tends to concentrate most of the available energy in the meson and leave the residual nucleus in a state of low excitation. Consider the final state of the nucleus (bound or unbound) after emitting the π but no nucleons. If the initial light nucleus had $T=0$ in its ground state ($N=Z$), the final nuclear state will have $T=\frac{1}{2}$, $\frac{3}{2}$ for π^+ emission, but $T=\frac{3}{2}$ only for π^- emission. For light, odd-A nuclei, however, the lowest $T=\frac{3}{2}$ state lies above the lowest $T=\frac{1}{2}$ state by $\Delta(\frac{3}{2},\frac{1}{2})\approx 10$ Mev.[9] Thus the low-lying residual states have a preponderance of $T=\frac{1}{2}$ levels, which will tend to inhibit π^- production and increase the value of ρ.

This mechanism of π^- inhibition should be most effective for production from deuterium: here $\Delta(\frac{3}{2},\frac{1}{2})$ presumably has a maximum value, and the center-of-gravity energy is the lowest for a given E_p. The observations agree with this expectation: ρ achieves its largest values for D; and, even more satisfactory, it increases rapidly with E_π for fixed E_p.[4]

The above argument for light nuclei is special to targets with $T=0$. For light, odd-A targets with $N=Z+1$, both π^+ and π^- emission lead to nuclear states with $T=1$, 2 that differ only by Coulomb energy. For a light nucleus the Coulomb difference should be small, so that ρ would approach the uninhibited value. This is in agreement with the observations on Be.

An Al target, on the other hand, shows a relatively high value of ρ. In this case the Coulomb energy difference between the final nuclear states is $\gtrsim 10$ Mev, even though isotopic spin appears not to have broken down seriously. This Coulomb energy difference is again in the direction to inhibit π^- production.

These arguments for light nuclei indicate that with protons on $N=Z+1$ target nuclei π^0 production should be considerably enhanced for $A=4n-1$ target but not for $A=4n+1$, because[9] the $\Delta(1, 0)$ of the final nuclei are of order 15 and 0 Mev, respectively.

For heavy nuclei isotopic spin is no longer a good quantum number, and a tendency to balance exists between the energies of nuclear symmetry and of Coulomb repulsion. The residual nuclei from π^+ and π^- emission have relatively small energy differences in their ground states—of order 4, 3, and 2 Mev for Cu, Ag, and Pb targets. Structural effects of heavy nuclei therefore provide relatively little π^- inhibition.

The inhibition of π^- production by protons on light nuclei may partly account for the observation that π^- production increases more steeply with target number A than π^+ production. Among light nuclei, π^- production should show strong isotope dependence (e.g., B^{10}, B^{11}); no such dependence is expected for π^+ production, and none has been found.[10]

[9] For notation see D. C. Peaslee, Phys. Rev. **95**, 717 (1954).

[10] J. Merritt, University of California Radiation Laboratory Report UCRL-2424, 1953 (unpublished).

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Phase Shift Analysis of the Scattering of Negative Pions by Hydrogen*

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A phase shift analysis of the scattering of negative pions by hydrogen in the range 115 to 215 Mev is presented. In the present paper two solutions are given that represent the data within the experimental error by two rather different sets of phase shifts. One of these solutions is excluded on the basis of some information on the scattering of positive pions. The other solution is compatible with all the experimental data known at present. It is pointed out, however, that this is not the only solution that has such properties. In addition, calculations carried out on scattering of positive and negative pions at 61.5 Mev are presented.

I. INTRODUCTION

This paper describes some systematic attempts to analyze in terms of phase shifts the experimental data on pion-hydrogen scattering.[1,2] Two essentially independent calculations are described. Part 1 is concerned with the analysis of the experimental results of B[1] on the scattering of negative pions in the energy range 115 to 215 Mev. These calculations that are now being published have been completed during the spring and summer of 1953 and have been circulated privately. In the intervening period there have been a number of other attempts at interpreting essentially the same data.[3]

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[1] Anderson, Fermi, Martin, and Nagle, Phys. Rev. **91**, 155 (1953), quoted as A; Fermi, Glicksman, Martin, and Nagle, Phys. Rev. **92**, 161 (1953), quoted as B.

