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# Two-Nucleon Problem When the Potential Is Nonlocal but Separable. I

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The two-nucleon problem is considered in terms of an interaction of the form $(\mathbf{p}|V|\mathbf{p}')=-(\lambda/M)g(p)g(p')$. In this case we can find exact solutions both for bound states and for continuum states, and prescribe arbitrarily and independently the interaction effective in states of different angular momenta. This important feature makes the analysis of scattering straightforward and unambiguous. An example for $g(p)=(p^2+\beta^2)^{-1}$ is presented and compared with the low-energy neutron-proton data. The photodisintegration of the deuteron in our model is also discussed.

## 1. INTRODUCTION

IN the past several years, much (experimental and theoretical) work[1–4] has been published on the two-nucleon system. However, our information about nuclear forces is still rather small. From the theoretical point of view, this is partly due to the difficulty in solving the scattering problem. Therefore, it seems to us worthwhile to discuss a special nuclear potential for which we can get readily an exact solution. This potential[5] is nonlocal and thus of a fundamentally different form than those usually discussed, but since no one knows the correct form there remains room to test new types. If we choose our nonlocal potential to be without a long tail, we can depend upon the validity of the so-called effective range theory to guarantee a fit to the low-energy data for the two-nucleon system. Furthermore, our wave functions described below are convenient for the examination of various phenomena involving two nucleons. To illustrate this we will treat the photodisintegration of the deuteron in the last section.

## 2. FORMULATION OF THE PROBLEM

Let us consider the two-body nonrelativistic Schrödinger equation

$$\left\{E+\frac{1}{M}\Delta\right\}\psi(\mathbf{r})=\int(\mathbf{r}|V|\mathbf{r}')d\mathbf{r}'\psi(\mathbf{r}'),\qquad(1)$$

$E$ and $M$ being, respectively, the total energy in the center-of-mass system and the nucleon mass. $(\mathbf{r}|V|\mathbf{r}')$ is the nuclear potential which, in general, is nonlocal.[6] Transforming to the momentum space representation, we have

$$(k^2/M-p^2/M)\psi(\mathbf{p})=\int(\mathbf{p}|V|\mathbf{p}')d\mathbf{p}'\psi(\mathbf{p}'),\qquad(2)$$

where $\mathbf{k}$ is the incident momentum and $k=|\mathbf{k}|$, $k^2/M=E$. If $(\mathbf{p}|V|\mathbf{p}')$ depends only on $\mathbf{p}-\mathbf{p}'$, then our potential is "local," i.e., $(\mathbf{r}|V|\mathbf{r}')$ must be the function of $\mathbf{r}$ multiplied by $\delta(\mathbf{r}-\mathbf{r}')$; otherwise our potential becomes necessarily nonlocal. As stated before, we want to assume a nonlocal but separable potential[7]

$$(\mathbf{r}|V|\mathbf{r}')=-(\lambda/M)v^*(\mathbf{r})v(\mathbf{r}'),\qquad(3)$$

or equivalently

$$(\mathbf{p}|V|\mathbf{p}')=-(\lambda/M)g^*(\mathbf{p})g(\mathbf{p}'),$$

where the asterisk designates complex conjugate. If we postulate time reversibility, $v(\mathbf{r})$ and $g(\mathbf{r})$ must be real in the sense defined by Wigner.[8] Equation (2) takes

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[1] G. F. Chew and M. L. Goldberger, Phys. Rev. **75**, 1637 (1949); J. M. Blatt and J. D. Jackson, Phys. Rev. **76**, 18 (1949); H. A. Bethe, Phys. Rev. **76**, 38 (1949); J. D. Jackson and J. M. Blatt, Revs. Modern Phys. **22**, 77 (1950); J. Schwinger, Phys. Rev. **78**, 135 (1950); E. E. Salpeter, Phys. Rev. **82**, 60 (1951); H. Feshbach and J. Schwinger, Phys. Rev. **84**, 194 (1951); Taketani, Nakamura, and Sasaki, Prog. Theoret. Phys. Japan **6**, 581 (1951); Taketani, Machida, and Onuma, Prog. Theoret. Phys. Japan **7**, 45 (1952); L. Hulthén and K. V. Laurikainen, Revs. Modern Phys. **23**, 1 (1951). For other references, see: L. Rosenfeld, *Nuclear Forces* (Interscience Publishers, Inc., New York—London, 1949); J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952).

[2] Li, Whaling, Fowler, and Lauritsen, Phys. Rev. **83**, 512 (1951).

[3] E. Melkonian, Phys. Rev. **76**, 1744 (1949); Burgy, Ringo, and Hughes, Phys. Rev. **84**, 1160 (1950).

[4] Bailey, Bennett, Bergstrahl, Nuckolls, Richards, and Williams, Phys. Rev. **70**, 583 (1946); D. H. Frisch, Phys. Rev. **70**, 589 (1946); Lampi, Freier, and Williams, Phys. Rev. **80**, 853 (1950); Poss, Salant, Snow, and Yuan, Phys. Rev. **87**, 11 (1952); G. Snow, Phys. Rev. **87**, 21 (1952); Hafner, Hornyak, Falk, Snow, and Coor, Phys. Rev. **89**, 204 (1953). For other references, see Rosenfeld, or Blatt and Weisskopf, reference 1.

[5] This type of nuclear potential was discussed by E. P. Wigner (unpublished), quoted in Blatt and Weisskopf (reference 1, p. 139), Bloch, Gell-Mann, and Goldberger (unpublished).

