Two-Nucleon Problem When the Potential Is Nonlocal but Separable. II

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(Received April 15, 1954)

The two-nucleon problem is considered in terms of an interaction of the form $(\mathbf{p}|V|\mathbf{p}')=-(\lambda/M)g(\mathbf{p})g(\mathbf{p}')$, with $g(\mathbf{p})=C(p)+(1/\sqrt{8})\{3p^{-2}(\boldsymbol{\sigma}^p\cdot\mathbf{p})(\boldsymbol{\sigma}^n\cdot\mathbf{p})-(\boldsymbol{\sigma}^p\cdot\boldsymbol{\sigma}^n)\}T(p)$, where the second term gives rise to a mixture of D state with the S state determined by the first term. After deriving general results valid for any form of $g(\mathbf{p})$, we discuss in detail the special case of $C(p)=(p^2+\beta^2)^{-1}$ and $T(p)=-tp^2(\gamma^2+p^2)^{-2}$ and compared it with observed data. The photodisintegration of the deuteron is also discussed and it is found that the D-wave part of the deuteron plays an important role at high photon energies leading to a larger cross section than given by other calculations to date.

1. INTRODUCTION

IN the preceding paper[1] (cited as I hereafter) we have shown that the two-body Schrödinger equation can be easily solved with a nonlocal but separable nuclear interaction. In continuation of I, we want to discuss here the special case of a separable neutron-proton interaction with spin-orbit coupling. Adopting the algebraic notation of Rarita-Schwinger,[2] our nuclear interaction for the triplet state takes the form

$$(\mathbf{p}|V|\mathbf{p}')=-(\lambda/M)g(\mathbf{p})g(\mathbf{p}'),$$
$$g(\mathbf{p})=C(p)+(1/\sqrt{8})S(\mathbf{p})T(p),$$

and

$$S(\mathbf{p})=(3/p^2)(\boldsymbol{\sigma}^p\cdot\mathbf{p})(\boldsymbol{\sigma}^n\cdot\mathbf{p})-(\boldsymbol{\sigma}^p\cdot\boldsymbol{\sigma}^n), \qquad (1)$$

where $C(p)$ and $T(p)$ are the functions of $p=|\mathbf{p}|$, and $\boldsymbol{\sigma}^p$ or $\boldsymbol{\sigma}^n$ is the usual spin matrix for the proton or neutron. $C(p)$ and $T(p)$, or equivalently $g(\mathbf{p})$, must be real in the sense of Wigner.[3] As will be shown, the two-body problem can easily be solved for this interaction both for the bound and the continuum states. It seems therefore worth while to report the detailed discussion based on this potential to see the effects of the so-called tensor force. In the case of the usual local form, these effects are so complicated that they cannot be seen in a simple manner, because a local potential including tensor force requires tedious numerical integration[4] for solution.

It is evident that our triplet potential (1) acts in the $(^3S_1+{}^3D_1)$ state only. Moreover, as will be seen in the next section, there is no room to find two independent solutions for $(^3S_1+{}^3D_1)$ states, because the wave function is uniquely determined by $g(\mathbf{p})$. In other words, our potential acts just in the "eigen-S" state, following the terminology of Blatt and Biedenharn.[5] This striking feature is of course a direct consequence of the special assumption for the form of $g(\mathbf{p})$. To get an interaction acting in other states than the eigen-S, one may adopt another form for $(\mathbf{p}|V|\mathbf{p}')$, which necessarily contains further dependence upon the direction of $\mathbf{p}$. Nevertheless, for the time being let us confine our discussion to the nuclear potential (1).

If we choose our nonlocal potential involving spin-orbit coupling to be without a long tail, then we can depend upon the validity of the effective range theory,[6] just as in the case of local spin-orbit forces,[7] to guarantee a fit to the low-energy two-nucleon data. Since one gets the deuteron and continuum wave functions in very simple forms, one can easily apply them to high-energy regions which are outside of the validity of the shape-independent theory. Thus we shall calculate on the basis of our model the neutron-proton scattering and the photodisintegration of the deuteron at high energy. It turns out that the neutron-proton scattering up to 100 Mev is fairly well explained by the force (20) for the triplet eigen-S state and the singlet potential fixed in I, and it is shown that the D wave in the deuteron plays an important role in the high-energy photodisintegration. The photodisintegration cross section will be expressed in such a way as to make clear the dependence on the assumed deuteron function.

2. GENERAL FORMULATION

In this section we want to give relevant formulas for the neutron-proton system, without specifying the form of $g(\mathbf{p})$. In the next section we shall examine the two-nucleon system, assuming a simple form for $g(\mathbf{p})$. As in I, our problem can be treated more conveniently if the momentum-space representation is adopted rather than the coordinate representation.

(a) Deuteron Problem

Let us start with the deuteron problem. The deuteron function must take the form

$$\psi(\mathbf{p},\text{spin})=\{u(p)+(1/\sqrt{8})S(\mathbf{p})w(p)\}\chi_1{}^m, \qquad (2)$$

[1] Y. Yamaguchi, preceding paper [Phys. Rev. **95**, 1628 (1954)], cited as I.
[2] W. Rarita and J. Schwinger, Phys. Rev. **59**, 436, 556 (1941).
[3] E. Wigner, Nachr. Ges. Wiss. Göttingen **31**, 546 (1932).
[4] T. Miyazima, Proc. Phys.-Math. Soc. Japan **22**, 188 (1940); H. Feshbach and J. Schwinger, Phys. Rev. **84**, 194 (1951); J. M. Blatt and M. H. Kalos, Phys. Rev. **92**, 156 (1953); Fujii, Iwadare, Otsuki, Takotani, Tani, and Watari, Progr. Theoret. Phys. Japan **10**, 478 (1953). Other references will be found in J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952).
[5] J. M. Blatt and L. C. Biedenharn, Phys. Rev. **86**, 399 (1952).
[6] E.g., see J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952).
[7] R. S. Christian, Phys. Rev. **75**, 1675 (1949); Y. Yamaguchi, Progr. Theoret. Phys. Japan **6**, 439 (1951).