[2] Bodansky, Sachs, and Steinberger, Phys. Rev. **90**, 997 (1953); **93**, 918, 1367 (1954).

[3] M. Glicksman, Phys. Rev. **94**, 1335 (1954); R. Martin, this issue [Phys. Rev. **95**, 1583 (1954)]; and in particular de Hoffmann, Metropolis, Alei, and Bethe, following paper [Phys. Rev. **95**, 1563 (1954)], where the calculations presented in this paper are extended and attempts are made to choose one out of several possible solutions.

We refer to Fig. 1 of B where the experimental results are summarized. From this figure one can see that the charge exchange scattering apparently goes through a maximum somewhat below 200 Mev, and also that at about the same energy the angular distribution of this process changes from being mostly backward to being mostly forward. It has been suggested[4] that some features of the pion-nucleon scattering and of the pion photoeffect may indicate the existence of a resonance level in a state of isotopic spin $\frac{3}{2}$ and ordinary spin $\frac{3}{2}$ at approximately this energy. The features of the negative pion scattering in the vicinity of 200 Mev might be due to the effect of this hypothetical resonance level. For this reason it appeared worthwhile to see whether a phase shift analysis of the data would support the hypothesis of the existence of such a level. The solutions obtained in the present paper do not show this resonance level; however, solutions that are compatible with the existence of such a resonance are quoted in reference 3.

In Sec. IV we describe the phase shift calculations carried out on the experimental data of Bodansky, Sachs, and Steinberger[2] at 61.5 Mev. These authors suggested that we analyze their data to confirm their earlier calculations and to search for other solutions, as well as to determine the sensitivity of the analysis to small deviations in the phase shifts. We are indebted to them for the preparation of the data and for several useful discussions. For these relatively low pion energies, it is necessary to include a term corresponding to the Coulomb scattering.

After this work was completed, a preprint of a paper by Homa, Goldhaber, and Lederman[5] was received. In this work some results on the scattering of positive pions by hydrogen at 151 and 188 Mev are given. Other results on total cross sections of positive pions have been reported by Ashkin, Blaser, Stern, Gorman, and Feiner.[6] The relationship of these results with the results of the present calculation will be discussed at the end of this paper.

There are two reasons why a phase shift analysis of the experimental data presented in B is not very reliable. One is that only experimental results on the scattering of negative pions were available when this calculation was undertaken. Therefore there is no check of a set of phase shifts on the joint behavior of positive and negative pions. The second is that with our present experimental information it is impossible to include in the phase shift analysis any contribution of the d levels. Neglecting the d-level phase shifts is probably allowable at low energies, but becomes less plausible at higher energies where the relative de Broglie wavelength of the pion-nucleon system becomes appreciably smaller than the Yukawa radius.

For these reasons, the phase shift analysis of the Chicago data here presented may be seriously in error. It should be noted that the ambiguity is somewhat reduced by the fact that some results on the positive pion cross section have now become available, but even when these data are included some considerable ambiguity remains in the interpretation. These ambiguities are discussed in the following paper[3] where an attempt is made to arrive at the choice between the various possibilities on the basis of certain theoretical hypotheses.

II. THE MATHEMATICAL PROBLEM

The general procedure followed in the computation of the phase shifts for the s- and p-waves has already been described in A.[1] We elaborate on that discussion and remark on two variants in the method; the simplification for the analysis of the Columbia data[2] at lower energy is also discussed. At a given energy the differential cross sections for all the scattering processes,

$$\pi^+ \to \pi^+, \quad \pi^- \to \pi^-, \quad \text{and} \quad \pi^- \to \pi^0 \to 2\gamma, \qquad (1)$$

are expressed in terms of the six phase shift angles of the s- and p-waves of isotopic spins $\frac{3}{2}$ and $\frac{1}{2}$. For these six angles, we use the same notation of A and we indicate them by α_3, α_1, α_{33}, α_{31}, α_{13}, α_{11}. In the phase shift calculations described in A, differential cross sections were available for the three processes (1), each measured at three different angles.

