Phase-Shift Analysis of the Scattering of Negative Pions on Hydrogen*

RONALD L. MARTIN
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York
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A phase-shift analysis of the scattering of negative pions on hydrogen in the energy range 120 to 217 Mev is presented. The usual assumptions that the scattering is charge-independent and that scattering in S and P states only is important are made and in addition the P-wave phase shifts of isotopic spin $\frac{1}{2}$ assumed to be zero. Of many approximate solutions obtained a solution showing a resonance in the P state of ordinary spin $\frac{3}{2}$ and isotopic spin $\frac{3}{2}$ in the neighborhood of 195 Mev is considered to have the highest probability of being correct.

I. INTRODUCTION

WITH the assumption that only S and P states are important in pion-proton scattering in the energy range analyzed, 120–217 Mev, and that the interaction is charge independent, the reactions

$$\pi^+ + p \rightarrow \pi^+ + p, \quad (1)$$

$$\pi^- + p \rightarrow \pi^- + p, \quad (2)$$

$$\pi^- + p \rightarrow \pi^0 + n, \quad (3)$$

can be described by a set of six phase shifts[1] α_3, α_1, α_{33}, α_{31}, α_{13}, and α_{11}. Reaction (1) is then described by the three phase shifts of isotopic spin $\frac{3}{2}$ while reactions (2) and (3) require all six of the phase shifts to describe the interaction.

The angular distributions of reactions (2) and (3) have been investigated[2–4] at several energies in the energy region being considered. The angular distribution of reaction (1) has been investigated[2,5,6] at several energies in this interval but presents some difficulties in interpretation under the above assumptions. In particular, the distribution measured by Homa, Goldhaber, and Lederman[5] at 188 Mev requires D-wave scattering for the best fit and if this is substantiated by further investigation would render the phase-shift analysis presented here invalid. However, within the errors of the experiment the distribution can be fitted with S- and P-wave scattering.

Recently Ashkin and Vosko[7] have presented a geometrical method for obtaining the six phase shifts from the experimental angular distribution of the 3 reactions. Using the scattering data of negative pions alone, however, only 5 of the 6 independent parameters are related to the geometrical plot so that the six phase shifts are not directly determined by this method. If one assumes that the two P-wave phase shifts of isotopic spin $\frac{1}{2}$, α_{13} and α_{11}, are zero one can use the geometrical diagrams to solve for the remaining four phase shifts and this procedure was followed in the present analysis.

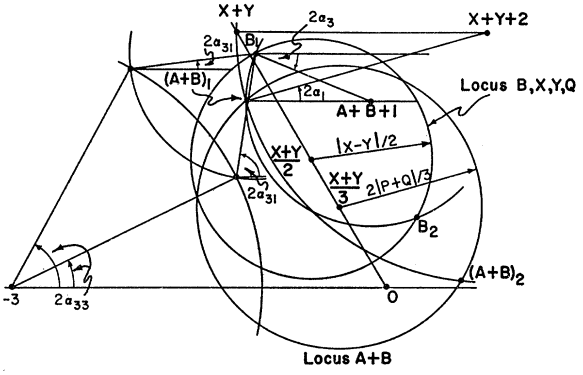

FIG. 1. Phase shift solution at 125 Mev by the goemetrical method of Ashkin and Vosko.

II. EXPLANATION OF METHOD

Following Ashkin and Vosko, but assuming $\alpha_{11}=\alpha_{13}=0$, leads to the following scattering formulas:[8]

$$\frac{d\sigma(+)}{d\Omega}=\frac{1}{4k^2}[|A+B\cos\theta|^2+|C\sin\theta|^2]$$
$$=a_+ + b_+\cos\theta + c_+\cos^2\theta, \quad (4)$$

$$\frac{d\sigma(-)}{d\Omega}=\frac{1}{36k^2}[|X+Y\cos\theta|^2+|Z\sin\theta|^2]$$
$$=a_- + b_-\cos\theta + c_-\cos^2\theta, \quad (5)$$

$$\frac{d\sigma(0)}{d\Omega}=\frac{1}{18k^2}[|P+Q\cos\theta|^2+|R\sin\theta|^2]$$
$$=a_0 + b_0\cos\theta + c_0\cos^2\theta, \quad (6)$$

* Assisted by the joint program of the U. S. Office of Naval Research and the U. S. Atomic Energy Commission.