[6] This nonlocal potential can be derived from the (unfamiliar) interaction between the nucleon field $\psi$ and the meson field $\varphi$:

$$-\frac{\lambda}{M}\frac{1}{(2\pi)^3}\frac{\mu^2}{\beta^4}\frac{1}{4!}\left(\int\bar{\psi}\varphi\psi d\mathbf{r}\right)^4,$$

providing that the nucleon has finite size:

$$\int\bar{\psi}\psi e^{-i\mathbf{p}\cdot\mathbf{r}}d\mathbf{r}=\frac{\beta}{(\mu)^{\frac{1}{2}}}\frac{(p^2+\mu^2)^{\frac{1}{4}}}{(p^2+\beta^2)^{\frac{1}{2}}}\quad(\mu=\text{meson mass}),$$

and provided that we limit ourselves within the lowest order adiabatic nuclear force.

[7] This potential leads to the saturation of nuclear binding energies.

[8] E. Wigner, Nachr. Ges. Wiss. Göttingen **31**, 546 (1932).

the form

$$(k^2-p^2)\psi(\mathbf{p})=-\lambda g(\mathbf{p})\int g(\mathbf{p}')d\mathbf{p}'\psi(\mathbf{p}'). \tag{4}$$

In such a case, one can easily find the exact solution. To see this, let us consider the simplest case, $g(\mathbf{p})=g(|\mathbf{p}|)$. For the bound state we must replace $-k^2/M$ by the binding energy $\alpha^2/M$. Thus

$$(\alpha^2+p^2)\psi(\mathbf{p})=\lambda g(p)\int g(p')d\mathbf{p}'\psi(\mathbf{p}'). \tag{5}$$

The desired solution must be[9]

$$\psi(\mathbf{p})=Ng(p)/(\alpha^2+p^2),\quad \frac{1}{N^2}=\int dq\frac{g^2(q)}{(\alpha^2+q^2)^2}; \tag{6}$$

and $\lambda=\lambda(\alpha)$ is now determined so as to fit the observed binding energy $\alpha^2/M$ of the deuteron:

$$\frac{1}{\lambda}=\int dq\frac{g^2(q)}{\alpha^2+q^2}. \tag{7}$$

Therefore, we see that the well-depth parameter $s$, defined by Blatt and Jackson,[1] is just

$$s=\lambda(\alpha)/\lim_{\alpha\to 0}\lambda(\alpha). \tag{7'}$$

For the scattering problem

$$\psi(\mathbf{p})=\delta(\mathbf{p}-\mathbf{k})-\frac{\lambda g(k)}{1+\lambda\int dq\frac{g^2(q)}{k^2-q^2+i\epsilon}}\cdot\frac{g(p)}{k^2-p^2+i\epsilon}, \tag{8}$$

where $\mathbf{k}$ is the incident momentum and $\epsilon$ is an infinitesimal positive quantity which makes the second term of the right-hand side of (8) an outgoing wave. The differential cross section turns out to be

$$\begin{aligned} d\sigma/d\omega&=|f|^2,\\ f&=e^{i\delta}\sin\delta/k=1/(-ik+k\cot\delta)\\ &=[2\pi^2\lambda g(k)g(p)]\Big/\left[1+\lambda\int dq\frac{g^2(q)}{k^2-q^2+i\epsilon}\right], \end{aligned} \tag{9}$$

where $\mathbf{p}$ is the momentum of scattered nucleon ($|\mathbf{p}|=k$).

In the more general case, where $g(p)$ depends not only on the magnitude of $\mathbf{p}$ but also on the direction, one can often find an exact solution. For example, if we assume the "central" plus "tensor" form:

$$g(\mathbf{p})=c(p)+1/\sqrt{8}\times\left\{\frac{3}{p^2}(\boldsymbol{\sigma}^{(1)}\cdot\mathbf{p})(\boldsymbol{\sigma}^{(2)}\cdot\mathbf{p})-(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})\right\}t(p), \tag{10}$$

where $c(p)$ and $t(p)$ are functions of $p=|\mathbf{p}|$, the exact solutions of the Schrödinger equation are readily obtainable. This case will be discussed in the following paper. A sufficient condition that the wave equation be soluble is as follows:

$$\int\frac{[g(\mathbf{k})\cdot g(\mathbf{p}')]d\mathbf{p}'[g(\mathbf{p}')\cdot g(\mathbf{p})]}{k^2-p'^2+i\epsilon}=[g(\mathbf{k})\cdot g(\mathbf{p})]\times(\text{function of } k=|\mathbf{k}|),$$

providing the integral exists.

If $g(p)$ depends on $p=|\mathbf{p}|$, we have just $s$ scattering. In order to get scattering in a state of orbital angular momentum (without spin-orbit coupling), we must assume the potential

$$(\mathbf{p}|V_l|\mathbf{p}')=-\frac{\lambda}{M}\sum_{m=-l}^{l}Y_l^{m*}(\theta_p,\varphi_p)Y_l^m(\theta_{p'},\varphi_{p'})g(p)g(p'), \tag{11}$$

where $g(p)$ is dependent on $p=|\mathbf{p}|$. It is evident that we can also find the exact solution for this potential (11). It is very important to see that $V_l$ causes scattering only for the $l$th partial wave, in other words, that the potential is completely "separable." We cannot have such separability in the case of a local potential. For example, the central local potential gives rise to scattering of all orders of spherical harmonics. This separability holds in terms of total angular momentum and parity also for the nonlocal potential involving spin-orbit coupling [e.g., (10)]. This fact makes the correlation of scattering phase shifts with the interaction quite definite and easy. Evidently this gain has been achieved by vastly extending the number of free parameters in the interaction.

