner similar to (35), but with six parameters instead of three. The integral equation could then be satisfied for six different values of p; this yields six simultaneous ordinary linear equations for the six parameters in $f(p)$.

Several simplifications in the kernel were made before attempting this procedure. The kernel is a sum of terms $K_1(C)$, $K_2(C)$, $K_1(B)$, and $K_2(B)$. The $K(B)$ terms are generally 1/20 of the $K(C)$ terms (i.e., the interaction with the one-nucleon, two-meson configuration predominates). The $K(B)$ terms were, as a result, drastically approximated. The K_1 terms (associated with spin-independent processes) and the K_2 terms (associated with spin-dependent processes) are of the same order of magnitude for momenta much bigger than M, but the latter are relatively very small for low momenta. Therefore, the K_2 terms were neglected below the momentum of integration $s=(3/4)M$.

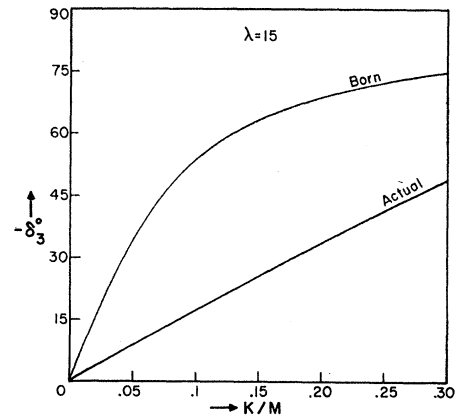


FIG. 8. Variation of $S_{\frac{1}{2}}$, $T=\frac{3}{2}$ phase shift (δ_3) with momentum k, in the c.m. system, for a coupling constant $\lambda=15$.

[16] It should be pointed out that this slow variation with energy may no longer hold for energies far in excess of the double meson production threshold energy, since the effect of the singularities, still small for 270 Mev, may become quite marked at these energies.

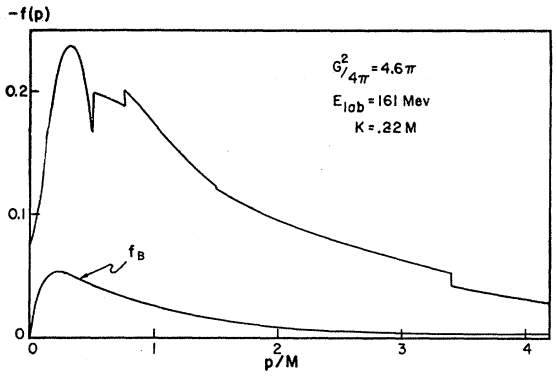


FIG. 9. Actual wave function together with the Born approximation wave function for the $P_{\frac{3}{2}}$, $T=\frac{3}{2}$ state.

The region of very high momenta ($p=10M$) was also neglected. This might be justified on the grounds that the error in the calculated phase shift associated with this neglect is less than about two percent. The singularities in the kernel associated with meson production were also neglected. Consider omission of the region of the integration variable from $[R/(\bar{E}+C)]=-\frac{1}{2}$ to $[R/(\bar{E}+C)]=\frac{1}{2}$ (which contains the singularities). It can be shown that the resulting error in the phase shift is $(\delta-\delta_{\text{calc}})/\delta\sim\{[\omega(k)-2\mu]/M\}^4$ for $k<M/2$. At these energies the error is then very small.