where $u(p)$ and $w(p)$ are the S and D wave functions, respectively, depending on $p=|\mathbf{p}|$; χ_1^m ($m=1, 0, -1$) is the triplet spin function. Inserting (2) into the Schrödinger equation,

$$(\alpha^2+p^2)\{u(p)+(1/\sqrt{8})S(\mathbf{p})w(p)\}\chi_1^m = \lambda\{C(p)+(1/\sqrt{8})S(\mathbf{p})T(p)\}\int d\mathbf{p}'\{C(p')u(p') + T(p')w(p')\}\cdot\chi_1^m, \quad (3)$$

where $\alpha^2/M=$ binding energy of the deuteron. One can readily find a solution

$$u(p)=NC(p)/(\alpha^2+p^2), \quad w(p)=NT(p)/(\alpha^2+p^2), \quad (4)$$

where N is a normalization constant fixed so that

$$\int d\mathbf{p}\{u^2(p)+w^2(p)\}=1,$$

or

$$N^{-2}=\int d\mathbf{p}\frac{C^2(p)+T^2(p)}{(\alpha^2+p^2)^2}. \quad (5)$$

One may determine the strength $\lambda=\lambda(\alpha)$ of our nuclear interaction so as to fit the observed binding energy α^2/M of the deuteron:

$$\frac{1}{\lambda}=\int d\mathbf{p}\frac{C^2(p)+T^2(p)}{\alpha^2+p^2}. \quad (6)$$

Thus the well-depth parameter s, defined by Blatt-Jackson,[7] is given by

$$s=\lambda(\alpha)/[\lim_{\alpha\to 0}\lambda(\alpha)].$$

The D-state probability P_D and the quadrupole moment of the deuteron Q are given by

$$P_D=\int d\mathbf{p}w^2(p)=4\pi N^2\int_0^\infty p^2dpT^2(p)(\alpha^2+p^2)^{-2}, \quad (7)$$

$$Q=-\frac{1}{4}\int d\mathbf{p}\left[\left\{u(p)+\frac{1}{\sqrt{8}}S(\mathbf{p})w(p)\right\}\chi_1^1\right]^* \times\left\{3\frac{\partial^2}{\partial p_z^2}-\frac{\partial^2}{\partial \mathbf{p}^2}\right\}\left\{u(p)+\frac{1}{\sqrt{8}}S(\mathbf{p})w(p)\right\}\chi_1^1$$
$$=\frac{\sqrt{8}\pi}{5}\int_0^\infty dp\left\{p\frac{\partial u}{\partial p}w-p^2\frac{\partial^2 u}{\partial p^2}w\right\} -\frac{\pi}{5}\int_0^\infty dp\left\{6w^2+p^2\left(\frac{\partial w}{\partial p}\right)^2\right\}. \quad (8)$$

(b) Neutron-Proton Scattering

Next let us examine the neutron-proton scattering. Our fundamental equation is

$$\psi_\mathbf{k}(\mathbf{p})=\varphi_\mathbf{k}(\mathbf{p})+\frac{M}{k^2-p^2+i\epsilon}\int(\mathbf{p}|V|\mathbf{p}')d\mathbf{p}'\psi_\mathbf{k}(\mathbf{p}'), \quad (9)$$

when $\mathbf{k}$ is the incident momentum, $k=|\mathbf{k}|$, $\varphi_\mathbf{k}(p)=\delta(\mathbf{p}-\mathbf{k})$ is the incident plane wave, and ϵ is a real positive infinitesimal quantity which makes the scattered part of $\psi_\mathbf{k}(\mathbf{p})$ only an outgoing wave. Introducing the T matrix,[8]

$$(\varphi_\mathbf{p}, V\psi_\mathbf{k})=(\mathbf{p}|T|\mathbf{k}), \quad (10)$$

Eq. (9) takes the form

$$(\mathbf{p}|T|\mathbf{k})=(\mathbf{p}|V|\mathbf{k})+\int\frac{(\mathbf{p}|V|\mathbf{p}')d\mathbf{p}'(\mathbf{p}'|T|\mathbf{k})}{(k^2-p'^2+i\epsilon)/M}. \quad (11)$$

If $(\mathbf{p}|V|\mathbf{k})$ is factorable [see (1)], our Eq. (11) is easily solved:

$$(\mathbf{p}|T|\mathbf{k})=-\frac{\lambda}{M}\cdot\frac{g(\mathbf{p})g(\mathbf{k})}{1+\lambda J(k)}, \quad (12)$$

where $J(k)$ is given by

$$J(k)=\int d\mathbf{p}'\frac{C^2(p')+T^2(p')}{k^2-p^2+i\epsilon}. \quad (13)$$

It should be noted that the wave function which describes a scattering process is

$$\psi_\mathbf{k}(\mathbf{p},\text{spin})=\left\{\delta(\mathbf{p}-\mathbf{k})-\frac{\lambda}{1+\lambda J(k)}\cdot\frac{g(\mathbf{p})g(\mathbf{k})}{k^2-p^2+i\epsilon}\right\}\chi_1^m. \quad (14)$$

This wave function is determined uniquely from the potential (1) assumed and describes the eigen-^3S scattering.

From the T matrix (12), we can readily write down the desired cross section; for example the differential cross section in the c.m. system for an unpolarized triplet beam is as follows:

$$\frac{d\sigma}{d\omega}=\frac{1}{3}\,\mathrm{Sp}\left[\frac{3+\boldsymbol{\sigma}^p\boldsymbol{\sigma}^n}{4}|2\pi^2M(\mathbf{p}|T|\mathbf{k})|^2\right], \quad (p=k), \quad (15)$$

or

$$\frac{d\sigma}{d\omega}=\frac{1}{3}\left|\frac{2\pi^2\lambda}{1+\lambda J(k)}\right|^2\mathrm{Sp}\left[\frac{3+\boldsymbol{\sigma}^p\boldsymbol{\sigma}^n}{4}g(\mathbf{k})g(\mathbf{p})g(\mathbf{p})g(\mathbf{k})\right], \quad (16)$$

where $(3+\boldsymbol{\sigma}^p\cdot\boldsymbol{\sigma}^n)/4$ is the projection operator for the triplet state and $\mathbf{p}$ is the relative momentum after the

[8] B. A. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950).

scattering. More explicitly,

$$\frac{d\sigma}{d\omega}=\frac{\sigma_{\text{tot}}}{4\pi}\left\{1+\epsilon(k)\cdot\frac{3\cos^2\theta-1}{2}\right\},\quad \cos\theta=(\mathbf{p}\cdot\mathbf{k})/pk,$$

$$\sigma_{\text{tot}}=4\pi\left|\frac{2\pi^2\lambda}{1+\lambda J(k)}\right|^2\{C^2(k)+T^2(k)\}^2, \tag{17}$$

and

$$\epsilon(k)=\left\{\frac{2C(k)T(k)-(1/\sqrt{2})T^2(k)}{C^2(k)+T^2(k)}\right\}^2. \tag{18}$$

Here it should be noted that the asymmetry coefficient $\epsilon(k)$ is always positive.