At a point in the six-dimensional space of the phase shifts, one can compute the above-mentioned nine cross sections using the formulas given in A (Sec. X). Then one can evaluate the function

$$M(\alpha_3, \alpha_1, \alpha_{33}, \alpha_{31}, \alpha_{13}, \alpha_{11}) = \sum_{i=1}^{9} (\Delta i/\epsilon_i)^2, \qquad (2)$$

where Δ_i is the difference between the experimentally measured and the computed cross section, and ϵ_i is the corresponding experimental error. The value of M is a measure of the approximation to the nine experimental data. The mathematical problem then is to explore the six-dimensional space for the minimum (or minima) of the function M.

With the use of the Los Alamos electronic computer, the MANIAC, two methods were used. In the first, one starts at some point and proceeds along the first coordinate axis, α_3, in steps of, say, $\frac{1}{2}°$ until a minimum is reached. Then one moves along the second axis, and so on, until no further improvement is found. The completed procedure is refined by using a smaller step of $\frac{1}{16}°$. In the second method, one first computes the gradient and then proceeds along that direction until a minimum is found. There the gradient is again computed and the process repeated until a (relative) minimum is found. In general, the second method is faster, although in certain instances this was not so, simply because the

[4] K. A. Brueckner, Phys. Rev. **86**, 106 (1952).
[5] Homa, Goldhaber, and Lederman, Phys. Rev. **93**, 554 (1954).
[6] Ashkin, Blaser, Stern, Gorman, and Feiner, quoted by Ashkin at the Fourth Annual Rochester Conference on High Energy Nuclear Physics, January, 1954 (University of Rochester Press, Rochester, to be published).

computation of the gradient is relatively time-consuming.

Under the assumptions stated in A, the differential cross sections may be written

$$d\sigma_n/d\omega = a_n + b_n \cos\chi + c_n \cos^2\chi, \qquad (3)$$

where n corresponds to the three processes (1), and χ is the scattering angle in the center-of-mass system. The coefficients a_n, b_n, c_n are the quantities that are first evaluated for a set of phase shifts. By means of Eq. (3) one finds then the cross section for the various values of χ. An alternate procedure, therefore, is to find a least-squares solution of the coefficients instead of the cross sections. The former may be regarded as the more fundamental quantities, but the experimental errors on them are larger.

For energies up to 135 Mev, experimental results were used for all three processes (1). However, for higher energies, only measurements for the scattering of negative pions were available. The differential cross sections for each of the two processes are measured at three angles. In this case, therefore, the six phase shift angles α_i are determined from six cross sections only, with no internal checking of the procedure being possible. However, for each solution, one can compute the total cross section for the scattering of positive pions and compare with the corresponding experimental value.

At each energy where a computation is made, it is found that, in general, more than one solution exists; i.e., several relative minima of Eq. (2) are reached. In order to correlate a solution at one energy with that at another, use may be made of a continuity principle. One may make a graphical interpolation of the experimental cross sections (or coefficients) and then find a set of solutions to the minimum problem at rather close intervals of energy. The starting point for the search of a minimum at each energy step may conveniently be taken as the solution in the preceding energy step. In this way one chooses a solution that is near the one obtained immediately before. This "tracking" procedure may enable one to select a set of solutions that

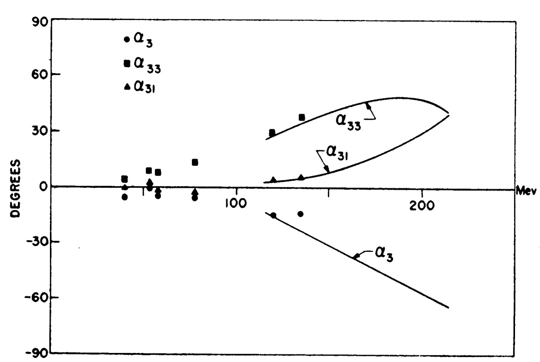

FIG. 1. Phase shifts of the states of isotopic spin $\frac{3}{2}$ plotted *versus* the energy of the primary pion for solution 1. For comparison, values of the same phase shifts at lower energies are also plotted.

