and

$$\Gamma(m,m)=\int \mathbf{\Gamma}(t,t')\exp[im(t-t')]dtdt'.$$

By using (A20), the same limit becomes

$$\mathop{\mathcal{L}}_{\substack{t_1\to+\infty\\ t_2\to-\infty}} \exp[im(t_1-t_2)]A_k=gZ_2{}^2\Gamma(m,m)(-\omega)^{-1}(2\omega\Omega)^{-\frac{1}{2}}, \tag{A22}$$

which, together with (A21), gives

$$Z_2\Gamma(m,m)=g^{-1}(-\omega)(2\omega\Omega)^{\frac{1}{2}}\langle\mathbf{N}|\alpha_k(0)|\mathbf{P}\rangle. \tag{A23}$$

The left-hand side of (A23) is the ratio of the renormalization constants, Z_2/Z_1. The k dependence of the right-hand side can be eliminated by using the identity:

$$\langle\mathbf{N}|[\alpha_k,H]|\mathbf{P}\rangle=0.$$

By commuting α_k with the total Hamiltonian, we have

$$\langle\mathbf{N}|\alpha_k|\mathbf{P}\rangle=-g\omega^{-1}(2\omega\Omega)^{-\frac{1}{2}}\langle\mathbf{N}|\tau_-|\mathbf{P}\rangle. \tag{A24}$$

By comparing (A24) with (A23), we can express the coupling constant renormalization (A14) as

$$g_c/g=Z_2/Z_1=\langle\mathbf{N}|\tau_-|\mathbf{P}\rangle.$$

Thus g_c/g, if real, must be less than unity.

These proofs can be obviously generalized to other renormalizable field theoretical problems. In the charged scalar theory these identities can be applied to calculate formally the values of Z_1 and Z_2 by using both the weak-coupling and strong-coupling solutions.[10] They are:

(i) weak-coupling solution:

$$Z_2=1-g_c{}^2\sum(2\omega^3\Omega)^{-1}+\cdots,$$
$$Z_2/Z_1=1-g_c{}^2\sum(2\omega^3\Omega)^{-1}+\cdots;$$

(ii) strong-coupling solution:

$$Z_2=\tfrac{1}{2}\exp[-g_c{}^2\sum(2\omega^3\Omega)^{-1}]+\cdots,\quad Z_2/Z_1=\tfrac{1}{2}.$$

[10] G. Wentzel, Helv. Phys. Acta **13**, 269 (1940); **14**, 633 (1941); R. Serber and S. Dancoff, Phys. Rev. **63**, 143 (1943); S. Tomonaga, Progr. Theoret. Phys. (Japan) **1**, 109 (1946).

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Fierz-Pauli Theory of Particles of Spin 3/2

SURAJ N. GUPTA
Department of Physics, Purdue University, Lafayette, Indiana
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The Fierz-Pauli field corresponding to particles of spin 3/2 is quantized, and its interaction with the electromagnetic field is investigated. It is also shown how the elements of the S matrix for collision processes, involving photons and charged particles of spin 3/2, can be obtained in a simple way.

1. INTRODUCTION

A THEORY of particles of arbitrary spin was first developed by Dirac,[1] Fierz and Pauli,[2] and since then several other theories have also been proposed.[3] Such theories are of special interest at the present time, because a number of new particles have been observed in recent years, and some of them may have a spin higher than one. However, except in the case of the gravitational field,[4] the interaction of a quantized field of spin higher than one with other fields has never been investigated.

We shall, therefore, discuss in some detail the Fierz-Pauli theory of particles of spin 3/2. We shall first carry out the quantization of the Fierz-Pauli field, and consider its interaction with the electromagnetic field. It will then be shown that in the present case, too, the contribution of any collision process can be obtained from the S matrix by means of simple rules, which are similar to the Feynman-Dyson[5] rules of quantum electrodynamics.

2. FIELD EQUATION FOR PARTICLES OF SPIN 3/2

According to Fierz and Pauli,[2] a field corresponding to particles of spin 3/2 is described by the symmetrical spinors

$$a^{\dot\alpha}{}_{\beta\nu}=a^{\dot\alpha}{}_{\nu\beta}\quad\text{and}\quad b_\nu{}^{\dot\alpha\dot\beta}=b_\nu{}^{\dot\beta\dot\alpha}, \tag{1}$$

and the auxiliary spinors c_α and $d^{\dot\alpha}$. The Lagrangian density for the field is