The eigen S-phase shift δ can easily be calculated from the relation

$$\frac{2\pi^2\lambda\{C^2(k)+T^2(k)\}}{1+\lambda J(k)}=\frac{e^{i\delta}\sin\delta}{k}=\frac{1}{-ik+k\cot\delta}. \tag{19}$$

Then by expanding $k\cot\delta$ in a power series in k^2, one can derive the analytic expressions for the scattering length a, the effective range r_0, etc. We shall show this in the next section where a simple form for $g(\mathbf{p})$ is adopted.

3. DETAILED DISCUSSION OF THE SPECIAL CASE

In this section we want to examine our problem with a special form for $g(\mathbf{p})$. It is evident that if we assume rational functions for $C(p)$ and $T(p)$, then we can perform all integrations required in solving the two-body problem. It is natural to choose the simplest function in this category. At first sight the simplest choice seems to be

$$C(p)=1/(\beta^2+p^2),\quad T(p)=-t/(\gamma^2+p^2)$$

where β, γ, and t are constant parameters. This is, however, not a reasonable choice, because it corresponds to a long-tail potential in the $\mathbf{r}$ representation, which does not seem physically plausible and in fact leads to undesirable results: (i) the asymmetry coefficient $\epsilon(k)$ is insensitive to the incident momentum k and in general $\epsilon(k)\neq 0$ even at $k=0$ and (ii) the electric dipole photodisintegration cross section of the deuteron becomes anomalously large at very low photon energy. Therefore we choose the next simplest function, namely,

$$C(p)=1/(\beta^2+p^2),\quad T(p)=-tp^2/(\gamma^2+p^2)^2, \tag{20}$$

where β, γ, and t are constant parameters. This choice is quite reasonable, being free from the objections mentioned previously, and for the special case of D interaction only, i.e., $C(p)=0$, the scattering phase shift δ goes to the form $k^5\cot\delta=(\text{const})+(\text{const})k^2+\cdots$, which is of course the normal behavior for pure D wave scattering (see I, Sec. 3; this can easily be seen from Eq. (27) below if t is made very large). The form (20) is a natural generalization of the special interaction $g(p)=(\beta^2+p^2)^{-1}$ discussed in I, Sec. 3. In fact, if we put $t=0$, all of our results found below reduce to those of the "central" case of I, Sec. 3. Note that if $t>0$ our tensor force is attractive for parallel nucleon spins and gives a positive value for the quadrupole moment of the deuteron.

(a) Deuteron Problem

With this choice of potential, the deuteron function is

$$u(p)=\frac{N}{(\alpha^2+p^2)(\beta^2+p^2)},\qquad w(p)=-\frac{Ntp^2}{(\alpha^2+p^2)(\gamma^2+p^2)^2}, \tag{21}$$

where

$$\frac{1}{N^2}=\pi^2\left\{\frac{1}{\alpha\beta(\alpha+\beta)^3}+t^2\frac{5\alpha+\gamma}{8\gamma(\alpha+\gamma)^5}\right\}. \tag{22}$$

In our case the S wave function is again of Hulthén form. λ is determined by

$$\frac{1}{\lambda}=\pi^2\left\{\frac{1}{\beta(\alpha+\beta)^2}+\frac{t^2}{8}\cdot\frac{(5\alpha^2+4\alpha\gamma+\gamma^2)}{\gamma(\alpha+\gamma)^4}\right\}, \tag{23}$$

so that the well-depth parameter s is

$$s=\frac{\lambda}{\beta^{-3}+(t^2/8\gamma^3)}. \tag{24}$$

The D-state probability P_D and the quadrupole moment Q are easily calculated:

$$P_D=\frac{t^2(5\alpha+\gamma)}{8\gamma(\alpha+\gamma)^5}\Bigg/\left[\frac{1}{\alpha\beta(\alpha+\beta)^3}+\frac{t^2(5\alpha+\gamma)}{8\gamma(\alpha+\gamma)^5}\right], \tag{25}$$

$$\begin{aligned}Q=\frac{\sqrt{2}\pi^2N^2t}{10(\alpha+\beta)^4}\Bigg[&\frac{1}{(\beta+\gamma)^2(\alpha+\gamma)^5}\{\alpha\beta^2(5\alpha^2+4\alpha\beta+\beta^2)\\&+\beta\gamma(10\alpha^3+33\alpha^2\beta+22\alpha\beta^2+5\beta^3)\\&+\gamma^2(5\alpha^3+22\alpha^2\beta+33\alpha\beta^2+10\beta^3)\\&+\gamma^3(\alpha^2+4\alpha\beta+5\beta^2)\}\\&+\frac{2}{(\beta+\gamma)^3(\alpha+\gamma)^4}\{\alpha(\beta+\gamma)^3+4(\alpha^2+\beta\gamma)\\&\times(\beta^2+3\beta\gamma+\gamma^2)+16\alpha\beta\gamma(\beta+\gamma)\}\\&+\frac{2}{(\beta+\gamma)^4(\alpha+\gamma)^3}\{\beta(\alpha+\gamma)^3+4(\beta^2+\alpha\gamma)\\&\times(\alpha^2+3\alpha\gamma+\gamma^2)+16\alpha\beta\gamma(\alpha+\gamma)\}\Bigg]\\&-\frac{\pi^2N^2t^2}{160}\frac{(7\alpha^3+49\alpha^2\gamma+91\alpha\gamma^2+33\gamma^3)}{\gamma^3(\alpha+\gamma)^7}.\end{aligned} \tag{26}$$

(b) Scattering Problem

Since we have solved the scattering problem without specifying a form for $g(\mathbf{p})$, we can readily write down the unpolarized triplet cross section corresponding to the potential (20)

$$\frac{d\sigma}{d\omega}=\left|\frac{1}{-ik+k\cot\delta}\right|^2\left\{1+\epsilon(k)\cdot\frac{3\cos^2\theta-1}{2}\right\},$$

$$k\cot\delta=\left(\frac{1}{(\beta^2+k^2)^2}+\frac{t^2k^4}{(\gamma^2+k^2)^4}\right)^{-1}\left[\frac{1}{2\pi^2\lambda}-\frac{\beta^2-k^2}{2\beta(\beta^2+k^2)^2}-\frac{t^2(\gamma^6+5\gamma^4k^2+15\gamma^2k^4-5k^6)}{16\gamma(\gamma^2+k^2)^4}\right], \quad (27)$$

and

$$\epsilon(k)=\frac{\left[\dfrac{2}{(\beta^2+k^2)}\cdot\dfrac{tk^2}{(\gamma^2+k^2)^2}+\dfrac{1}{\sqrt{2}}\left\{\dfrac{tk^2}{(\gamma^2+k^2)^2}\right\}^2\right]^2}{\left[\left(\dfrac{1}{\beta^2+k^2}\right)^2+\left\{\dfrac{tk^2}{(\gamma^2+k^2)^2}\right\}^2\right]^2}. \quad (28)$$