One should note the seeming resemblance between the form of our wave function (8) and the second Born approximation to the usual local potential.

### 3. DETAILED DISCUSSION OF SPECIAL CASE $g(p)=(p^2+\beta^2)^{-1}$

If we assume, say, a rational function of $p=|\mathbf{p}|$ for $g(p)$, we can easily perform all integrations required in solving the two-body problem. In this section we consider in detail the simplest case,[10] where

$$g(p)=1/(p^2+\beta^2). \tag{12}$$

This interaction acts in the $S$ state only.

[9] The factorable potential can have at most one bound state.

[10] This interaction was independently examined by F. Bloch and M. Gell-Mann (to be published). Their results are in agreement with ours.

### (a) Bound State—Deuteron Problem

The Schrödinger equation takes the form

$$(\alpha^2+p^2)\psi(\mathbf{p})=\lambda\frac{1}{p^2+\beta^2}\int\frac{d\mathbf{p}'\psi(\mathbf{p}')}{p'^2+\beta^2}, \tag{13}$$

and its solution can readily be written down:

$$\psi(\mathbf{p})=N/[(\alpha^2+p^2)(\beta^2+p^2)],\quad N^2=\pi^{-2}\alpha\beta(\alpha+\beta)^3; \tag{14}$$

where $N$ is the normalization constant. Here $\lambda$ must be determined so as to fit the observed binding energy $\alpha^2/M$ of the deuteron. We get

$$\lambda=\pi^{-2}\beta(\alpha+\beta)^2=s\pi^{-2}\beta^3;\quad s=(\alpha+\beta)^2/\beta^2. \tag{15}$$

Curiously our deuteron function has exactly the same form as that for the well-known local potential of the Hulthén type; i.e., the deuteron function in the coordinate space is

$$\psi(\mathbf{r})=\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{N}{\beta^2-\alpha^2}\frac{e^{-\alpha r}-e^{-\beta r}}{r},\quad r=|\mathbf{r}|. \tag{14'}$$

### (b) Continuum State—Neutron-Proton Scattering

Our fundamental equation is

$$(k^2-p^2+i\epsilon)\psi(\mathbf{p})=-\lambda\frac{1}{\beta^2+p^2}\int d\mathbf{p}'\frac{1}{\beta^2+p'^2}\psi(\mathbf{p}'). \tag{16}$$

Putting

$$\psi(\mathbf{p})=\delta(\mathbf{p}-\mathbf{k})-\frac{k^2+p^2}{2\pi^2}f(k)\cdot\frac{1}{\beta^2+p^2}\cdot\frac{1}{k^2-p^2+i\epsilon} \tag{17}$$

and inserting this into (16), we can find $f(k)$:

$$f(k)=1\bigg/\left[-ik+\left(-\beta+\frac{\beta^2+k^2}{2\beta}+\frac{(\beta^2+k^2)^2}{2\pi^2\lambda}\right)\right]. \tag{17'}$$

If we transform (17) into the $\mathbf{r}$ representation,

$$\psi(\mathbf{r})=e^{i\mathbf{k}\mathbf{r}}+f(k)\frac{e^{ikr}-e^{-\beta r}}{r}$$
$$=e^{i\delta}\left\{\frac{\sin(kr+\delta)-e^{-\beta r}\sin\delta}{kr}\right\}+\left(e^{i\mathbf{k}\mathbf{r}}-\frac{\sin kr}{kr}\right) \tag{17''}$$

(this form has often been used by many authors[1]), we can see that $f(k)$ is equal to the scattering amplitude;

$$f(k)=\frac{e^{i\delta}\sin\delta}{k}=\frac{1}{-ik+k\cot\delta},\quad (\delta=\text{phase shift}).$$

Thus we find

$$k\cot\delta=-1/a+\tfrac{1}{2}r_0k^2-Pr_0^3k^4+Qr_0^5k^6+\cdots \tag{18}$$
$$=-\alpha+\tfrac{1}{2}r_d(\alpha^2+k^2)-P_dr_d^3(\alpha^2+k^2)^2+Q_dr_d^5(\alpha^2+k^2)^3+\cdots; \tag{19}$$

and we can determine the scattering length $a$, the effective range $r_0$, etc.:

$$\frac{1}{a}=\frac{\beta}{2}\left(1-\frac{\beta^3}{\pi^2\lambda}\right),\quad r_0=\frac{1}{\beta}\left(1+\frac{2\beta^3}{\pi^2\lambda}\right),$$
$$Pr_0^3=-\frac{1}{2\pi^2\lambda},\quad Qr_0^5=0,\quad\cdots. \tag{20}$$

For the triplet neutron-proton system, using (15), we find

$$a=\frac{2(\alpha+\beta)^2}{\alpha\beta(\alpha+2\beta)},\quad r_0=\frac{(\alpha+\beta)^2+2\beta^2}{\beta(\alpha+\beta)^2},$$
$$r_d=\frac{3\beta-\alpha}{\beta(\alpha+\beta)},\quad Pr_0^3=P_dr_d^3=-\frac{1}{2\beta(\alpha+\beta)^2}, \tag{20'}$$
$$Q=Q_d=0,\quad\cdots.$$