After the above approximations, the detailed method of the solution of the equation was as follows. The interval of integration was divided into four regions: 0 to $0.5M$, $0.5M$ to $1.5M$, $1.5M$ to $3.5M$, $3.5M$ to $10M$. The kernel was expanded in each of the regions as a function of s in a manner similar to (33), with the criterion that the error in the expansion be less than 10 percent at the boundaries of the regions. This was done for six values of p altogether. The form of the expansion for the solution $f(p)$ was chosen following roughly the indications from the expansion of the kernel; $f(p)$ was then taken to be a polynomial with 3, 1, 1, 1 undetermined coefficients in each of the four regions, respectively. The shape of this trial function was taken to be quadratic in $E-E(p)-\omega(p)$ in the region $p<M/2$. From $M/2$ to M, after some experimentation, it was taken as essentially constant. For $p>M$, the first trial function had a shape roughly of $1/p$, or more accurately $pf_B(p)$. The integrals were performed and the set of six simultaneous equations solved. The shape of the solution was then adjusted in the upper regions to improve the fit at the boundaries of the regions, and the equations were solved again. The solution thus obtained at 161 Mev for $G^2/4\pi=16$ is shown in Fig. 9. The solutions in Fig. 9 are represented very well (from about k to $5M$) by

$$f(p)=a(1+bp)f_B(p), \quad (40)$$

with $b=3/2M$ for $G^2/4\pi=16$. For $p\gg5M$, it is found that the behavior of $f(p)$ is independent of the energy. At the coupling constant in question ($G^2/4\pi=16$) we

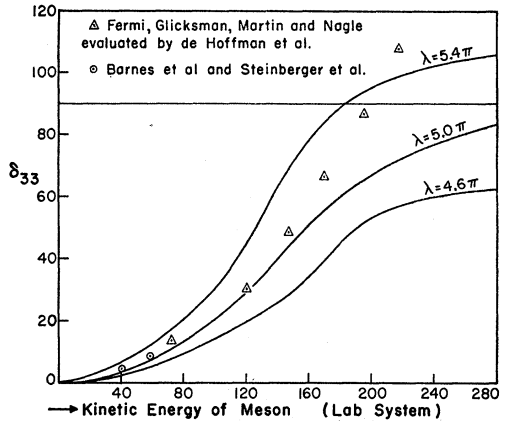


FIG. 10. Variation of $P_{\frac{3}{2}}$, $T=\frac{3}{2}$ phase shift (δ_{33}) with kinetic energy of meson in laboratory system.

find

$$f(p)\approx p^{-1.4}\approx p^{0.6}f_B(p). \tag{41}$$

Solutions were obtained at four energies, 113, 161, 186, and 272 Mev in the laboratory system, corresponding to c.m. momenta $k=0.18M$, $0.22M$, $0.24M$, and $0.30M$. In addition, the slope $f(k)/k^3$ was roughly determined by using Eq. (39) at zero scattering energy. In this case the coefficient bM was 1.5, 2.0, for $G^2/4\pi=4.6\pi$, 5.0π, respectively. The calculated $P_{\frac{3}{2}}$, $T=\frac{3}{2}$ phase shift is plotted as a function of energy for several coupling constants in Fig. 10.

The accuracy of these solutions can be estimated very crudely in the following manner. Let the integral equation be written

$$f(p)=f_B(p)+\left[\int L'(p,s)\frac{f(s)}{f(p)}ds\right]f(p).$$

Then,

$$\Delta f(p)=\left\{\int\Delta[L'(p,s)]\frac{f(s)}{f(p)}ds + \int L'(p,s)\Delta\left[\frac{f(s)}{f(p)}\right]ds\right\}f(p)+\frac{f(p)-f_B(p)}{f(p)}\Delta f(p), \tag{42}$$

so that

$$\frac{\Delta f(p)}{f(p)}=\frac{f(p)}{f_B(p)}\{\epsilon_1+\epsilon_2\}, \tag{43}$$

where $\{\epsilon_1+\epsilon_2\}$ represents the two terms in curly brackets in Eq. (42). We have

$$\int dsL'(p,s)\frac{f(s)}{f(p)}\sim 1 \quad \text{for} \quad \frac{f(p)}{f_B(p)}\gg 1.$$