[1] The notation is that of reference 2.

[2] Anderson, Fermi, Martin, and Nagle, Phys. Rev. **91**, 155 (1953).

[3] Fermi, Glicksman, Martin, and Nagle, Phys. Rev. **92**, 161 (1953).

[4] M. Glicksman, Phys. Rev. **94**, 1335 (1954).

[5] Homa, Goldhaber, and Lederman, Phys. Rev. **93**, 554 (1954).

[6] R. A. Grandey and A. F. Clark, Phys. Rev. **94**, 766 (1954).

[7] J. Ashkin and S. H. Vosko, Phys. Rev. **91**, 1248 (1953).

[8] Our notation differs from that used in the preceding paper by de Hoffmann *et al.* as follows: Our lower case letters, in equations (4), (5), and (6), are related to the notation of de Hoffmann *et al.* by $a_+ \rightarrow \lambda^2 A_+$, etc. Our upper case letters in (7) through (13) are written as lower case letters by de Hoffmann *et al.*

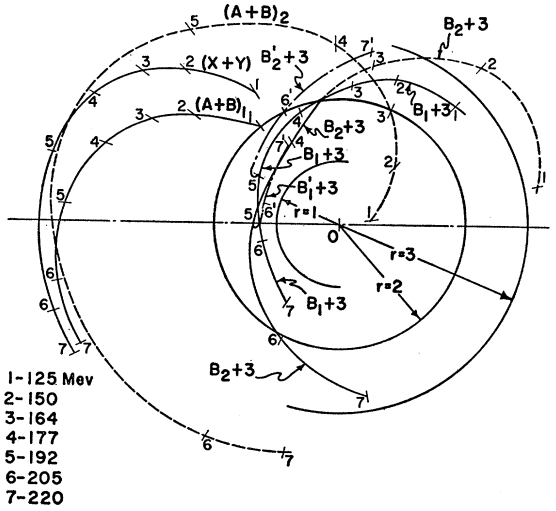

FIG. 2. Pertinent points of the geometrical solutions at different energies for an interpolation of the negative pion-proton scattering data.

where k is the wave number in the c.m. system.