Here we can easily check the internal consistency:

$$1/a=\alpha-\tfrac{1}{2}r_d\alpha^2-P_dr_d^3\alpha^4,\quad r_0=r_d-4P_dr_d^3\alpha^2,$$
$$r_0=2\int_0^\infty dr[(1-r/a)^2-(1-r/a-e^{-\beta r})^2], \tag{20''}$$

and

$$r_d=2\int_0^\infty dr[(e^{-\alpha r})^2-(e^{-\alpha r}-e^{-\beta r})^2].$$

Note that $P$ is negative for attractive potential $\lambda>0$. The intrinsic range $b$ (see Blatt and Jackson[1]) is

$$b=\lim_{s\to1}r_0=\lim_{\alpha\to0}\left\{\frac{1}{\beta}+\frac{2\beta^2}{\pi^2\lambda(\alpha)}\right\}=\frac{3}{\beta}. \tag{21}$$

It is important to see that the expansion (18) does not contain terms higher than $k^6$. This feature is a result of our special assumption, (12), for the form of $g(p)$. In general, if we take $g(p)^{-1}$ as a polynomial in $p^2$ of order $n$ (here we assume $g(p)^{-1}=0$ has no real roots), we find that our expansion (18) is a polynomial in $k^2$ of order $2n$. Furthermore, if $g(\mathbf{p})\propto p^l$ for small $p=|\mathbf{p}|$, we can prove that

$$k^{2l+1}\cot\delta=(\text{const})+(\text{const})k^2+\cdots.$$

### (c) Comparison with Low-Energy Neutron-Proton Data

Let us distinguish the various quantities $\alpha$, $\beta$, $r_0$, and so on, corresponding to the triplet or singlet state by the suffix $t$ or $s$. We start with the triplet neutron-proton system. Adopting the value 938.9 Mev for $M$ (this is equal to the average of the proton and neutron mass), and using $\alpha_t^2/M=2.225$ Mev,[2] we get

$$\alpha_t=2.316\times10^{12}\ \text{cm}^{-1}. \tag{22}$$

Then $\beta_t$ can be determined from the experimental value[3] $a_t = 5.378\times10^{-13}$ cm for the triplet scattering length $a_t$. We find

$$\beta_t = 6.255\alpha_t = 14.488\times10^{12}\ \text{cm}^{-1}. \tag{23}$$

Inserting these values into (20′) we can calculate the triplet effective range $r_{0t}$ and the shape-dependent factor $P_t$:

$$r_{0t} = 1.716\times10^{-13}\ \text{cm}, \tag{24}$$

$$P_t(r_{0t})^3 = -0.122\times10^{-39}\ \text{cm}^3 = -0.024(r_{0t})^3. \tag{25}$$

These values may be compared with the corresponding parameters in Table I. Also the frequently used quantity $\rho_t = r_{0t} + 2\alpha_t^2 P_t r_{0t}^3 = 1.703\times10^{-13}$ cm. We get a very small value for $P_t$, because our nuclei potential does not have a long tail (the range $\beta_t^{-1}$ is small compared with that of usually accepted local Yukawa well).

Secondly, let us examine the singlet neutron-proton scattering. For simplicity, we want to assume $\beta_s = \beta_t$. Then the singlet scattering length[3] $a_s = -23.69\times10^{-13}$ cm is sufficient to fix the value of strength $\lambda_s$ of singlet potential. Using (20) we find the effective range $r_{0s}$ and the shape-dependent factor $P_s$,

$$r_{0s} = 2.151\times10^{-13}\ \text{cm} \tag{26}$$

and

$$P_s(r_{0s})^3 = -0.174\times10^{-39}\ \text{cm}^3 = -0.017(r_{0s})^3. \tag{27}$$

At present experimental information about the singlet neutron-proton system is rather poor, and the results (27) and (26), based on choosing $\beta_s = \beta_t$, are in agreement with existing knowledge[4] [see Table II]. If we want to have a different value for $r_{0s}$, we may take a different value for $\beta_s$ [see Table II]. Figure 1 compares our theoretical total neutron-proton cross section with experimental results.[4]

We have not yet examined proton-proton scattering, because the combination of the local Coulomb potential with our nonlocal interaction presents a complicated problem.

## 4. PHOTODISINTEGRATION OF THE DEUTERON

We must consider the problem of gauge invariance when we take into account the electromagnetic field in our two-nucleon system, because we have assumed a nonlocal and therefore a velocity-dependent nuclear potential. That is to say, we must replace our nuclear potential (3) by a modified potential which is compatible with the gauge transformation. The most reasonable and simplest choice will be given by

$$(-\lambda/M)v(\mathbf{r}_p-\mathbf{r}_n)v(\mathbf{r}_p'-\mathbf{r}_n') \times\exp\left\{ie\int_{\mathbf{R}}^{\mathbf{r}_p}\mathbf{A}\cdot d\mathbf{s}+ie\int_{\mathbf{r}_p'}^{\mathbf{R}}\mathbf{A}\cdot d\mathbf{s}\right\}, \tag{28}$$