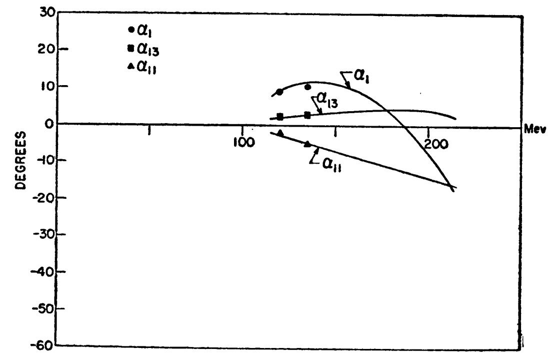

FIG. 2. Phase shifts of the states of isotopic spin $\frac{1}{2}$ plotted *versus* the energy of the primary pion for solution 1. Values previously obtained at 120 and 135 Mev are also plotted.

represents the data within the experimental errors and is consistent with the total positive pion cross sections. It has the advantage to average out in part the very sizable experimental errors.

III. CALCULATIONAL RESULTS FOR THE ENERGY RANGE 115 TO 215 MEV

Results on the phase shifts for the energies of 120 and 135 Mev were given in A. Both positive- and negative-pion measurements were used there. It was found that two sets of almost equivalent solutions, the "first solution" and the "Yang solution" existed.[7] A third minimum was found with a very large value of M, hence a very poor least-squares solution of the problem.

For the higher energies, where only negative-pion cross sections were used, two rather different solutions have been found that correspond to two somewhat different interpolations of the experimental results.

In both calculations we have chosen for the lowest energy a solution close to the "first solution" obtained in A. No attempt has been made in the work described in this paper to investigate the high-energy behavior of the Yang solution. This point is discussed, among others, in the next paper.[3]

The two sets of solutions obtained are presented in Tables I and II, and the first set is shown graphically in Figs. 1 and 2. The first of these solutions actually was obtained by following a somewhat mixed procedure. First, two sets were computed starting from two different graphic interpolations of the experimental data. The corresponding phase shifts were rather similar. They were combined and smoothed out in order to obtain the phase shifts and it was verified that they represent the experimental cross sections within the experimental error.

The second set was obtained from a different interpolation of the experimental data. In this case an analytical interpolation formula was used. This calculation was carried out twice with different analytic interpola-

[7] E. Fermi and N. Metropolis, Los Alamos unclassified report LA-1492, 1952 (unpublished).

TABLE I. First solution.

Energy (Mev)	Phase shifts (degrees)						Computed cross sections in the c. m. system corresponding to the laboratory scattering angles 45°, 90°, 135° (10^{-27} cm²/sterad)									Total cross section of π^+ (10^{-27} cm²)
	α_3	α_1	α_{33}	α_{31}	α_{13}	α_{11}	$-\to-$			$-\to\gamma$			$+\to+$			
115	−13.4	8.2	26.0	2.7	1.6	−2.1	0.90	0.41	0.85	1.69	2.86	5.01	3.34	4.42	12.69	75
125	−18.4	10.3	30.6	3.7	2.0	−3.5	1.13	0.53	0.98	2.20	3.74	6.59	4.29	5.70	15.89	95
135	−23.5	11.3	34.9	5.0	2.4	−4.9	1.35	0.65	1.15	2.79	4.50	7.88	5.57	6.84	18.59	115
145	−28.5	11.3	38.9	6.7	2.8	−6.3	1.52	0.76	1.32	3.44	5.03	8.70	7.10	7.67	20.46	133
155	−33.6	10.3	42.6	9.2	3.2	−7.7	1.66	0.85	1.53	4.17	5.35	9.13	8.84	8.16	21.73	149
165	−38.6	8.2	45.8	12.3	3.6	−9.1	1.78	0.92	1.76	4.96	5.48	9.16	10.72	8.36	22.38	163
175	−43.7	5.1	48.3	16.0	4.0	−10.5	1.86	1.00	2.01	5.70	5.36	8.73	12.45	8.16	22.23	173
185	−48.7	0.9	49.8	20.8	4.2	−12.0	1.94	1.09	2.30	6.47	5.12	8.16	14.10	7.72	21.92	181
195	−53.8	−4.2	49.4	26.5	4.2	−13.4	2.05	1.22	4.68	7.02	4.68	7.37	15.19	7.01	21.16	184
205	−58.8	−10.5	46.9	32.6	3.6	−14.8	2.23	1.43	2.74	7.31	4.11	6.45	15.65	6.25	19.81	181
215	−63.9	−17.7	41.0	39.8	2.0	−16.2	2.53	1.70	2.66	7.20	3.46	5.44	15.21	5.68	17.60	170

tion and gave in both cases rather similar results. Only one of them is reported here in detail.