The asymmetry factor has following features:

$$\epsilon(k)\simeq(4t^2\beta^4/\gamma^8)k^4, \quad (k\ll\beta,\gamma);$$

and

$$\epsilon(k)\rightarrow\left\{\frac{(2+t/\sqrt{2})t}{1+t^2}\right\}^2, \quad (k\rightarrow+\infty);$$

thus

$$\epsilon(k)=0 \quad \text{at} \quad k=0.$$

If we expand $k\cot\delta$, Eq. (27), in a power series in k^2, one can easily find the desired coefficients, the scattering length a, the effective range r_0, and the shape-dependent factor P;

$$\frac{1}{a}=-\frac{\beta^4}{2\pi^2\lambda}+\frac{\beta}{2}+\frac{t^2\beta^4}{16\gamma^3}$$
$$=\frac{\alpha\beta(\alpha+2\beta)}{2(\alpha+\beta)^2}+\frac{t^2}{16}\alpha\cdot\frac{\beta^4}{\gamma^4}\cdot\frac{\alpha\gamma(\alpha^2+4\alpha\gamma+\gamma^2)}{(\alpha+\gamma)^4}, \quad (29)$$

$$r_0=\frac{2\beta^2}{\pi^2\lambda}+\frac{1}{\beta}-\frac{t^2}{8}\beta^2\left(\frac{\beta^2}{\gamma^5}+\frac{2}{\gamma^3}\right)$$
$$=\frac{2\beta^2+(\alpha+\beta)^2}{\beta(\alpha+\beta)^2}+\frac{t^2}{8}\beta^2\left\{\frac{2(5\alpha^2+4\alpha\gamma+\gamma^2)}{\gamma(\alpha+\gamma)^4}-\frac{\beta^2}{\gamma^5}-\frac{2}{\gamma^3}\right\}, \quad (30)$$

$$Pr_0^3=-\frac{1}{2\pi^2\lambda}+\frac{t^2}{2}\frac{\beta^2}{\gamma^2}\left(\frac{1}{\pi^2\lambda}-\frac{1}{\beta^3}-\frac{t^2}{8\gamma^3}\right)+\frac{t^2}{16}\frac{1}{\gamma^3}\left(1+\frac{2\beta^2}{\gamma^2}+\frac{5\beta^4}{\gamma^4}\right)$$
$$=\frac{-1}{2\beta(\alpha+\beta)^2}-\frac{1}{2}t^2\left[\frac{5\alpha^2+4\alpha\gamma+\gamma^2}{8\gamma(\alpha+\gamma)^4}-\frac{1}{8\gamma^3}\left(1+\frac{2\beta^2}{\gamma^2}+\frac{5\beta^4}{\gamma^4}\right)+\frac{\beta^8}{\gamma^8}\left(\frac{1}{\beta^3}-\frac{1}{\beta(\alpha+\beta)^2}\right)\right]$$
$$-t^4\frac{\beta^8}{2\gamma^8}\left(\frac{1}{8\gamma^3}-\frac{5\alpha^2+4\alpha\gamma+\gamma^2}{8\gamma(\alpha+\gamma)^4}\right). \quad (31)$$

Thus the intrinsic range[6] b is given by

$$b=\frac{3}{\beta}-\frac{t^2\beta^4}{8\gamma^5}, \quad (32)$$

where we have used Eq. (23).

(c) Comparison with Experiments

Now we are in a position to compare our results with experimental data. To do this we must first find values of the parameters, α, β, γ, and t, which fit the low-energy neutron-proton data. This procedure can be performed conveniently as follows. First of all α can be fixed from the deuteron binding energy,[9] $\alpha^2/M=2.225$ Mev:

$$\alpha=2.316\times10^{12}\ \text{cm}^{-1}. \quad (33a)$$

Next, from formula (29) for the scattering length a and Eq. (25) for the D-state probability P_D, eliminating t, we get

$$\frac{2(1-P_D)}{P_D}\left(\frac{\alpha+\beta}{\beta}\right)^3\left[\frac{1}{a\alpha}-\frac{\beta(\alpha+2\beta)}{2(\alpha+\beta)^2}\right]=\frac{\alpha+\gamma}{\gamma(5\alpha+\gamma)}\left[2\alpha+\frac{(\alpha+\gamma)^2}{\gamma}\right].$$

TABLE I. Comparison with experiment of various quantities calculated with the parameters given in Eq. (33b).

Calculated	Experimental
$P_D=4.000$ percent	2–8 percent (due to Miyzawa's argument)[a]; 5–10 percent (due to Machida's argument)[b]
$Q=2.7394\times10^{-27}$ cm^2	$(2.74\pm0.02)\times10^{-27}$ cm^2
$s=1.291_5$	
$b=2.122\times10^{-13}$ cm	
$a=5.378\times10^{-13}$ cm	$(5.378\pm0.021)\times10^{-13}$ cm
$r_0=1.704\times10^{-5}$	

[a] See reference 12.
[b] See reference 13.

[9] Li, Whaling, Fowler, and Lauritsen, Phys. Rev. **83**, 512 (1951).

Inserting P_D and the observed value[10] of $a=5.378\times10^{-13}$ cm in this equation, we can find a value of γ for an assumed value for β. Then, for any set of values for β and γ, we can calculate the value of t either from Eq. (29) for the scattering length [or equivalently from Eq. (25)] or from the Eq. (26) for the quadrupole moment. If these two values of t coincide for some set of β and γ, this set (β,γ,t) is the desired one. Unfortunately the D-state probability P_D has not been accurately determined up to the present time. Many authors, however, have adopted the value 4 percent,[11] and we shall tentatively do the same here. For a complete discussion of the D-state probability we refer the reader to existing articles.[12,13] Then we determine the values of parameters to be as follows:

$$\beta=5.759\alpha,\quad \gamma=6.771\alpha,\quad t=1.784. \tag{33b}$$

For the sake of completeness, we list in Table I the values of various quantities (re-)calculated with these values for β, γ, and t. From these results it is evident that our model fits the low-energy neutron-proton data very well. As is seen from the form (20) for $T(p)$, the effect of tensor force is not appreciable at low energies [also see the formula just under (28)]. The total cross section for the triplet neutron-proton scattering is in agreement with that of the "central" case of I, Sec. 3 up to 20 Mev. However the tensor part $T(p)$ plays an important role at higher energies.