where $\mathbf{r}_p$ and $\mathbf{r}_n$ are the coordinates of proton and neutron, respectively, $\mathbf{r}=\mathbf{r}_p-\mathbf{r}_n$, $\mathbf{r}'=\mathbf{r}_p'-\mathbf{r}_n'$, $\mathbf{R}=(\mathbf{r}_p+\mathbf{r}_n)/2=(\mathbf{r}_p'+\mathbf{r}_n')/2$, and $\mathbf{A}$ is the electromagnetic potential acting on the proton. The integral over $ds$ from $\mathbf{B}$ to $\mathbf{C}$ means the integral along the straight line from $\mathbf{B}$ to $\mathbf{C}$. Of course the most general choice should have a form $(28)\times F+G$, where $F$ and $G$ are gauge-invariant quantities which approach limits $F\rightarrow1$ and $G\rightarrow0$ as the electromagnetic field vanishes (also see Sachs[11] and Osborne and Foldy[11]). For the sake of simplicity, however, we confine our discussion to an interaction of the form (28). Furthermore we want to adopt the special potential which is "spherically symmetric" (i.e., acts in $S$ states only) when the electromagnetic potential vanishes. For this case, if we make a power series expansion in $\mathbf{A}$ and keep only zeroth- and first-order terms,

$$-\left\{\frac{\lambda_t}{M}v_t(r)v_t(r')\tfrac{1}{4}(3+\boldsymbol{\sigma}^p\cdot\boldsymbol{\sigma}^n)+\frac{\lambda_s}{M}v_s(r)v_s(r')\tfrac{1}{4}(1-\boldsymbol{\sigma}^p\cdot\boldsymbol{\sigma}^n)\right\} \times\left\{1+ie\int_0^1\tfrac{1}{2}\mathbf{r}\cdot\mathbf{A}(\mathbf{R}+\tfrac{1}{2}\mathbf{r}s)ds -ie\int_0^1\tfrac{1}{2}\mathbf{r}'\cdot\mathbf{A}(\mathbf{R}+\tfrac{1}{2}\mathbf{r}'(1-s))ds\right\}, \tag{29}$$

TABLE I. Parameters for the $n$-$p$ triplet interaction.

| | $r_t$ ($10^{-13}$ cm) | $P_t$ | $s_t$ | $b_t$ ($10^{-13}$ cm) |
|---|---|---|---|---|
| Square well[a] | 1.724 | −0.040 | 1.440 | 2.040 |
| Exponential well[a] | 1.687 | +0.029 | 1.416 | 2.346 |
| Yukawa well[a] | 1.637 | +0.137 | 1.419 | 2.913 |
| Nonlocal | | | | |
| $\beta_t = 1.4488\times10^{13}$ cm$^{-1}$ | 1.716 | −0.024 | 1.345 | 2.071 |

[a] G. Snow, Phys. Rev. **87**, 21 (1952).

TABLE II. Parameters for the $n$-$p$ singlet interaction.

| | $r_s$ ($10^{-13}$ cm) | $P_s$ | $s_s$ | $b_s$ ($10^{-13}$ cm) |
|---|---|---|---|---|
| Square well[a] | 2.47 ±0.20 | −0.03 | 0.926 ±6 | 2.58 ±6 |
| Exponential well[a] | 2.30 ±0.21 | +0.010 | 0.937 ±6 | 2.51 ±6 |
| Yukawa well[a] | 2.03 ±0.23 | +0.055 | 0.953 ±6 | 2.47 ±6 |
| Nonlocal $\beta_s/\alpha_t$ | | | | |
| 6.2547 | 2.151 | −0.01748 | 0.94494 | 2.071 |
| 6.25 | 2.153 | −0.01747 | 0.94490 | 2.072 |
| 6.00 | 2.246 | −0.01744 | 0.9427 | 2.159 |
| 5.75 | 2.348 | −0.01739 | 0.9404 | 2.252 |
| 5.50 | 2.459 | −0.01734 | 0.9379 | 2.345 |
| 5.25 | 2.581 | −0.01729 | 0.9351 | 2.467 |
| 5.00 | 2.716 | −0.01723 | 0.9321 | 2.590 |

[a] E. M. Hafner *et al.*, Phys. Rev. **89**, 204 (1953). Also see reference 1.

[11] R. G. Sachs, Phys. Rev. **74**, 433 (1948); R. K. Osborne and L. L. Foldy, Phys. Rev. **79**, 795 (1948).

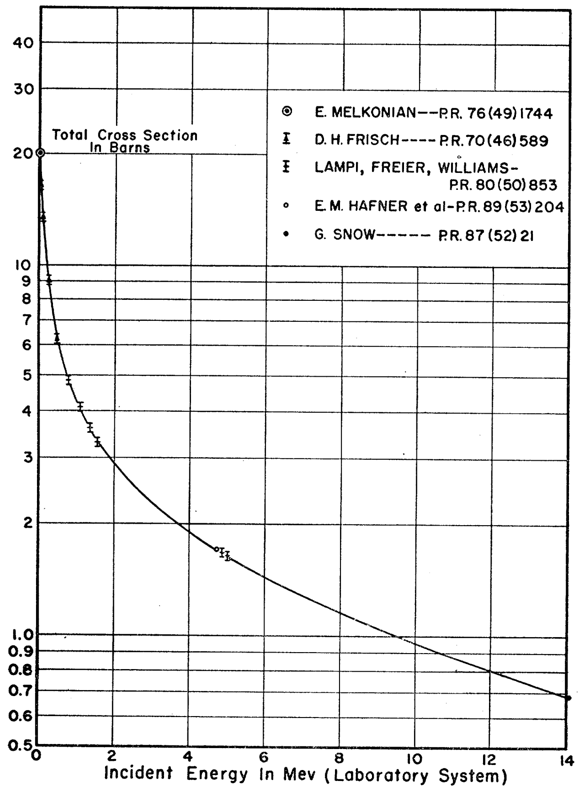


FIG. 1. Calculated $n$-$p$ total cross section (triplet $S$ plus singlet $S$ only) is compared with observation. Here the $S$ phase shifts are evaluated on the basis of following parameters (see Eqs. (24)–(27)):