IV. ANALYSIS OF COLUMBIA EXPERIMENT AT 61.5 MEV

Bodansky, Sachs, and Steinberger[2] provided us with the experimental measurements given in Table III. The data represent extrapolations of their measured values to 61.5 Mev from 58 Mev for the positive pions and from 65 Mev for the negative pions. This extrapolation was done in order to have a complete set for both positive and negative pions at a single energy. Thus a least-squares procedure was used for fourteen cross sections. At this relatively low energy it is possible to use somewhat simpler expressions for the cross sections obtained by expanding the exponentials of A, formula (10), and keeping first-order terms. This expansion is permissible because all phase shifts should be small at this energy. It is necessary, however, to include a term for the Coulomb scattering, that is important at low energy.

Two computations were made starting at two points corresponding to approximate solutions of the problem previously obtained by Bodansky, Sachs, and Steinberger. In addition to these, twenty-two computations were made starting at randomly selected points. Seven different solutions were found. These are given in Table IV. The first two are similar to those found by Bodansky, Sachs, and Steinberger.

It is known from purely algebraic considerations that the multiplicity of solutions may be large. With our assumption of small phase shifts, whenever the p-phase shifts for one of the two isotopic spin states, say $T=\frac{1}{2}$, are approximately equal, there exists a twofold multiplicity of p-phase shifts for $T=\frac{3}{2}$.[8] These latter are related as follows:

$$2\alpha_{33}+\alpha_{31}=2\alpha_{33}'+\alpha_{31}',\qquad \alpha_{33}-\alpha_{31}=\alpha_{31}'-\alpha_{33}', \tag{4}$$

corresponding to the two multiple sets. These conditions hold approximately for minima 1,2; and 6,7 of Table IV for $T=\frac{3}{2}$; and for 3,4 for $T=\frac{1}{2}$. Minimum 5 shows no

TABLE II. Second solution.

Energy (Mev)	Phase shifts (degrees)						Computed cross sections in the c. m. system corresponding to the laboratory scattering angles 45°, 90°, 135° (10^{-27} cm²/sterad)									Total cross section of π^+ (10^{-27} cm²)
	α_3	α_1	α_{33}	α_{31}	α_{13}	α_{11}	$-\to-$			$-\to\gamma$			$+\to+$			
115	−10.9	8.5	23.2	9.8	3.0	−8.2	0.99	0.50	0.73	1.39	2.49	5.03	2.69	2.82	12.26	64
125	−14.0	9.7	33.1	5.1	1.3	−2.8	1.21	0.47	1.04	2.47	3.76	6.76	5.19	5.36	16.62	102
135	−16.5	10.5	39.7	3.8	1.0	−0.4	1.42	0.50	1.32	3.38	4.66	7.86	7.08	6.88	19.29	126
143	−18.5	10.8	44.3	2.8	1.1	0.9	1.56	0.55	1.52	3.99	5.14	8.33	8.40	7.80	20.53	141
155	−17.9	10.7	51.8	1.9	1.5	1.4	1.74	0.66	1.78	4.77	5.47	8.70	10.32	8.53	21.61	159
167	−8.9	10.0	61.6	0.3	1.6	1.4	1.91	0.81	2.01	5.46	5.47	8.50	12.26	8.78	22.00	173
173	−3.3	9.4	66.6	−0.2	1.8	1.1	1.98	0.89	2.11	5.73	5.34	8.27	13.09	8.76	21.81	177
184	6.9	7.3	74.8	3.2	3.3	−3.4	2.11	1.06	2.29	6.22	5.01	7.74	14.60	8.42	21.25	184
188	9.4	6.6	69.6	10.6	4.0	−14.9	2.16	1.13	2.36	6.38	4.86	7.50	13.48	6.87	19.93	174
192	11.0	6.4	61.7	15.1	3.1	−25.5	2.21	1.20	2.43	6.57	4.80	7.27	13.47	5.03	17.66	156
196	12.7	5.9	51.3	22.2	1.3	−35.9	2.24	1.26	2.47	6.67	4.48	6.94	11.87	2.56	14.60	128
200	14.4	5.3	46.9	25.6	0.0	−40.5	2.28	1.34	2.53	6.81	4.26	6.62	11.37	1.62	12.87	117
208	18.1	3.7	40.3	31.5	−3.1	−46.7	2.35	1.48	2.63	7.05	3.78	5.93	10.89	0.57	10.03	103
216	20.7	0.3	35.4	38.6	−5.5	−49.7	2.42	1.63	2.73	7.27	3.26	5.17	10.74	0.28	8.49	97