Let us compare our results with high-energy nucleon-nucleon collision experiments.[14,15] Figures 1 and 2 show the energy dependence of $\sigma_{\text{tot}}/4\pi$ and $\epsilon(k)$, based on the parameters (33). We have determined in I, Sec. 3(c) the singlet neutron-proton interaction. Using this singlet interaction combined with the triplet interaction obtained here, we can calculate the high-energy neutron-proton scattering cross sections which, of course, contain contributions only from ^1S and eigen-^3S states (see Fig. 3). The deviation from the isotropic angular dependence is caused by the D wave in the eigen-^3S state. Furthermore, assuming charge dependence of nuclear interactions, one can compare the singlet S phase shift without Coulomb force with the observed one,[15] 50.2°, for proton-proton scattering at 30 Mev. (See Fig. 4.) As is seen from these figures, our calculated cross sections are in fair agreement with observations

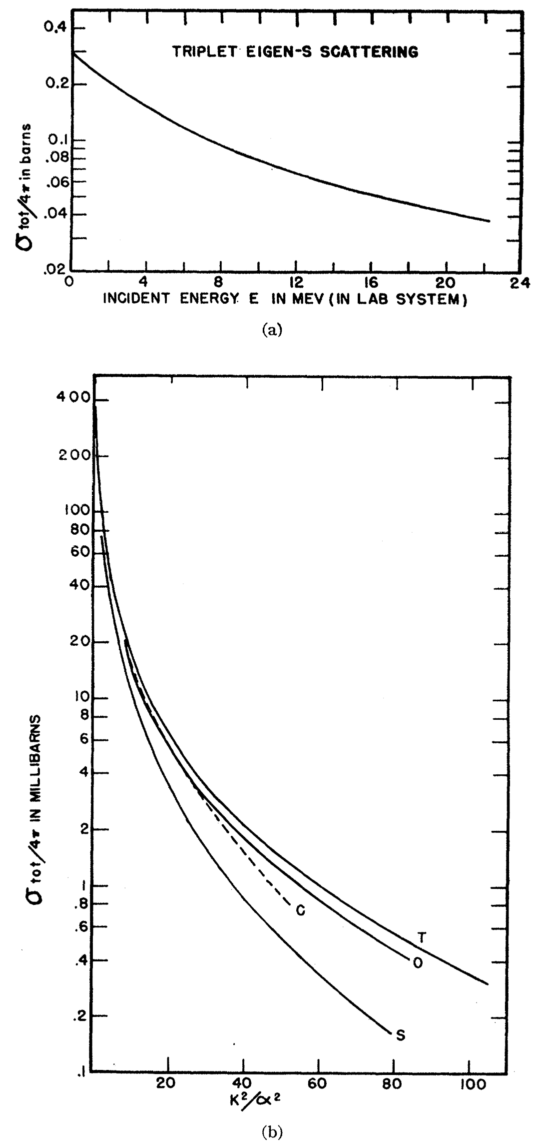

FIG. 1. (a) The n-p total cross section σ_{tot} divided by 4π for the triplet eigen-S scattering *vs* incident neutron energy E (in the laboratory system). This curve is in good agreement with that of the "central" case discussed in I, Sec. 3 [also see Fig. 1(b)]. (b) Curve T is the n-p total cross section σ_{tot} divided by 4π for the triplet eigen-S scattering [calculated from (27) and (33)]. Note that the triplet eigen-S scattering cross section is $d\sigma/d\omega=(\sigma_{\text{tot}}/4\pi)\{1+\epsilon(k)(3\cos^2\theta-1/2)\}$. For comparison, we illustrated by curves C and S the triplet and singlet S-cross sections on the basis of the "central" interaction described in I, Sec. 3. The incident energy E in the laboratory system is given by $E=2k^2/M=(k^2/\alpha^2)\times4.45$ Mev. The curve O shows the average cross section of the triplet eigen-S and the singlet S scattering.

up to 100 Mev for the neutron-proton case and up to ~40 Mev for the proton-proton case. However, at still higher energies the calculated cross sections for both neutron-proton and proton-proton scattering are definitely smaller than the experimental values, indicating interactions in P or higher-order partial waves. This is of course the conclusion reached by Christian and

[10] E. Melkonian, Phys. Rev. **76**, 1744 (1949); Burgy, Ringo, and Hughes, Phys. Rev. **84**, 1160 (1951).

[11] This value is correct under two doubtful assumptions: (i) that the relativistic correction to the magnetic moment is small, and (ii) that there is complete additivity of the proton and the neutron moments.

[12] H. Miyazawa, Progr. Theoret. Phys. **7**, 207 (1952).

[13] S. Machida, Progr. Theoret. Phys. **9**, 683 (1953).

[14] Hardley, Kelly, Leith, Segrè, Wiegand, and York, Phys. Rev. **75**, 351 (1949); Kelly, Leith, Segrè, and Wiegand, Phys. Rev. **79**, 96 (1950); Randle, Tayler, and Wood, Proc. Roy. Soc. (London) **A213**, 392 (1952).

[15] W. K. H. Panofsky and F. L. Fillmore, Phys. Rev. **79**, 96 (1950); Cork, Johnston, and Richman, Phys. Rev. **79**, 57 (1950).

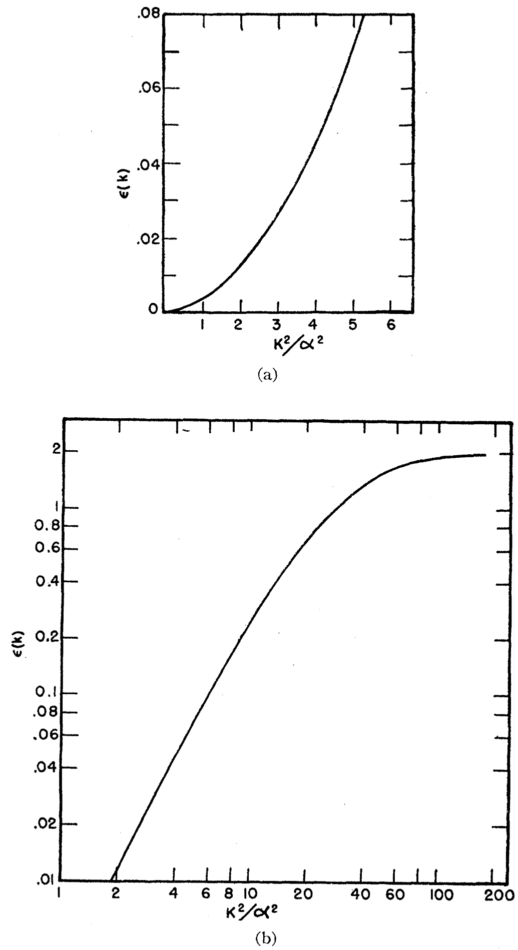

FIG. 2. The energy dependence of the asymmetry factor, $\epsilon(k)$, calculated from (28) and (33). Incident energy in laboratory system $E=(k^2/\alpha^2)\times 4.45$ Mev.

others.[16] Actually there is already some indication of a P wave interaction even in 90-Mev neutron-proton scattering, as is seen from the asymmetry around 90°. More detailed analysis of high-energy nucleon-nucleon scattering based on separable potentials is in progress.