| | triplet | singlet |
|---|---|---|
| scattering length (in $10^{-13}$ cm) | 5.378 | −23.69 |
| effective range (in $10^{-13}$ cm) | 1.716 | 2.151 |
| shape-dependent parameter | −0.024 | −0.017 |

where $\boldsymbol{\sigma}_p$ and $\boldsymbol{\sigma}_n$ are the spins of proton and neutron, respectively; $-(\lambda_t/M)v_t(r)v_t(r')$ and $-(\lambda_s/M)v_s(r)v_s(r')$ are the nonlocal potentials for triplet and singlet states, respectively. This interaction then contributes to the radiative processes in neutron-proton system such as the photodisintegration of the deuteron,[12] aside from the usual electromagnetic interactions,

$$-(e/M)\mathbf{A}\cdot\mathbf{p}_p, \tag{30}$$

$$-[\mu_p\boldsymbol{\sigma}^p\cdot\operatorname{curl}\mathbf{A}(\mathbf{r}_p)+\mu_n\boldsymbol{\sigma}^n\cdot\operatorname{curl}\mathbf{A}(\mathbf{r}_n)]\mu_0,$$

and

$$\mu_p=2.793,\quad \mu_n=-1.913,\quad \mu_0=e/2M. \tag{31}$$

Since we assume no interactions except in $^3S$ and $^1S$ states and we know the correct $S$ wave functions, we can easily calculate the photodisintegration cross section of the deuteron including all multipoles,

$$d\sigma/d\omega=d\sigma_e/d\omega+d\sigma_m/d\omega. \tag{32}$$

[12] The singlet part of (29) does not contribute to the deuteron photoeffect.

Here $d\sigma_e/d\omega$ is the spin-independent part [due to the interactions (29), (30)],

$$\frac{d\sigma_e}{d\omega}=\pi^2\frac{e^2}{4\pi}\frac{p_f^3}{M\nu}N^2\left[\left\{\frac{g_t(p)}{\alpha^2+p^2}\right\}_{p=|\mathbf{p}_f-\frac{1}{2}\mathbf{v}|}-\int_0^1 ds\left\{\frac{\partial g_t(p)}{2p\partial p}\right\}_{p=|\mathbf{p}_f-(s/2)\mathbf{v}|}\right]^2\sin^2\theta, \tag{32e}$$

and

$$g_t(p)=\frac{1}{(2\pi)^{\frac{3}{2}}}\int v_t(r)e^{-i\mathbf{p}\cdot\mathbf{r}}d\mathbf{r},\ N^2,\ \lambda_t\text{: see Eqs. (6), (7).}$$

$[g_t(p)/(\alpha^2+p^2)]$ comes from the usual interaction $\mathbf{A}\mathbf{p}_p$, (30), and $(1/2p)\partial g/\partial p$ comes from the modified nuclear interaction (29); while the spin-dependent cross section due to the magnetic interaction (31), which does not interfere with the spin-independent transition, is

$$\frac{d\sigma_m}{d\omega}=\frac{\pi^2}{6}\cdot\frac{e^2}{4\pi}\cdot\frac{p_f\nu}{M}\cdot N^2\{2|T|^2+|S|^2\}. \tag{32m}$$

In these expressions $\mathbf{p}_f(p_f=|\mathbf{p}_f|)$ is the final proton momentum, $\mathbf{v}(\nu=|\mathbf{v}|)$ is the incident photon momentum, $\theta=(\mathbf{p}_f\mathbf{v})/(p_f\nu)$; thus the energy balance is

$$\nu+\nu^2/4M=p_f^2/M+\alpha^2/M;$$

$\alpha^2/M$ is, as before, the binding energy of deuteron. All quantities are measured in the center-of-mass system. Note that $e^2/4\pi=1/137$. The quantities $T$ and $S$ are given by ($T$: $^3S\to$triplet, $S$: $^3S\to$singlet)

$$T=\mu_p\frac{g_t(p_-)}{\alpha^2+p_-^2}+\mu_n\frac{g_t(p_+)}{\alpha^2+p_+^2}-\frac{(\mu_p+\mu_n)\lambda_t g_t(p_f)}{1+\lambda_t J_t(p_f)}\times\int\frac{g_t(q)q_t(|\mathbf{q}-\frac{1}{2}\mathbf{v}|)}{(p_f^2-q^2+i\epsilon)(\alpha^2+|\mathbf{q}-\frac{1}{2}\mathbf{v}|^2)}d\mathbf{q}, \tag{32'}$$

$$S=\mu_p\frac{g_t(p_-)}{\alpha^2+p_-^2}-\mu_n\frac{g_t(p_+)}{\alpha^2+p_+^2}-\frac{(\mu_p-\mu_n)\lambda_s g_s(p_f)}{1+\lambda_s J_s(p_f)}\times\int\frac{g_s(\mathbf{q})g_t(|\mathbf{q}-\frac{1}{2}\mathbf{v}|)}{(p_f^2-q^2+i\epsilon)(|\mathbf{q}-\frac{1}{2}\mathbf{v}|^2+\alpha^2)}d\mathbf{q};$$

$$p_\pm=|\mathbf{p}_f\pm\tfrac{1}{2}\mathbf{v}|,$$

$$J_j(p_f)=\int d\mathbf{q}\frac{g_j^2(q)}{p_f^2-q^2+i\epsilon}\quad(j=t\text{ or }s).$$