Then ϵ_1 is roughly given by the percentage error resulting from our approximations of L'. This error can be taken as about ± 3 percent. We may estimate ϵ_2 by examination of the deviation of the obtained solution, $f(p)$, from a smooth curve at the boundaries of the regions (see Fig. 9). This deviation is of the order of ± 20 percent. We shall take the average error in $f(s)/f(p)$, optimistically, as quadratic in this deviation, or $\epsilon_2\sim\pm 4$ percent. The ratio to the Born approximation $f(k)/f_B(k)\sim 6$ at 161 Mev. Then an error $\Delta f(k)/f(k)$ of about $\pm\frac{1}{3}$ is expected for a given coupling constant. As the ratio to the Born approximation increases, this error in $\tan\delta$ increases without limit. The error in the angle δ becomes the more meaningful quantity. For $\delta\approx 1$, $\tan\delta>1$, we can write

$$\frac{\Delta\delta}{\delta}=\frac{1}{\delta}\left(\frac{\Delta\tan\delta}{1+\tan^2\delta}\right)\sim-\frac{1}{\pi f_B(k)}\{\epsilon_1+\epsilon_2\}. \tag{44}$$

In the region of interest this error is also about $\pm\frac{1}{3}$. Since the coupling constant is varied in practice so that the solution obtained at one energy agrees with experiment, the question is how accurate is the relative phase shift determined at another energy? It might be expected that if the energy difference is of the order of the characteristic length of the solution obtained in the region 0 to $0.5M$ (say $\Delta k=0.2M$) then the error at the new energy would be of the order of that indicated in Eqs. (43) and (44).

XI. DISCUSSION

The numerical results for the two states investigated in detail, using the approximate equation of motion derived above, can be summarized as follows. For the $P_{\frac{3}{2}}$, $T=\frac{3}{2}$ state the calculated phase shift is appreciably larger than the Born approximation and is a very sensitive function of coupling constant $\lambda=G^2/4\pi$ and of energy. Rough agreement with experiment is obtained for a value of λ slightly larger than 15 (see Fig. 8 and below). For the $S_{\frac{1}{2}}$, $T=\frac{3}{2}$ state the ratio of phase shift to its Born approximation value is considerably less than unity, for $\lambda=15$ and decreases slowly with λ. Since the Born approximation itself increases slowly with λ, the calculated phase shift is a very insensitive function of λ indeed. This ratio to Born approximation, as well as the Born approximation phase shift itself, are also slowly varying functions of the energy, and so is the calculated phase shift.

For the other two states which were not calculated in detail but are not subject to renormalization difficulties, *viz.*, $P_{\frac{1}{2}}$ for $T=\frac{3}{2}$ and $P_{\frac{3}{2}}$ for $T=\frac{1}{2}$, the Born approximation phase shift is small (of order 5°) and is further reduced in the Tamm-Dancoff theory.

A direct comparison between the calculated phase shifts and the observed one for the S state of $T=\frac{3}{2}$, which we shall call δ_3, does not give good agreement. For example, at 136 Mev, the observed value is about $-15°$, while the calculated one is $-32°$ and $-34°$ for the two values of the coupling constant. This would indicate that the reduction as compared with the Born approximation is not nearly large enough. However, it is well known that the phase shift δ_3 has experimentally

a very complicated behavior at low energy which presumably cannot be explained by our theory, but which requires probably the introduction of an additional weak, long-range interaction[17,18] between nucleon and meson. This potential might be due to an interaction between the incident meson and the "bound" mesons in the nucleon, and one of us (M.R.) has found that a phenomenological theory based on this assumption accounts reasonably well for the behavior of δ_3 at low energies.