4. PHOTODISINTEGRATION OF THE DEUTERON

Since we have found a nice form (21) for the deuteron function, it is interesting to apply it to the photodisintegration problem. As was shown in the preceding section, the neutron-proton interaction up to 100 Mev can be well expressed in terms of our separable potentials (20) for the eigen-^3S state and the singlet S potential described in I, and, as is well known, the forces acting in states higher than P are not important in this energy reaction. Thus it may be concluded that we are in a position to make a more accurate calculation of the electric dipole disintegration cross section up to 50-Mev photon energy (in the c.m. system) than has been reported so far.[17] The magnetic transition is partly due to an unknown effect, the so-called interaction magnetic moment.[18] However, neglecting this small effect, we can also calculate the magnetic transition using our wave functions. Therefore, keeping within the low and intermediate photon energies, we want to discuss here the dipole photodisintegration of the deuteron.

(a) Electric Dipole Transition

As stated previously, we want to use free ^3P waves and the deuteron function (4). According to Siegert's theorem,[19] the transition matrix for the electric dipole transition is given by $\int(^3P \text{ function})\mathbf{r}\cdot\boldsymbol{\varepsilon}(\text{deuteron function})$, where $\mathbf{r}$ is the relative coordinate between the proton and the neutron, and $\boldsymbol{\varepsilon}$ is the polarization vector of electromagnetic field. This matrix element can also be calculated using the argument of gauge invariance as was done in I, Sec. 4. These two methods naturally give the same results. Thus one can find the electric dipole disintegration cross section in the c.m. system

$$\frac{d\sigma_e}{d\omega}=\frac{e^2}{4\pi}\frac{\alpha p^3}{(\alpha^2+p^2)^3}\frac{\pi^2 N^2}{\alpha}[(E_1+E_2)\sin^2\theta+E_3(1+\cos^2\theta)],$$

$$E_1=\left\{C(p)-\frac{\alpha^2+p^2}{2p}\frac{\partial C}{\partial p}\right\}^2,$$

$$E_2=\left\{T(p)-\frac{\alpha^2+p^2}{2p}\frac{\partial T}{\partial p}\right\}^2, \tag{34}$$

and

$$E_3=\frac{3}{4}\left\{\frac{\alpha^2+p^2}{p^2}T(p)\right\}^2,$$

where θ is the angle between the photoproton and the incident photon, $\mathbf{p}$ is the relative momentum of the final neutron-proton system, E_1 represents the transition $^3S\rightarrow{}^3P$, and E_2 and E_3 correspond to the transition $^3D\rightarrow{}^3P$. It may be seen from this result that if $T(p)$ is constant for small p, the electric dipole cross section becomes proportional to $1/p$ at very low photon energy. In contrast to this, if $T(p)\propto p^2$ at small p as in the case of (20), the contribution from $^3D\rightarrow{}^3P$ is reasonably small at low photon energy.

Next let us compare our result (34) with that of the "central" potential. For this purpose, it is convenient

[16] R. S. Christian and E. W. Hart, Phys. Rev. **77**, 441 (1950); R. S. Christian and H. P. Noyes, Phys. Rev. **79**, 85 (1950); K. M. Case and A. Pais, Phys. Rev. **80**, 203 (1950); R. Jastrow, Phys. Rev. **81**, 165 (1951).

[17] A. Sugie and S. Yoshida, Progr. Theoret. Phys. **10**, 236 (1953); W. Rarita and J. Schwinger, Phys. Rev. **59**, 556 (1941); N. Austern, Phys. Rev. **85**, 283 (1952); T. M. Hu and H. S. W. Massey, Proc. Roy. Soc. (London) **A196**, 135 (1949). Other references will be found in I.

[18] N. Austern and R. G. Sachs, Phys. Rev. **81**, 710 (1951); N. Austern, reference 17.

[19] A. J. F. Siegert, Phys. Rev. **52**, 787 (1937); R. G. Sachs and N. Austern, Phys. Rev. **81**, 705, 710 (1951).

to introduce the "associated central potential" $-(\lambda_a/M)g_a(p)g_a(p')$, which is defined from the original interaction (1) including spin-orbit coupling as follows.

(i) The form of associated central potential is the same as the central part of the original potential; $g_a(p)=C(p)$.

(ii) The strength λ_a of the associated central potential is now determined so as to give the same value for the binding energy of the deuteron as that of the original interaction (1). Therefore the associated central potential does not exactly fit the low-energy data, even if the original potential (1) does fit. An example of this is given by the potential (20) with (33) fitted to low-energy neutron-proton data. The associated central potential has the value 5.759α for β, but the best-fit central potential has a different value, 6.255α, for β (see I, Sec. 3). It should be noted that the relation between the normalization constants N and N_a for the deuteron functions corresponding to the potential (1) and its associated central potential, respectively, is given by $N^2=N_a^2(1-P_D)$.

If we adopt the associated central potential, the electric dipole cross section turns out to be

$$\frac{d\sigma_e}{d\omega}=\frac{e^2}{4\pi}\frac{\alpha p^3}{(\alpha^2+p^2)^3}\frac{\pi^2N_a^2}{\alpha}E_1\sin^2\theta. \qquad (35)$$

This form is, of course, the same as (33 E.D.) in I. Therefore, if in (34) we put $E_2=E_3=0$ and replace N by N_a [this fact is simply expressed by $t=0$, keeping in mind Eqs. (5), (6), and (7)], we find the result (35).

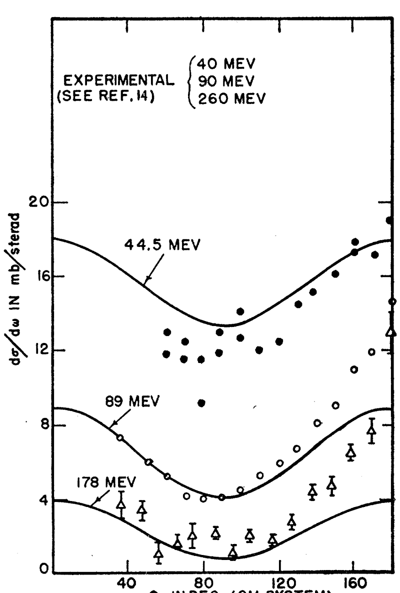

FIG. 3. The calculated average cross sections (triplet eigen-S plus singlet S scattering only) are compared with observation.