If we are interested in the case where the incident photon energy is sufficiently small, (32) will reduce to the dipole cross sections

$$\frac{d\sigma_e}{d\omega}\to\frac{d\sigma(\text{E.D.})}{d\omega}=\frac{e^2}{4\pi}\frac{p_f^3}{\alpha^2+p_f^2}\pi^2N^2\sin^2\theta\times\left\{\frac{g_t(p_f)}{\alpha^2+p_f^2}-\frac{1}{2p_f}\frac{\partial g_t(p_f)}{\partial p_f}\right\}^2 \tag{33 E.D.}$$

and

$$\frac{d\sigma_m}{d\omega}\rightarrow\frac{d\sigma(\text{M.D.})}{d\omega}=\frac{e^2}{4\pi}\cdot\frac{p_f}{M^2(\alpha^2+p_f^2)}(\mu_p-\mu_n)^2\frac{N^2}{24\pi^2}\times\left\{\sin\delta_s\frac{\lambda_t-\lambda_s}{\lambda_t\lambda_s}\cdot\frac{1}{g(p_f)}\right\}^2; \quad (33\text{ M.D.})$$

where $\delta_s$ = singlet phase shift (note that $T=0$). In (33 M.D.) we have assumed the same shape for the triplet and singlet nuclear forces:

$$g_t(p)=g_s(p)=g(p), \quad (34)$$

and used Eq. (7) for $\lambda_t$.

### Special Case

As in the preceding section, let us assume

$g_t(p)=1/(\beta_t^2+p^2)$ for the triplet potential,

and

$g_s(p)=1/(\beta_s^2+p^2)$ for the singlet potential,

where $\beta_t$ and $\beta_s$ are constants which determine the ranges of nuclear potential. Then (32e) takes the form:

$$\frac{d\sigma_e}{d\omega}=\frac{e^2}{4\pi}\frac{\alpha p_f^3}{M\nu}\beta_t(\alpha+\beta_t)^3\sin^2\theta\left[\frac{1}{(\alpha^2+p_-^2)(\beta^2+p_-^2)}-\left\{\frac{\partial}{2q\partial q}\left(\frac{1}{\frac{1}{2}\nu(\beta_t^2+q^2)^{\frac{1}{2}}}\times\tan^{-1}\frac{\frac{1}{2}\nu(\beta_t^2+q^2)^{\frac{1}{2}}}{\beta_t^2+q^2-\{\frac{1}{2}\nu^2-(\mathbf{p}_f\mathbf{\nu})\}(\mathbf{p}_f\mathbf{\nu})\nu^{-2}}\right)\right\}_{q^2=p_f^2-\nu^{-2}(\mathbf{p}_f\mathbf{\nu})^2}\right]^2. \quad (35e)$$

For $d\sigma_m/d\omega$, we find

$$T=\frac{\mu_p}{(\alpha^2+p_-^2)(\beta_t^2+p_-^2)}+\frac{\mu_n}{(\alpha^2+p_+^2)(\beta_t^2+p_+^2)}+\frac{(\mu_p+\mu_n)2f_t/\nu}{\beta_t^2-\alpha^2}\left[\tan^{-1}\left\{\frac{(\beta_t-\alpha)\frac{1}{2}\nu}{(\beta_t-ip_f)(\alpha-ip_f)+\frac{1}{4}\nu^2}\right\}-\tan^{-1}\left\{\frac{(\beta_t-\alpha)\frac{1}{2}\nu}{2\beta_t(\alpha+\beta_t)+\frac{1}{4}\nu^2}\right\}\right] \quad (35')$$

$$S=\frac{\mu_p}{(\alpha^2+p_-^2)(\beta_t^2+p_-^2)}-\frac{\mu_n}{(\alpha^2+p_+^2)(\beta_t^2+p_+^2)}+\frac{(\mu_p-\mu_n)2f_s/\nu}{\beta_t^2-\alpha^2}\left[\tan^{-1}\left\{\frac{(\beta_t-\alpha)\frac{1}{2}\nu}{(\beta_t-ip_f)(\alpha-ip_f)+\frac{1}{4}\nu^2}\right\}-\tan^{-1}\left\{\frac{(\beta_t-\alpha)\frac{1}{2}\nu}{(\beta_t+\beta_s)(\alpha+\beta_s)+\frac{1}{4}\nu^2}\right\}\right];$$

where as before

$$p_\pm=|\mathbf{p}_f\pm\tfrac{1}{2}\mathbf{\nu}|,$$
$$f_j=1/(-ip_f+p_f\cot\delta_j)\quad (j=t\text{ or }s),$$
$$\delta_t=\text{triplet phase shift},$$

and

$$\delta_s=\text{singlet phase shift}.$$

Here we have *not* assumed $g_t(p)=g_s(p)$, i.e., $\beta_t=\beta_s$.