$$\begin{aligned}L=&-(a^{*\nu}{}_{\dot\alpha\dot\beta}\,p^{\dot\beta\rho}\,a^{\dot\alpha}{}_{\nu\rho}+b^*{}_{\dot\nu}{}^{\alpha\beta}\,p_{\alpha\dot\rho}\,b_\beta{}^{\dot\nu\dot\rho})\\ &+\kappa(a^{*\nu}{}_{\dot\alpha\dot\beta}\,b_\nu{}^{\dot\alpha\dot\beta}+b^*{}_{\dot\nu}{}^{\alpha\beta}\,a^{\dot\nu}{}_{\alpha\beta})+(p_{\dot\nu}{}^{\beta}\,d^{*\alpha}\,a^{\dot\nu}{}_{\alpha\beta}\\ &+p_{\dot\beta}{}^{\nu}\,c^*{}_{\dot\alpha}\,b_\nu{}^{\dot\alpha\dot\beta}-a^{*\nu}{}_{\dot\alpha\dot\beta}\,p_\nu{}^{\dot\beta}\,d^{\dot\alpha}-b^*{}_{\dot\nu}{}^{\alpha\beta}\,p_\beta{}^{\dot\nu}\,c_\alpha)\\ &+3(d^{*\alpha}\,p_{\alpha\dot\beta}\,d^{\dot\beta}+c^*{}_{\dot\alpha}\,p^{\dot\alpha\beta}\,c_\beta)+6\kappa(d^{*\alpha}\,c_\alpha+c^*{}_{\dot\alpha}\,d^{\dot\alpha}),\end{aligned} \tag{2}$$

[1] P. A. M. Dirac, Proc. Roy. Soc. (London) **A155**, 447 (1936).

[2] M. Fierz and W. Pauli, Proc. Roy. Soc. (London) **A173**, 211 (1939).

[3] A specially interesting field equation for particles with two different mass states has recently been given by H. J. Bhabha, Phil. Mag. **43**, 33 (1952).

[4] S. N. Gupta, Proc. Phys. Soc. (London) **A65**, 161, 608 (1952).

[5] F. J. Dyson, Phys. Rev. **75**, 486, 1736 (1949).

$$\psi=\begin{pmatrix} a_{11}^{\dot{1}} \\ \sqrt{2}\, a_{12}^{\dot{1}} \\ a_{22}^{\dot{1}} \\ a_{11}^{\dot{2}} \\ \sqrt{2}\, a_{12}^{\dot{2}} \\ a_{22}^{\dot{2}} \\ 3\sqrt{2}\, d^{\dot{1}} \\ 3\sqrt{2}\, d^{\dot{2}} \\ b_1^{\dot{1}\dot{1}} \\ \sqrt{2}\, b_1^{\dot{1}\dot{2}} \\ b_1^{\dot{2}\dot{2}} \\ b_2^{\dot{1}\dot{1}} \\ \sqrt{2}\, b_2^{\dot{1}\dot{2}} \\ b_2^{\dot{2}\dot{2}} \\ 3\sqrt{2}\, c_1 \\ 3\sqrt{2}\, c_2 \end{pmatrix}$$

FIG. 1. Components of the matrix ψ.

whence we obtain the field equations:

$$\begin{aligned} -p^{\dot{\beta}\rho}\, a^{\dot{\alpha}}{}_{\nu\rho}-p^{\dot{\alpha}\rho}\, a^{\dot{\beta}}{}_{\nu\rho}-p_\nu{}^{\dot{\beta}}\, d^{\dot{\alpha}}-p_\nu{}^{\dot{\alpha}}\, d^{\dot{\beta}}+2\kappa\, b_\nu{}^{\dot{\alpha}\dot{\beta}}&=0,\\ -p_{\alpha\dot{\rho}}\, b_\beta{}^{\dot{\nu}\dot{\rho}}-p_{\beta\dot{\rho}}\, b_\alpha{}^{\dot{\nu}\dot{\rho}}-p_\beta{}^{\dot{\nu}}\, c_\alpha-p_\alpha{}^{\dot{\nu}}\, c_\beta+2\kappa\, a^{\dot{\nu}}{}_{\alpha\beta}&=0,\\ -p_{\dot{\nu}}{}^{\beta}\, a^{\dot{\nu}}{}_{\alpha\beta}+3p_{\alpha\dot{\beta}}\, d^{\dot{\beta}}+6\kappa\, c_\alpha&=0,\\ -p_{\dot{\beta}}{}^{\nu}\, b_\nu{}^{\dot{\alpha}\dot{\beta}}+3p^{\dot{\alpha}\beta}\, c_\beta+6\kappa\, d^{\dot{\alpha}}&=0. \end{aligned} \tag{3}$$