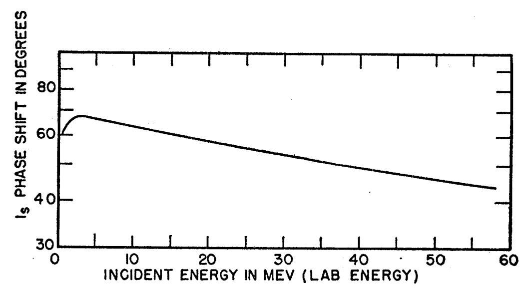

FIG. 4. ^1S phase shift is shown in this figure. This is calculated from Eqs. (18), (26), and (27) in I.

Example

If we adopt (20) for $g(\mathbf{p})$, the electric dipole cross section takes the form

$$\frac{d\sigma_e}{d\omega}=\frac{e^2}{4\pi}\frac{\alpha p^3}{(\alpha^2+p^2)^3}\frac{\beta(\alpha+\beta)}{(\beta-\alpha)^2}(1-P_D)$$
$$\times\Bigg[\left\{1-\left(\frac{\alpha^2+p^2}{\beta^2+p^2}\right)^2\right\}^2\sin^2\theta$$
$$+t^2\frac{(\beta^2-\alpha^2)^2(-\alpha^2\gamma^2+\alpha^2p^2+2p^4)^2}{(\gamma^2+p^2)^6}\sin^2\theta$$
$$+\frac{3}{4}t^2\frac{(\beta^2-\alpha^2)^2(\alpha^2+p^2)^2}{(\gamma^2+p^2)^4}(1+\cos^2\theta)\Bigg]$$
$$\equiv a_e+b_e\sin^2\theta, \qquad (36)$$

where we use (22) and (25). If we put $t=0$, Eq. (36) has just the same form as Eq. (36 E.D.) in I.

(b) Magnetic Dipole Transition

As mentioned previously we shall not discuss here the effect of the interaction moment. The magnetic transition can then be evaluated unambiguously by using our wave functions. For future reference let us write down the singlet S potential $-(\lambda_s/M)g_s(p)g_s(p')$, and define $J_s(p)$ by

$$J_s(p)=\int d\mathbf{q}\frac{g_s^2(q)}{p^2-q^2+i\epsilon}.$$

The magnetic dipole cross section is

$$\frac{d\sigma_m}{d\omega}=\frac{1}{6}\frac{e^2}{4\pi}\left(\frac{\mu_p-\mu_n}{M}\right)^2\pi^2N^2p(\alpha^2+p^2)$$
$$\times\left[|M_1|^2+|M_2|^2+M_3\frac{3\cos^2\theta-1}{2}\right], \qquad (37)$$

where θ is again the angle between the photonproton and the incident photon, M_1 and M_2 correspond to the $^3S\rightarrow{}^1S$ and $^3D\rightarrow{}^1S$ transitions, respectively, and M_3 is the interference term:

$$M_1=\frac{C(p)}{\alpha^2+p^2}-\frac{\lambda_s g_s(p)}{1+\lambda_s J_s(p)}\int d\mathbf{q}\frac{g_s(q)C(q)}{(q^2+\alpha^2)(p^2-q^2+i\epsilon)},$$

$$M_2=T(p)/(\alpha^2+p^2),$$

and

$$M_3=\sqrt{2}\,\mathrm{Re}(M_1M_2)-\tfrac{1}{2}M_2^2.$$

If we replace our triplet interaction by the associated central interaction, we get the result (37) with $t=0$, i.e., $N^2=N_a^2$, $(M_2)^2=M_3=0$. If $T(p)=0$ at $p=0$, the effect of the D wave deuteron function is very small near threshhold. Therefore our magnetic dipole cross section is equal to that corresponding to the associated triplet central potential except for the factor $1-P_D$. This holds also in the case of thermal-neutron capture by hydrogen.

Example

Let us again use the deuteron function (21) and the singlet S wave function described in I, Sec. 3. To avoid confusion, let us write

$$g_s(p)=1/(\beta_s^2+p^2) \quad\text{and}\quad C(p)=1/(\beta_t^2+p^2). \tag{38}$$

Then the magnetic dipole cross section is

$$\begin{aligned}\frac{d\sigma_m}{d\omega}=&\frac{1}{6}\frac{e^2}{4\pi}\left(\frac{\mu_p-\mu_n}{M}\right)^2\alpha\beta_t(\alpha+\beta_t)^3(1-P_D)p(\alpha^2+p^2)\\ &\times\left[\frac{\sin^2\delta_s}{p^2}[\quad]^2+\frac{t^2p^4}{(\alpha^2+p^2)^2(\gamma^2+p^2)^4}\right.\\ &-\left\{\frac{\sqrt{2}\sin\delta_s\cos\delta_s}{p}[\quad]+\frac{tp^2}{2(\alpha^2+p^2)(\gamma^2+p^2)^2}\right\}\\ &\left.\times\frac{tp^2}{(\alpha^2+p^2)(\gamma^2+p^2)^2}\frac{3\cos^2\theta-1}{2}\right]\\ \equiv& a_m+b_m\sin^2\theta,\end{aligned}\tag{39}$$

where δ_s is the singlet S phase shift and

$$[\quad]=\frac{p\cot\delta_s}{(\alpha^2+p^2)(\beta_t^2+p^2)}+\frac{1}{\beta_t+\alpha}\left\{\frac{\alpha\beta_t-p^2}{(\alpha^2+p^2)(\beta_t^2+p^2)}-\frac{1}{(\alpha+\beta_s)(\beta_t+\beta_s)}\right\}.$$

For thermal neutron capture by hydrogen we can neglect the ${}^3D\rightarrow{}^1S$ transition and obtain the total cross section σ_{cap} as

$$\begin{aligned}\sigma_{\mathrm{cap}}v_n=&2\pi\frac{e^2}{4\pi}\left(\frac{\mu_p-\mu_n}{M}\right)^2\left(\frac{\alpha}{M}\right)^3\frac{\alpha+\beta_t}{\beta_t}(1-P_D)(a_s\beta_t)^2\\ &\times\left[-\frac{1}{a_s\beta_t}\cdot\frac{\alpha+\beta_t}{\beta_t}+\frac{\alpha}{\beta_t}-\frac{1}{(\alpha+\beta_s)(\beta_t+\beta_s)}\right],\end{aligned}\tag{40}$$

where v_n is the incident neutron velocity in the laboratory system and a_s is the singlet scattering length at zero energy.