$$A=\exp(2i\alpha_3)-1, \tag{7}$$

$$B=2\exp(2i\alpha_{33})+\exp(2i\alpha_{31})-3, \tag{8}$$

$$C=\exp(2i\alpha_{33})-\exp(2i\alpha_{31}), \tag{9}$$

$$X=A+2\exp(2i\alpha_1)-2, \tag{10}$$

$$Y=Q=B, \tag{11}$$

$$Z=R=C, \tag{12}$$

$$P=3A/2-X/2, \tag{13}$$

$$|X+Y|^2=36k^2(a_-+b_-+c_-), \tag{14}$$

$$|X-Y|^2=36k^2(a_--b_-+c_-), \tag{15}$$

$$\mathrm{Re}(X+Y)=-6k^2(a_-+\tfrac{1}{3}c_-+a_0+\tfrac{1}{3}c_0), \tag{16}$$

$$|P+Q|^2=18k^2(a_0+b_0+c_0), \tag{17}$$

$$|P-Q|^2=18k^2(a_0-b_0+c_0), \tag{18}$$

$$(A+B)=2(P+Q)/3+(X+Y)/3, \tag{19}$$

$$(A+B)=(X+Y)+2-2\exp(2i\alpha_1), \tag{20}$$

$$\mathrm{Re}(A+B)=-2k^2\sigma_+(\mathrm{total})/4\pi. \tag{21}$$

Figure 1 shows a representative solution by the method applied. The point $X+Y$ is located from the experimental data through (14) and (16). A circle with center at $(X+Y)/2$ and radius $|X-Y|/2$, given by (15), is the locus of both points X and Y on opposite sides of a diagonal. This same circle is also the locus of points B and Q from (11) (this is true only for $\alpha_{11}=\alpha_{13}=0$). Two circles are obtained from (19) and (20) as the locus of point $(A+B)$, namely a circle of radius 2 from center $X+Y+2$ and a circle of radius $2|P+Q|/3$, given by (17), from center $(X+Y)/3$. Two points $A+B$ are obtained by the intersections of these circles and each defines a value of α_1 through (20). One of these points $(A+B)$ can always be excluded as will be seen later. A is a vector from the origin to a circle of radius 1 and center -1 as given by (7) so that a second locus of point B is obtained from $B=(A+B)-A$ as a circle of radius 1 with center at point $A+B+1$. Therefore two points B are located, each defining an angle α_3 as shown. Angles α_{33} and α_{31} are determined from (8) as shown and exist as two sets for each point B. The set of α_{33} and α_{31} with the larger value of α_{33} is commonly called the Fermi type solution and the other the Yang type solution. At a given energy then one finds two values of α_3 with the same value of α_1, and for each value of α_3 a Fermi and a Yang set of α_{33} and α_{31}, a total of four sets of phase shifts.

III. PHASE-SHIFT SOLUTIONS

In order to determine the continuity of solutions a smooth interpolation of the experimental cross sections of the scattering of negative pions on hydrogen in the energy interval 120–217 Mev was chosen as the input data for the graphical solutions. The pertinent points of the Ashkin diagrams at different energies are shown in Fig. 2. In locating these points many ambiguities exist to give rise to numerous solutions and each must be considered. One could as well have chosen the initial point $X+Y$ in the opposite imaginary plane. This would have the effect of reflecting all points on Fig. 2 through the real axis which would correspond to a change of sign of all phase shifts. However, at 125 Mev the point $X+Y$ was chosen in the positive imaginary plane to make the value of α_3 negative[9] at this energy, so that the signs of the phase shifts are then determined from continuity. With increasing energy the point $X+Y$ moves toward the real axis and may or may not cross the real axis, as shown by Fig. 3, so that both possibilities must be considered.

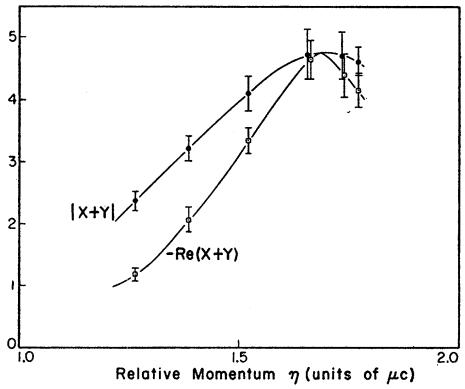

FIG. 3. Graph indicating possibility that $X+Y$ becomes real.

[9] Bodansky, Sachs, and Steinberger, Phys. Rev. **90**, 997 (1953); Phys. Rev. **93**, 918 (1954); J. Orear, Phys. Rev. **93**, 918 (1954). Barnes, Angell, Perry, Miller, Ring, and Nelson, Phys. Rev. **92**, 3128 (1953).

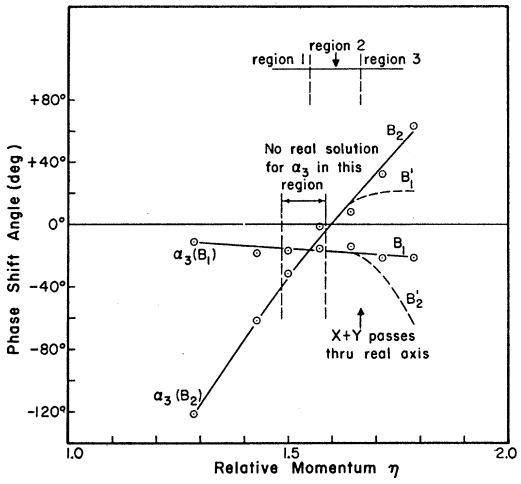

FIG. 4. Possible behavior of α_3.