We therefore consider it more reasonable to compare our theory merely with the higher energy experiments. Above 40 Mev, the experimental phase shift δ_3 can be represented quite well by a straight line of the form[19]

$$\delta_3 = 11° - 130°(k/M). \tag{45}$$

Our theory, for $G^2/4\pi = 13$, gives very nearly

$$\delta_3 = -160°(k/M). \tag{46}$$

Therefore, the *slope* of the straight line δ_3 *vs* k is given quite well by the theory, as is the straight-line relation itself. By the way, this relation indicates that the interaction corresponds to a very strong repulsive core whose radius is given by the coefficient of k (in radians), and is therefore

$$2.3\hbar/Mc = 4.8\times10^{-14} \text{ cm from the experimental relation;} \tag{47}$$

$$2.8\hbar/Mc = 5.9\times10^{-14} \text{ cm from the theoretical relation.} \tag{48}$$

The *constant* term in (47) is attributed to the long-range attraction mentioned above.

Regarding the $P_{\frac{3}{2}}$ state there has been considerable controversy on the behavior of the phase shift above 120 Mev. It is now most likely[20] that there is a resonance at about 195 Mev, and that the phase shift goes nearly linearly with energy near the resonance. These experimental phase shifts are shown in Fig. 10. At energies up to about 150 Mev, these follow rather closely the theoretical curve for $G^2/4\pi = 5.1\pi = 16.0$ (not drawn). However, at higher energy, the theoretical curve does not increase steeply enough. This may be improved by including the renormalization terms as has been shown by Visscher.[21] It is also possible that higher configurations, involving more mesons, play a more important role and increase the phase shifts.

The work reported in this paper is to be considered purely as a pilot study for the following reasons. As discussed above, an unsophisticated three-dimensional approach was used throughout and hence all renormalization terms had to be omitted. It is hoped that work relating this formalism to a fully covariant approach will remove this difficulty.[9] More important still is the approximation of restricting ourselves to states directly coupled to the one-meson one-nucleon state. Nothing is known about the rate of convergence of the Tamm-Dancoff expansion in terms of the number of virtual mesons and nucleon pairs. A helpful fact in this respect is the increase of the energy denominators with the complexity of intermediate states. But even if the hope for convergence is justified, one will certainly need more terms of this expansion than were carried in the present paper to get quantitatively meaningful results. Higher terms could be included without essential modification in the method of deriving the equations of motion, but the solution of these coupled integral equations would be very lengthy, probably requiring the use of electronic computing machines. It was not felt worthwhile to carry such a program through until the renormalization difficulties have been removed.

One therefore should not expect any quantitative agreement between the present results and experiment, but we hope that the terms omitted in this paper will at least be qualitatively similar to those carried. In particular we expect that the distinction between effectively attractive and effectively repulsive interactions will remain, so that the phase shift for the $P_{\frac{3}{2}}$, $T=\frac{3}{2}$ state will continue to be the largest, and considerably enhanced compared with the Born approximation, while all other phase shifts will continue to be less than their Born approximation.

To sum up, the present work has by no means proved that pseudoscalar meson theory agrees with experiment, nor even that it is a self-consistent formalism. But at least it removes the impression given by perturbation theory calculations, that pseudoscalar meson theory with pseudoscalar coupling is incompatible with experiment.

ACKNOWLEDGMENTS

It is a great pleasure to express our hearty thanks to Dr. G. Chew and to Dr. S. D. Drell and Dr. E. Henley for showing us their work before publication. We thank Mr. A. N. Mitra for making available to us the results on the effects of meson-meson interaction. One of us (E.E.S.) is indebted to the Australian National University, where part of this work was completed.

APPENDIX

In the Schrödinger equation

$$(H_0+H_I)\Psi = E\Psi,$$

H_0 represents the free-field Hamiltonian and can be written in terms of the number operators of various field quanta in the form

$$H_0 = \sum_{\mathbf{p},u} N_u(\mathbf{p})E_u(\mathbf{p}) + \sum_{\mathbf{k},\alpha} N_\alpha{}^M(\mathbf{k})\omega_\alpha(\mathbf{k}),$$