(c) Discussion

We have seen that the photodisintegration cross section can be expressed in a tractable form, and our approach has the particular advantage of showing the effect of tensor force. Although the results obtained above were derived on the assumption of a separable potential, some aspects are valid quite generally. For example, our electric dipole cross section (34) depends only on the form of the deuteron function (4), which may be regarded as free from the assumption of a separable nuclear potential, and as the solution for a local potential if preferable. [Note that the Hulthén function is the bound-state solution for the local Hulthén potential and also simultaneously for the separable potential $g(p)=(p^2+\beta^2)^{-1}$.] We can also generalize the concept of an associated central potential for any type of potential with spin-orbit coupling. The photodisintegration cross sections based on such potentials will obey the relationships stated above. We are now interested in comparison with experiment.

Table II summarizes our calculation on thermal neutron capture by hydrogen. The calculated values are definitely smaller than the experimental value,[20] $\sigma_{\mathrm{cap}}v_n=7.30\times10^{-20}$ cm³/sec, for reasonable values for the singlet effective range and shows the evidence for interaction moment in agreement with Austern.[21]

The dipole cross section is shown in Fig. 5. We can see that the contribution from ${}^3S\rightarrow{}^3P$ is essentially in agreement with those reported on the basis of a local Hulthén force,[22] apart from minor difference coming

TABLE II. The thermal-neutron capture cross sections calculated from (40) and (33).

	Singlet effective range	Capture cross section multiplied by neutron velocity
β_s/α	r_{0s} (10^{-13} cm)	$\sigma_{\mathrm{cap}}v_n$ (10^{-20} cm³/sec)
7.00	1.914	7.05
6.50	2.067	6.95
6.25	2.153	6.93
6.00	2.246	6.88
5.75	2.348	6.83
5.50	2.459	6.78

[20] Hamermesh, Ringo, and Wexler, Phys. Rev. **90**, 603 (1953); Harris, Muehlhause, Rose, Schroeder, Thomas, and Wexler, Phys. Rev. **91**, 125 (1953); G. von Dardel and A. W. Waltner, Phys. Rev. **91**, 1284 (1953).

[21] N. Austern, Phys. Rev. **92**, 670 (1953).

[22] See I, where other references will be found.

from difference choices for the values of β_t and the factor $1-P_D$. The ${}^3D\to{}^3P$ transition becomes, however, more important with increasing photon energy and causes a rapid increase of the isotropic part $a=a_e+a_m$ in the angular distribution of photoprotons; see Fig. 5. In the magnetic dipole transition the D wave $w(p)$ of the deuteron gives some contribution. Thus we may conclude that the isotropic component $a=a_e+a_m\simeq a_e$ is a direct measure of $w(p)$ at a moderate photon energy. Up to the present there are no definite experimental results[23] about the isotropic part up to 20 Mev, and thus our results naturally do not contradict observation (the calculated total cross sections are in agreement with observation). At higher energies precise experiments by the Illinois group are now available.[24] Their total cross sections at 20–60 Mev photon energy are in reasonable agreement with our results but the angular distributions do not agree with our theoretical calculations. But one can expect to find larger values for a/b if one adopts a *larger* value for the D-state probability P_D than 4 percent.

At still higher energies the experimental cross sections are definitely larger than our prediction. However, at these energies we must certainly take into account effects disregarded so far; namely, (i) higher multipole transitions, (ii) forces in P or higher-angular-momentum states (see Austern[17]), and (iii) interaction moments or mesonic effects, and so on. One can hardly say anything without a careful examination of each of these points.

It should be noted that the smallness of P_D or Q does not necessarily mean that the D wave function $w(p)$ of the deuteron must be small at large p. Thus the possible and probable importance of D wave at large p remains one of our conclusions which is valid generally, because this statement follows directly from the form of deuteron function itself and is not dependent upon the assumption of a separable force.

Finally we should like to emphasize again the ease with which the approach presented in this paper handles the two-nucleon problem. The ability of the interaction (20) to fit observed results, in spite of its simple form, is remarkable. A slightly different separable nuclear potential with spin-orbit coupling was considered independently by Gell-Mann, Goldberger, and Bloch.[25] These authors started with the radial wave equations, in which the nuclear forces were assumed to be factorable.

We are deeply indebted to Professor G. F. Chew for his valuable discussions throughout this work. We also wish to thank Dr. A. O. Hanson, Mr. L. Allen, and Mr. E. A. Whalin for their discussion of the photodisintegration of the deuteron and for informing us of their results before publication.

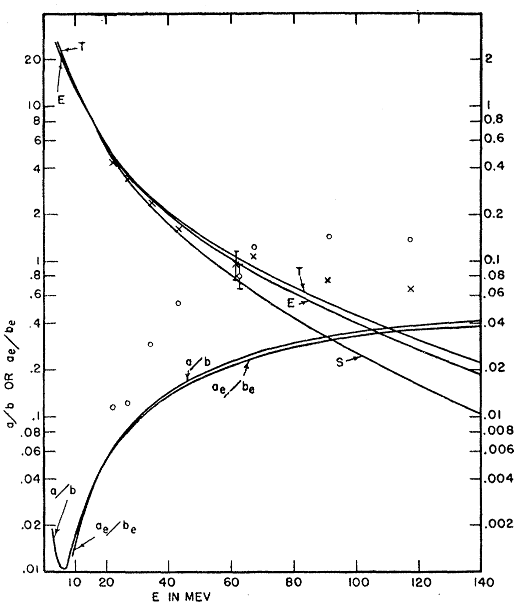

FIG. 5. The dipole cross section for $d(\gamma,p)n$ takes the form $d\sigma/d\omega=a+b\sin^2\theta$, $a=a_e+a_m$, $b=b_e+b_m$ [see Eqs. (36) and (39)]. The curves labeled T, E and S are T: the total cross section (E.D. plus M.D.), E: the total cross section for E.D. transition, and S: the contribution from ${}^3S\to{}^3P$ in the E.D. transition. The ratios a/b and a_e/b_e are also shown in this figure. It is noted that our calculations are based on the D-state probability P_D of 4 percent. Experimental results are shown by ×: total cross section, and o: a/b. The abscissa is the total energy E; $E=$ (photon energy)+(kinetic energy of the incident deuteron) in the c.m. system.

[23] V. E. Krohn and E. F. Shrader, Phys. Rev. **86**, 391 (1952); H. Waffler and S. Younis, Helv. Phys. Acta **24**, 483 (1951); J. Halpern and E. V. Weinstock, Phys. Rev. **91**, 934 (1953).

[24] Allen, Hanson, and Whalin (private communication).

[25] Bloch, Gell-Mann, and Goldberger (unpublished).