The point labeled $(A+B)_2$ can be rejected at each energy by one or more arguments. At low energies the point is rejected by its representation of a negative total cross section through (21). In addition, at all energies except those represented by points 5 and 6 it does not give rise to acceptable solutions in that the circle for the locus of B derived from $(A+B)_2$ does not intersect the other circle of locus B. Both arguments are valid at 125 Mev as shown in Fig. 1. The point $(A+B)_2$ does give acceptable solutions at energies represented by points 5 and 6 but these are rejected on the basis of continuity and the points B derived from this $A+B$ are not included on Fig. 2. The acceptable point $(A+B)_1$ follows closely the point $X+Y$ and so crosses or does not cross the real axis with $X+Y$.

As pointed out by Ashkin and Vosko[7] some points A (here the corresponding point B) do not give rise to acceptable solutions because the point $B+3$ is more than 3 units from the origin, violating (8). This is the

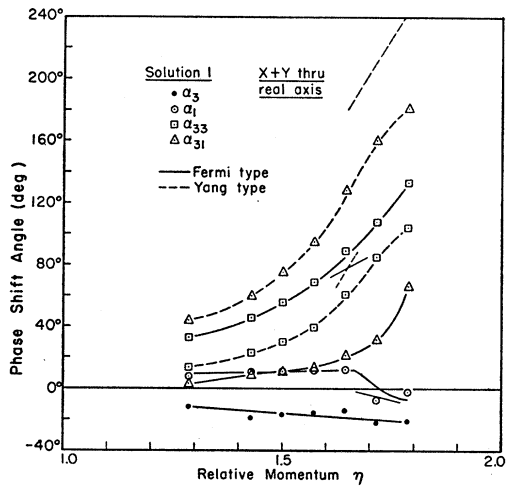

FIG. 5. Phase-shift solution 1.

case for point B_2, as shown in Fig. 2. at the lowest two energies analyzed. Hence, later solutions arising from point B_2 must be rejected on the basis of continuity until that energy is reached at which the two points B coincide. For the two points B to coincide requires the two circles representing the locus of B to become exactly tangent, and they must always intersect since a true solution must exist. With the input data used these two circles did not quite intersect at 164 Mev nor at 177 Mev, which is not surprising in view of the relatively large experimental errors and the assumption that $\alpha_{11}=\alpha_{13}=0$. The fact that the circles do not intersect at these two energies is regarded as evidence that the points B do coincide in the true solution. Perhaps stronger evidence for this conclusion is provided by the resultant solutions for α_3 as shown by the solid lines of Fig. 4. It is noted that if one allows point $X+Y$ to pass through the real axis, which is highly probable, then one can draw nearly straight lines to connect each set of

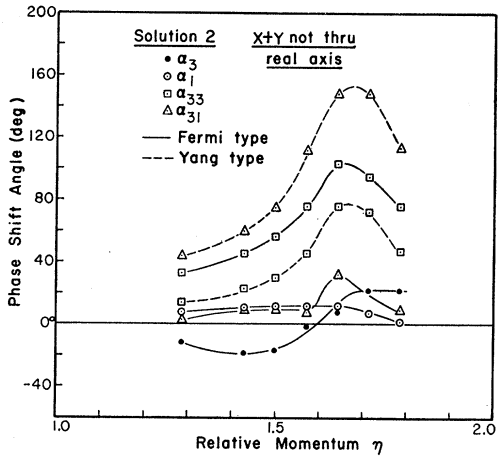

FIG. 6. Phase-shift solution 2.

values of α_3 at high and low energies, indicating a very regular behavior of the α_3 which is the only solution at lower energies. To find the solutions at 164 and 177 Mev, points B_1 and B_2 were chosen to be equidistant from the point of closest approach of the two circles defining the locus of B and giving values of α_3 which best fit the solid lines of Fig. 4. These points B gave real solutions of α_{33} and α_{31}. The procedure is justified in that cross sections predicted by the phase shifts with these assumed values of α_3 give reasonable agreement with the experimental data.