[17] R. E. Marshak, Phys. Rev. **88**, 1208 (1952).
[18] J. Tinlot and A. Roberts, Phys. Rev. **90**, 951 (1953).
[19] Bethe, de Hoffman, and Schweber, *Introduction to Meson Theory* (Row, Peterson, & Company, Evanston, to be published).
[20] De Hoffmann, Metropolis, Alei, and Bethe, Phys. Rev. (to be published); M. Glicksman, Phys. Rev. **95**, 1335 (1954); R. L. Martin, Phys. Rev. (to be published).
[21] W. M. Visscher, Cornell University, thesis, 1953 (unpublished).

where $E(p)=+(M^2+p^2)^{\frac{1}{2}}$ and $\omega(k)=+(k^2+\mu^2)^{\frac{1}{2}}$, M and μ being the masses of nucleon and meson, respectively. The suffix α in the second term refers to the charge state of the meson. $N_u(\mathbf{p})$ is defined in terms of creation operators $[a_u^*(\mathbf{p})]$ and destruction operators $a_u(\mathbf{p})$ of nucleons in the form

$$N_u(\mathbf{p})=a_u^*(\mathbf{p})a_u(\mathbf{p}),$$

if u is a suffix referring to positive-energy states of the nucleon field. If u refers to negative-energy states, then

$$N_u(\mathbf{p})=a_u(\mathbf{p})a_u^*(\mathbf{p}).$$

The $a_u(p)$'s satisfy the relation

$$[a_u(\mathbf{p}),\ a_{u'}^*(\mathbf{p}')]_+=\delta_3(\mathbf{p}-\mathbf{p}')\delta_{uu'}.$$

Similarly the meson number operator is defined in terms of meson creation and destruction operators, $c_\alpha^*(\mathbf{k})$ and $c_\alpha(\mathbf{k})$ respectively, in the form

$$N_\alpha{}^M(\mathbf{k})=c_\alpha^*(\mathbf{k})c_\alpha(\mathbf{k}),$$

with the c's satisfying the commutation relations

$$[c_\alpha(\mathbf{k}),\ c_{\alpha'}^*(\mathbf{k}')]=\delta_3(\mathbf{k}-\mathbf{k}')\delta_{\alpha\alpha'}.$$

The nucleon field operator can be expanded in the form

$$\psi(\mathbf{x})=\frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3p\left(\frac{M}{E(p)}\right)^{\frac{1}{2}}\times[\textstyle\sum_+ a_u(\mathbf{p})u(\mathbf{p})\exp(ip_\mu x_\mu)+\sum_- a_u(\mathbf{p})u(\mathbf{p})\exp(-ip_\mu x_\mu)].$$

The summation in the two terms extends over positive energy spinors and negative energy spinors, respectively. In the exponentials appears the four-dimensional scalar product of the vector p_u and x_u. The 0 component of p_μ, i.e., p_0, is defined as $+E(p)$. With the above decomposition for $\psi(\mathbf{x})$, $\bar{\psi}(\mathbf{x})$ takes the form

$$\bar{\psi}(\mathbf{x})=\frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3p(M/E(p))^{\frac{1}{2}}\times[\textstyle\sum_+ a_u^*(\mathbf{p})\bar{u}(\mathbf{p})\exp(-ip_\mu x_\mu)+\sum_- a_u^*(\mathbf{p})\bar{u}(\mathbf{p})\exp(ip_\mu x_\mu)].$$

The meson operator $\phi_\alpha(\mathbf{x})$ is decomposed in the form

$$\phi_\alpha(\mathbf{x})=\frac{1}{[2(2\pi)^3]^{\frac{1}{2}}}\int d^3k\frac{1}{[\omega(k)]^{\frac{1}{2}}}[c_\alpha(\mathbf{k})\exp(ik_\mu x_\mu)+c_\alpha^*(\mathbf{k})\exp(-ik_\mu x_\mu)].$$