At very low photon energy, we can approximate (35) by the dipole cross sections

$$\frac{d\sigma(\text{E.D.})}{d\omega}=\frac{e^2}{4\pi}\frac{\alpha p_f^3}{(\alpha^2+p_f^2)^3}\cdot\frac{\beta_t(\alpha+\beta_t)}{(\beta_t-\alpha)^2}\times\left\{1-\left(\frac{\alpha^2+p_f^2}{\beta^2+p_f^2}\right)^2\right\}^2\sin^2\theta \quad (36\text{ E.D.})$$

and

$$\frac{d\sigma(\text{M.D.})}{d\omega}=\frac{1}{6}\frac{e^2}{4\pi}\frac{\alpha^2+p_f^2}{M^2}\cdot p_f(\mu_p-\mu_n)^2\alpha\beta_t(\alpha+\beta_t)^2\frac{\sin^2\delta_s}{p_f^2}\times\left[\frac{p_f\cot\delta_s}{(\alpha^2+p_f^2)(\beta_t^2+p_f^2)}+\frac{1}{\beta_t+\alpha}\left\{\frac{\alpha\beta_t-p_f^2}{(\alpha^2+p_f^2)(\beta^2+p_f^2)}-\frac{1}{(\alpha+\beta_s)(\beta_t+\beta_s)}\right\}\right]^2. \quad (36\text{ M.D.})$$

In the special case where $\beta_t=\beta_s=\beta$, we get, using Eq. (15),

$$\frac{d\sigma(\text{M.D.})}{d\omega}=\frac{1}{24}\frac{e^2}{4\pi}\left(\frac{\mu_p-\mu_n}{M}\right)^2\sin^2\delta_s\frac{\alpha p_f}{\beta(\alpha+\beta)}\times\left(\frac{\lambda_t-\lambda_s}{\lambda_s}\right)^2\frac{\alpha^2+p_f^2}{p_f^2}\left(\frac{\beta^2+p_f^2}{\alpha^2+p_f^2}\right)^2.$$

Here we also put $M\nu$ equal to $\alpha^2+p_f^2$.

As is well-known, the electric dipole matrix element is always given by ($\boldsymbol{\varepsilon}$ is the polarization vector of the electromagnetic field)

$$\frac{i}{2}e\nu\int(^3P\text{ function})^*\boldsymbol{\varepsilon}\cdot\mathbf{r}(\text{deuteron function}).$$

Therefore, if we assume (i) a form of deuteron function, $Ng_t(p)/(\alpha^2+p^2)$, and (ii) no forces in $^3P$ states, the dipole cross section is just given by (33 E.D.), which is, in this sense, independent on a specific assumption of nonlocal nuclear potential. This fact holds for all electric multipole transitions (see an article of Sachs and Austern[13]) (for low photon energy).

[13] A. J. F. Siegert, Phys. Rev. **52**, 787 (1937); R. G. Sachs and N. Austern, Phys. Rev. **81**, 705 (1951).

Equation (36 E.D.) has very familiar form and in fact agrees with that reported by other authors[14] on the basis of the local Hulthén potential. In the magnetic dipole cross section (36 M.D.), we use familiar forms of wave functions both for the deuteron and the $^1S$ state, and so obtain a well-known result.

TABLE III. The thermal-neutron capture cross sections $\sigma_{\text{cap}}$ by hydrogen, calculated from the formula (33 M.D.) multiplied by $(3/2)\nu^2/p_f^2$. For $\alpha_t$ and $\beta_t$ we used Eqs. (22) and (23), while we assumed several values for $\beta_s$ but fixed the value of the singlet scattering length as $-23.69\times10^{-13}$ cm.

| $\beta_s/\alpha_t$ | $\sigma_{\text{cap}}v_n$ ($10^{-20}$ cm$^3$/sec) | $\sigma_{\text{cap}}$ (barns) |
|---|---|---|
| 6.2547 | 7.069 | 0.32130 |
| 6.25 | 7.068 | 0.32126 |
| 6.00 | 7.019 | 0.3191 |
| 5.75 | 6.967 | 0.3167 |
| 5.50 | 6.909 | 0.3141 |
| 5.25 | 6.847 | 0.3112 |
| 5.00 | 6.778 | 0.3081 |

[14] H. A. Bethe and C. Longmire, Phys. Rev. **77**, 647 (1950); J. S. Levinger, Phys. Rev. **76**, 699 (1949); L. I. Schiff, Phys. Rev. **78**, 733 (1950); J. F. Marshall and E. Guth, Phys. Rev. **78**, 738 (1950); E. E. Salpeter, Phys. Rev. **82**, 60 (1951); J. G. Brennan and R. G. Sachs, Phys. Rev. **88**, 824 (1952); L. Hulthén and B. C. H. Nagel, Phys. Rev. **90**, 62 (1953).

Therefore, the content of Eqs. (35) and (36) is no greater than that of equations reached by other authors except that (35) includes *all multipoles*. For example, our cross sections cannot explain: (i) The experimental result (0.330 barn) for thermal neutron capture by hydrogen.[15] [In this case we must modify (36 M.D.) but in an obvious way (see Table III).] (ii) The large cross section at photon energy $\gtrsim$100 Mev. We do not repeat here, therefore, a detailed discussion based on the cross sections which we have derived.

In conclusion, we should like to emphasize again the usefulness of the wave functions examined in this paper. As was illustrated in Section 4 for the photodisintegration of deuteron, our wave functions are very convenient for the examination of various phenomena involving the two-nucleon system.

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[15] N. Austern, Phys. Rev. **92**, 670 (1953); Hamermesh, Ringo, and Wexler, Phys. Rev. **90**, 603 (1953); Harris, Muehlhause, Rose, Schroeder, Thomas, and Wexler, Phys. Rev. **91**, 125 (1953); G. von Dardel and A. W. Waltner, Phys. Rev. **91**, 1284 (1953).