For the case that point $X+Y$ is not allowed to cross the real axis the points of Fig. 2 obtained at the last two energies should be reflected through the real axis. The continuity of points then requires that B_1 be continuous with the reflected points of B_2, which will be called B_2', and *vice versa*. The resultant continuous curves of $B+3$ are shown on Fig. 2. One can easily see by constructing α_{33} and α_{31} through (8) from points $B+3$ that

this reflection of points reflects the values of α_{33} and α_{31} through 90° so that what was the Fermi solution of one point becomes the Yang solution of the reflected point, and that this involves no loss of continuity. The labels 1 and 2 on points $A+B$ cannot be interchanged when these points are reflected through the real axis since only one of these points (here labeled $A+B_1$) is acceptable at both high and low energies and so must be continuous whether or not it crosses the real axis.

The phase shift solutions obtained are shown in Figs. 5 through 8. Each figure shows two solutions which are the Fermi and Yang type sets of α_{33} and α_{31} derived from the same point B and with the same values of α_3 and α_1. At the lowest energy investigated only one point B is acceptable so that all solutions arise continuously from the two solutions at 125 Mev. The two solutions at this energy are very close to those reported by Anderson.[2] The various solutions may be identified from Fig.

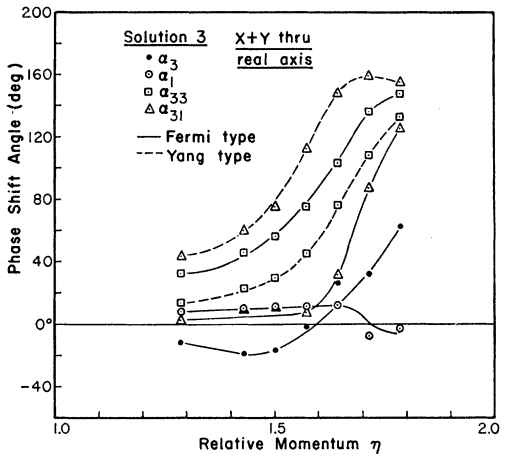

FIG. 7. Phase-shift solution 3.

2 by their relation to curves $B+3$, which is labeled in 3 intervals for this reason. All solutions arise from point B_1 from the lowest energy investigated to the point of intersection of B_1 and B_2, and from this point continue as follows: Solution 1 arises from B_1 and B_1, solution 2 from B_2 and B_1', solution 3 from B_2 and B_2, and solution 4 from B_1 and B_2'. Here the first letter indicates the curve followed in the energy interval from the energy at which points B intersect to the energy at which point $X+Y$ either crosses the real axis or turns back into the positive imaginary plane, and the second letter indicates the curve followed in the higher energy interval.

The Fermi-type solution 4 is similar in characteristics to a solution found by Fermi and Metropolis.[10] Although the phase shifts are considerably different in the neighborhood of 190 Mev, it will be seen later that the data allow large ranges of phase shifts in this energy region.

[10] Fermi, Metropolis, and Alei, this issue [Phys. Rev. **95**, 1581 (1954)].

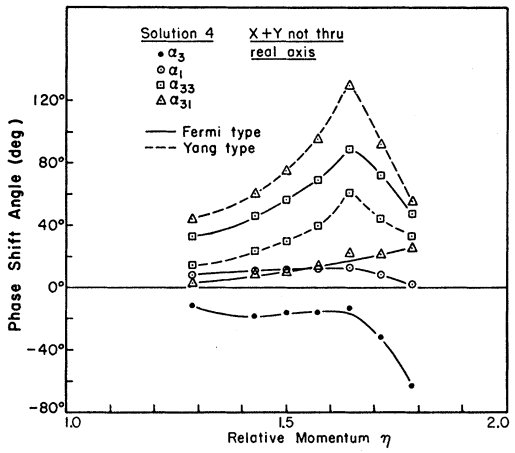

FIG. 8. Phase-shift solution 4.

The Yang-type solution 2 is similar to that reported by de Hoffmann *et al.*[11] In addition, by their tracking method, they obtained the solution corresponding to the Fermi-type solution 2, but chose to reverse signs at the higher energies and thereby obtained an entirely new solution.