[8] This is exactly the multiplicity of p-phase shifts found in an analysis of p-He^4 elastic scattering angular distribution by C. L. Critchfield and D. C. Dodder, Phys. Rev. **76**, 602 (1949).

multiplicity because for both isotopic spins, the *p*-wave shifts are neither small nor equal.

Finally, we took minimum 1 of Table IV and varied separately each of the phase shifts in turn until the value of M of Eq. (2) increased from its minimum value of 86 to 128. The results are summarized in Table V. This gives a measure of the sensitivity of the solution to small changes in the phase shifts. Both positive and negative changes are shown.

TABLE III. Extrapolated data of Columbia experiments.

$\pi^+ \longrightarrow$	π^+
θ	$d\sigma/d\omega$
36°	0.27±0.12
47	0.53±0.09
64	0.74±0.07
101	1.39±0.08
129	2.36±0.12
155	3.13±0.17
$\pi^- \longrightarrow$	π^-
θ	$d\sigma/d\omega$
42°	0.83±0.12
53	0.47±0.07
70	0.28±0.05
101	0.21±0.05
151	−0.01±0.10
$\pi^- \longrightarrow$	2γ
θ	$d\sigma/d\omega$
50°	0.69±0.05
97	1.76±0.07
150	2.88±0.16

V. CONCLUSIONS

We first observed that solution 2 is almost certainly to be discarded. This was not evident when the solution was first obtained in the summer of 1953 but appears now well established because it seems to be incompatible with the data on positive-pion cross sections that have since become available.[5,6,9] This, of course, does not prove that solution 1 is correct, because there are other solutions that have been found subsequently[3] that are about equally compatible with the experimental data. About solution 1, one may remark that it does not show a resonance of the state of isotopic spin $\frac{3}{2}$ and angular momentum $\frac{3}{2}$. This can be seen because the corresponding phase shift α_{33} does not reach 90° but starts decreasing after going through a maximum of approximately 50°. It has already been pointed out that other solutions quoted in reference 3 show the resonance feature. Otherwise they are about equivalent to solution 1 in accuracy in which they represent the experimental data, so that a decision between them on this basis does not appear possible at the present time. A decision can be obtained only by introducing theoretical arguments as has been done, for example, in the following paper. At the present status of the theory, such arguments must of necessity be somewhat hypothetical.

[9] Fowler, Lea, Shephard, Shutt, Thorndike, and Whittemore, Phys. Rev. **92**, 832 (1953).

TABLE IV. Analysis of Columbia data.

	α_3	α_1	α_{33}	α_{31}	α_{13}	α_{11}	Relative minimum
1	−5.4	9.4	8.3	−1.8	−2.0	0.1	86
2	−5.4	9.4	1.6	11.6	−0.7	−2.7	86
3	13.0	−3.6	−2.7	−1.1	6.3	−1.2	143
4	13.0	−3.6	−1.6	−3.1	1.3	9.0	143
5	−11.2	0.9	4.0	0.4	−8.3	0.5	250
6	8.5	−2.5	−0.7	−10.3	5.9	5.2	324
7	8.5	−2.5	−6.9	2.7	5.4	5.8	324

TABLE V. Change in phase angles separately to produce a given change in the minimum function M.

	$+\delta$	$-\delta$
α_3	1.1°	−1.4°
α_1	0.9	−1.0
α_{33}	0.6	−0.8
α_{31}	0.9	−1.1
α_{13}	0.9	−0.8
α_{11}	1.7	−1.7