The Dirac spinors $u(p)$ are normalized in such a way that

$$\textstyle\sum_+ \bar{u}(\mathbf{p})u(\mathbf{p})=+1,\quad\text{and}\quad\sum_- \bar{u}(\mathbf{p})u(\mathbf{p})=-1.$$

The u's satisfy the Dirac equation $(\not{p}-iM)u(\mathbf{p})=0$ for positive energies, and $(\not{p}+iM)u(\mathbf{p})=0$ for negative energies. Substituting these expressions for the decomposition of the operators $\psi(\mathbf{x})$, $\bar{\psi}(\mathbf{x})$, and $\phi(\mathbf{x})$ in the expression for H_I, the interaction Hamiltonian, we can write it as the sum of the following eight terms:

$$\begin{aligned}H_I=iG(16\pi^3)^{-\frac{1}{2}}\int d^3p'\int d^3p\int d^3k\left(\frac{M^2}{E(p)E(p')\omega(k)}\right)^{\frac{1}{2}}&\cdot[\textstyle\sum_{+u'}\sum_{+u}\bar{u}(\mathbf{p}')\gamma_5\tau_\alpha u(\mathbf{p})a_{u'}^*(\mathbf{p}')\\&\cdot c_\alpha(\mathbf{k})a_u(\mathbf{p})\delta(\mathbf{p}-\mathbf{p}'+\mathbf{k})\\+\textstyle\sum_{+u'}\sum_{+u}\bar{u}(\mathbf{p}')\gamma_5\tau_\alpha u(\mathbf{p})a_{u'}^*(\mathbf{p}')&\cdot c_\alpha^*(\mathbf{k})a_u(\mathbf{p})\delta(\mathbf{p}-\mathbf{p}'-\mathbf{k})\\+\textstyle\sum_{+u'}\sum_{-u}\bar{u}(\mathbf{p}')\gamma_5\tau_\alpha u(\mathbf{p})a_{u'}^*(p')&\cdot c_\alpha(k)a_u(\mathbf{p})\delta(-\mathbf{p}-\mathbf{p}'+\mathbf{k})\\+\textstyle\sum_{+u'}\sum_{-u}\bar{u}(\mathbf{p}')\gamma_5\tau_\alpha u(\mathbf{p})a_{u'}^*(\mathbf{p}')&\cdot c_\alpha^*(\mathbf{k})a_u(\mathbf{p})\delta(-\mathbf{p}-\mathbf{p}'-\mathbf{k})\\+\textstyle\sum_{-u'}\sum_{+u}\bar{u}(\mathbf{p}')\gamma_5\tau_\alpha u(\mathbf{p})a_{u'}^*(\mathbf{p}')&\cdot c_\alpha(\mathbf{k})a_u(\mathbf{p})\delta(\mathbf{p}+\mathbf{p}'+\mathbf{k})\\+\textstyle\sum_{-u'}\sum_{+u}\bar{u}(\mathbf{p}')\gamma_5\tau_\alpha u(\mathbf{p})a_{u'}^*(\mathbf{p}')&\cdot c_\alpha^*(\mathbf{k})a_\mu(\mathbf{p})\delta(\mathbf{p}+\mathbf{p}'-\mathbf{k})\\+\textstyle\sum_{-u'}\sum_{-u}\bar{u}(\mathbf{p}')\gamma_5\tau_\alpha u(\mathbf{p})a_{u'}^*(\mathbf{p}')&\cdot c_\alpha(\mathbf{k})a_u(\mathbf{p})\delta(-\mathbf{p}+\mathbf{p}'+\mathbf{k})\\+\textstyle\sum_{-u'}\sum_{-u}\bar{u}(\mathbf{p}')\gamma_5\tau_\alpha u(\mathbf{p})a_{u'}^*(\mathbf{p}')&\cdot c_\alpha^*(\mathbf{k})a_u(\mathbf{p})\delta(-\mathbf{p}+\mathbf{p}'-\mathbf{k})].\end{aligned}$$