The author was aware of this solution proposed by de Hoffmann *et al.*, and also independently by Glicksman[4] when it was realized that within the errors of the experimental data the points $B+3$ on Fig. 2 could be shifted to the extent that the line $B+3$ becomes tangent to the unit circle. When the point $B+3$ lies on the unit circle, the two values of α_{33} of the Fermi and Yang type solutions become equal and the two values of α_{31} differ by 180° (they must differ by this amount because of continuity although the vectors representing these phase shifts are equal). The construction shown in Fig. 9 indicates that it then becomes possible at higher energies to interchange the values of α_{33} between the Fermi and Yang type solutions in Figs. 5 through 8. This re-

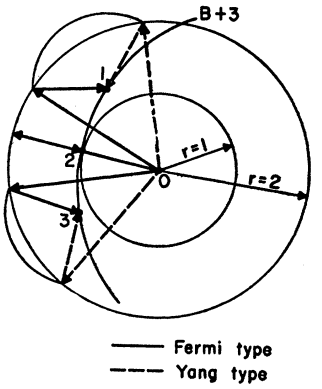

FIG. 9. Construction to show possibility of interchange of Fermi and Yang type solutions for α_{33}. Vectors with magnitude 2 have phase angles $2\alpha_{33}$ and vectors with magnitude 1 have phase angles $2\alpha_{31}$.

[11] See accompanying paper by de Hoffmann, Metropolis, Alei, and Bethe. Their track IIa, Table IX, corresponds to the Yang solution 2 presented here. Their solution 1 corresponds to the Fermi-type solution 2 presented here while this solution with signs reversed at 217 Mev is reported as Track I, Table VIII, and is the new solution spoken of above.

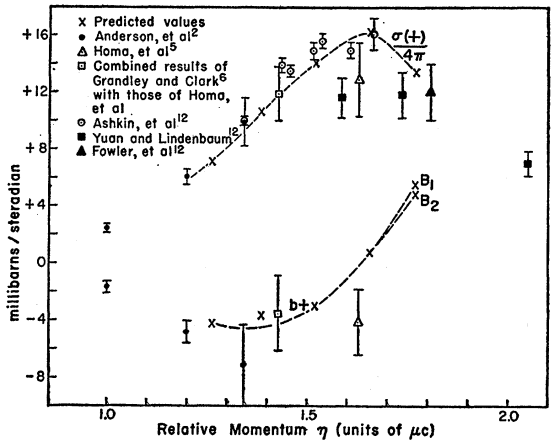

FIG. 10. Positive pion-proton scattering. Observed and predicted values of $\sigma(+)/4\pi$ and b.

quires also the interchange of the values of α_{31} with addition or subtraction of 180° to make this phase shift continuous. The interchange is indicated in Fig. 5 by the short lines and the resultant Fermi-type solution with α_{31} small is that solution found by de Hoffmann *et al.*, and by Glicksman. The interchange is possible in all of the other pairs of solutions but is not shown for the sake of clarity.

In order to check the accuracy of the method, solutions were also obtained using the experimental data[2–4] at 120, 144, 169, 194, and 217 Mev. These solutions are listed in Table I along with a criterion of fit to the data in the last column through the relation

$$M=\sum_{i=1}^{6}\left(\frac{\Delta_i}{\epsilon_i}\right)^2 \tag{22}$$

where Δ_i is the difference between the measured and predicted cross sections, ϵ_i is the error quoted for the measured cross sections, and the summation is over the cross-section measurements at 3 angles for reaction (2), and the γ cross-section measurements of reaction (3), also at 3 angles (except at 217 Mev where the cross sections of both reactions (2) and (3) were measured at 6 angles). With α_{11} and α_{13} equal to zero, the predicted angular distributions for reactions 1–3 of the corresponding Fermi and Yang type solutions are identical so that only one M value is listed in the last column for both sets. The solutions listed in this table are those corresponding to allowing point $X+Y$ of Fig. 2 to pass through the real axis (solutions 1 and 3) but the other two solutions may be obtained as before by suitable reversals of sign and addition of 180°. Here again, at 169 Mev, the two circles for locus of point B did not intersect so that the phase shifts listed at 169 Mev are those derived from assumed values of α_3 in the same manner as the solutions at 164 and 177 Mev from the interpolated data as previously discussed.

The phase-shift solutions given in Table I are not very different from the solutions obtained from the interpolated data, except at 194 Mev. The solution from the interpolated data at 194 Mev is also listed in Table I for comparison. It is noted that while the two solutions are reasonably different at 194 Mev they give about the same fit to the experimental data, and this energy region can be seen to be very critical in that a small change in data results in a relatively large change in phase shifts. This is mainly due to the relative insensitivity of the largest contributions to the predicted cross sections, such as $\sin^2\alpha_{33}$ and $\sin^2(\alpha_{33}-\alpha_3)$, when α_{33} is near 90°. It is also evident from the Ashkin diagrams, since point $X+Y$ is very close to the real axis at this energy so that any change of either $|X+Y|$ or $\mathrm{Re}(X+Y)$ makes a relatively large change in point $X+Y$.

Also at this energy the solutions obtained directly from the diagrams from point B_2 were very poor in that the phase shifts were not very continuous with those at lower and higher energy, gave a very poor fit to the experimental data, and predicted unreasonable structure in the parameters a, b, and c of reaction (1). They were therefore discarded and another point B_2 constructed on the diagrams (from points $A+B_1$ and B_1) which gave the same predictions for reaction (1) as did solutions from point B_1. These phase shifts were found to be much more acceptable, gave predictions in better agreement with the experimental data, and are those listed in Table I at 194 Mev.

Figures 10 and 11 show the values of a_+, b_+, c_+, and $\sigma(+)_{\text{total}}$ of reaction (1) predicted by the above phase shifts.

IV. DISCUSSION

The phase shifts presented here are only approximate solutions to the data due to graphical inaccuracies and the assumption that $\alpha_{11}=\alpha_{13}=0$, which latter implies the use of only four out of the six independent data. No attempt has been made to refine the values of the phase shifts obtained from the diagrams to give a better fit to the data, except as previously discussed at 194 Mev. To do this with any confidence requires a somewhat better knowledge of the angular distribution of reaction (1) than is presently available. While the actual

TABLE I. Approximate phase-shift solutions of experimental data.

E (Mev)	Rel. mom. η	Solution from point	α_3	α_1	Fermi solution α_{33}	Fermi solution α_{31}	Yang solution α_{33}	Yang solution α_{31}	M
120	1.26	B_1	−11.6	+8.0	30.3	4.3	13.6	39.7	5.1
144	1.39	B_1	−12.8	11.5	43.4	1.3	18.1	60.2	6.7
169	1.52	B_1	−16.0	11.4	59.3	11.5	31.7	79.3	6.4
194	1.66	B_1	−9.2	16.3	101.4	29.1	75.7	148.0	1.9
		B_1[b]	−16	10	92	23	66	135	2.3
		B_2	+14.6	16.3	119.2	50.6	91.0	159.6	3.9
217[a]	1.77	B_1	−19.8	−2.7	132.0	66.7	102.8	168.2	6.8
		B_2	+59.1	−2.7	145.6	123.2	131.2	153.6	12.4

[a] M values summed over 12 measurements.
[b] Solution from interpolated data.

values of the phase shifts presented here are not considered accurate it is felt that accurate solutions could be found with the characteristic behavior of all of the solutions 1–4.

It is possible that solutions to the negative pion scattering data alone could exist in which point $X+Y$ of Fig. 2 were on one side of the real axis and point $A+B$ on the other. Such solutions would give large values of one or more of α_1, α_{13}, and α_{11} and would not be detected in this analysis. If these solutions did exist they would be excluded by an examination of the total cross sections[12] which are related by

$$3\sigma(-)=\sigma(+)+8\pi\lambda^2(\sin^2\alpha_1+2\sin^2\alpha_{13}+\sin^2\alpha_{11}), \quad (23)$$

where $\sigma(-)$ represents the total π^- scattering cross section including charge exchange. The experimental values are such that these 3 phase shifts must be small in the energy region investigated.

An exact solution with α_{13} and α_{11} not equal to zero may cause an additional multiplicity of solutions because of the multiplicity involved in selecting α_{11} and α_{13}, but since the values of these two phase shifts are required to be small by (23) such additional solutions will not be very different from those shown. It can be seen that the discussion of Fig. 2 would be highly confused for an exact solution, including α_{11} and α_{13}, since in this case the Fermi and Yang type solutions would not give identical cross section predictions, would not be derived from the same point B, and would not have identical values of α_3 and α_1. However, a similar discussion of points of Fig. 2 obtained from positive pion scattering data would be valid and would lead to solutions with characteristics of all of the solutions presented here. One then would have the freedom of choosing values of α_1, α_{11}, and α_{13} for each of the solutions in order to best fit the negative pion scattering data. That a good fit could be obtained for each solution seems certainly possible in view of the results obtained here of almost identical fits without using any of the freedom of choice of these 3 phase shifts.

It is quite possible that solutions 3 and 4 may disappear at a slightly higher energy than the highest energy investigated. This would occur if point B_2+3 of Fig. 2 were to become more than three units from the origin, which violates (8). At the energy at which these solutions disappear α_{31} would be equal to α_{33} and the Fermi and Yang solutions would be identical. It is seen from Figs. 7 and 8 that these phase shifts appear to be converging, so that the disappearance of these 2 sets of solutions at higher energy is considered probable.

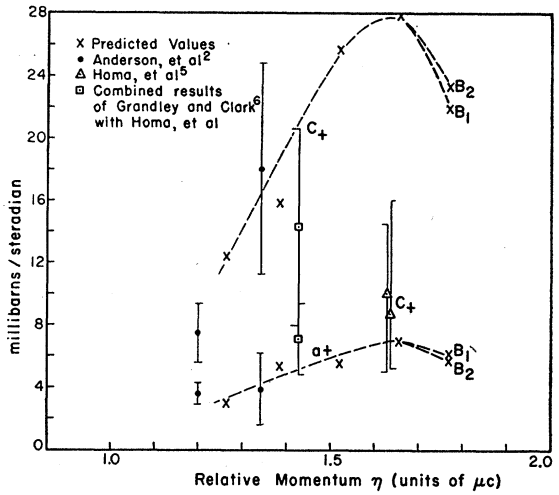

FIG. 11. Positive pion-proton scattering. Observed and predicted values of a and c.

It appears that it may be impossible to decide experimentally between the resonance solution 1 and the nonresonance solution 2 since cross sections predicted by these sets of solutions differ only in the narrow energy region of about 170–200 Mev, and here only by such small amounts as to be indetectable experimentally. In addition it is this energy region in which the errors on the phase shifts are largest for a given experimental error. The resonance solution 1 implies that the two points B coincide and that points $X+Y$ and $A+B$ each pass through the real axis, all of which appear probable from the existing data. It is, therefore, felt that the resonance solution is the correct choice since it would involve less abrupt changes of the phase shifts.

There are strong theoretical arguments[13] in favor of the resonance solution proposed by de Hoffmann *et al.*, and by Glicksman. In addition to the above conditions required by the resonance solution, their solution requires that, at some energy in the neighborhood of 190 Mev, point B_1+3 be exactly one unit from the origin. In view of the large uncertainty in the phase shifts at this energy, such a behavior of point B_1 is certainly allowed and this solution is probably the true solution.

I am indebted to Professor H. A. Bethe for many informative discussions on this subject, and I am grateful to Dr. F. de Hoffmann, Dr. N. Metropolis, Dr. E. F. Alei, and Dr. H. A. Bethe and to Dr. M. Glicksman for making their analysis of the phase shifts available to me before publication.

[12] References 2, 3, 4, 5, and 6; Fowler, Lea, Shephard, Shutt, Thorndike, and Whittemore, Phys. Rev. **92**, 832 (1953) (their pion energy is now stated as 225 Mev); Ashkin, Blaser, Fiener, Gorman, and Stern, post-deadline paper at the New York meeting of the American Physical Society, 1954; L. C. L. Yuan and S. J. Lindenbaum (private communication).

[13] de Hoffmann, Metropolis, Alei, and Bethe, preceding paper [Phys. Rev. **95**, 1586 (1954)].