The field Eqs. (3) can easily be expressed in a form analogous to that of the Dirac equation.[6] For, using the relations

$$\begin{aligned} p_{1\dot{1}}=-p_{\dot{1}}{}^2=-p_1{}^{\dot{2}}=p^{\dot{2}2}&=-i\partial/\partial x_3+\partial/\partial x_4,\\ p_{1\dot{2}}=-p_{\dot{2}}{}^2=p_1{}^{\dot{1}}=-p^{\dot{1}2}&=-i\partial/\partial x_1+\partial/\partial x_2,\\ p_{2\dot{1}}=p_{\dot{1}}{}^1=-p_2{}^{\dot{2}}=-p^{\dot{2}1}&=-i\partial/\partial x_1-\partial/\partial x_2,\\ p_{2\dot{2}}=p_{\dot{2}}{}^1=p_2{}^{\dot{1}}=p^{\dot{1}1}&=i\partial/\partial x_3+\partial/\partial x_4, \end{aligned} \tag{4}$$

we can express (3) as a set of sixteen equations involving only the ordinary space and time derivatives of the field quantities. We can then write these sixteen equations as

$$\alpha_\mu(\partial\psi/\partial x_\mu)+\kappa\psi=0, \tag{5}$$

where ψ is the one-column matrix shown in Fig. 1, and the α_μ are given in Fig. 2.

By actual multiplication of the matrices α_μ, it can be verified that they satisfy the relation

$$\sum(\alpha_\mu\alpha_\nu-\delta_{\mu\nu})\alpha_\lambda\alpha_\rho=0, \tag{6}$$

where $\sum$ denotes a sum over all possible permutations of the indices μ, ν, λ, and ρ. Multiplying (5) by $\alpha_\rho\alpha_\lambda\alpha_\nu\partial^3/\partial x_\rho\partial x_\lambda\partial x_\nu$, we get

$$\alpha_\rho\alpha_\lambda\alpha_\nu\alpha_\mu(\partial^4\psi/\partial x_\rho\partial x_\lambda\partial x_\nu\partial x_\mu) +\kappa\alpha_\rho\alpha_\lambda\alpha_\nu(\partial^3\psi/\partial x_\rho\partial x_\lambda\partial x_\nu)=0 \tag{7}$$

which gives, on using (6),

$$\alpha_\nu\alpha_\mu(\partial^4\psi/\partial x_\lambda{}^2\partial x_\nu\partial x_\mu)+\kappa\alpha_\rho\alpha_\lambda\alpha_\nu(\partial^3\psi/\partial x_\rho\partial x_\lambda\partial x_\nu)=0. \tag{8}$$

Further, using field equation (5), we obtain from the above equation

$$\kappa^2(\partial^2\psi/\partial x_\lambda{}^2)-\kappa^4\psi=0$$

or

$$(\Box^2-\kappa^2)\psi=0, \tag{9}$$

which shows that ψ satisfies the second-order wave equation.

3. LAGRANGIAN FORMALISM

In order to derive field equation (5) from a Lagrangian density, it is necessary to define an adjoint of ψ in the usual way. For this, we note that the nonsingular Hermitian matrix η of Fig. 3 satisfies the relations

$$\eta^{-1}\alpha_i{}^*\eta=-\alpha_i,\quad \eta^{-1}\alpha_4{}^*\eta=\alpha_4, \tag{10}$$

where an asterisk denotes the Hermitian conjugate. Further, taking the Hermitian conjugate of (5), we get

$$-(\partial\psi^*/\partial x_4)\alpha_4{}^*+(\partial\psi^*/\partial x_i)\alpha_i{}^*+\kappa\psi^*=0, \tag{11}$$

which, on being multiplied by η, can be written as

$$-(\partial\psi^*/\partial x_4)\eta(\eta^{-1}\alpha_4{}^*\eta)+(\partial\psi^*/\partial x_i)\eta(\eta^{-1}\alpha_i{}^*\eta) +\kappa\psi^*\eta=0. \tag{12}$$

Hence, using (10), we have

$$(\partial\bar{\psi}/\partial x_\mu)\alpha_\mu-\kappa\bar{\psi}=0, \tag{13}$$

where the adjoint $\bar{\psi}$ denotes the quantity

$$\bar{\psi}=\psi^*\eta. \tag{14}$$

The field equations (5) and (13) can be obtained by means of the usual variational principle from the Lagrangian density

$$L=-c\hbar[\bar{\psi}\alpha_\mu(\partial\psi/\partial x_\mu)+\kappa\bar{\psi}\psi]. \tag{15}$$

[6] The above procedure for writing the Fierz-Pauli equation in the Dirac form is due to K. K. Gupta, Proc. Indian Acad. Sci. **A35**, 255 (1952). However, our choice of the components of ψ and the representation for the α's is slightly different.

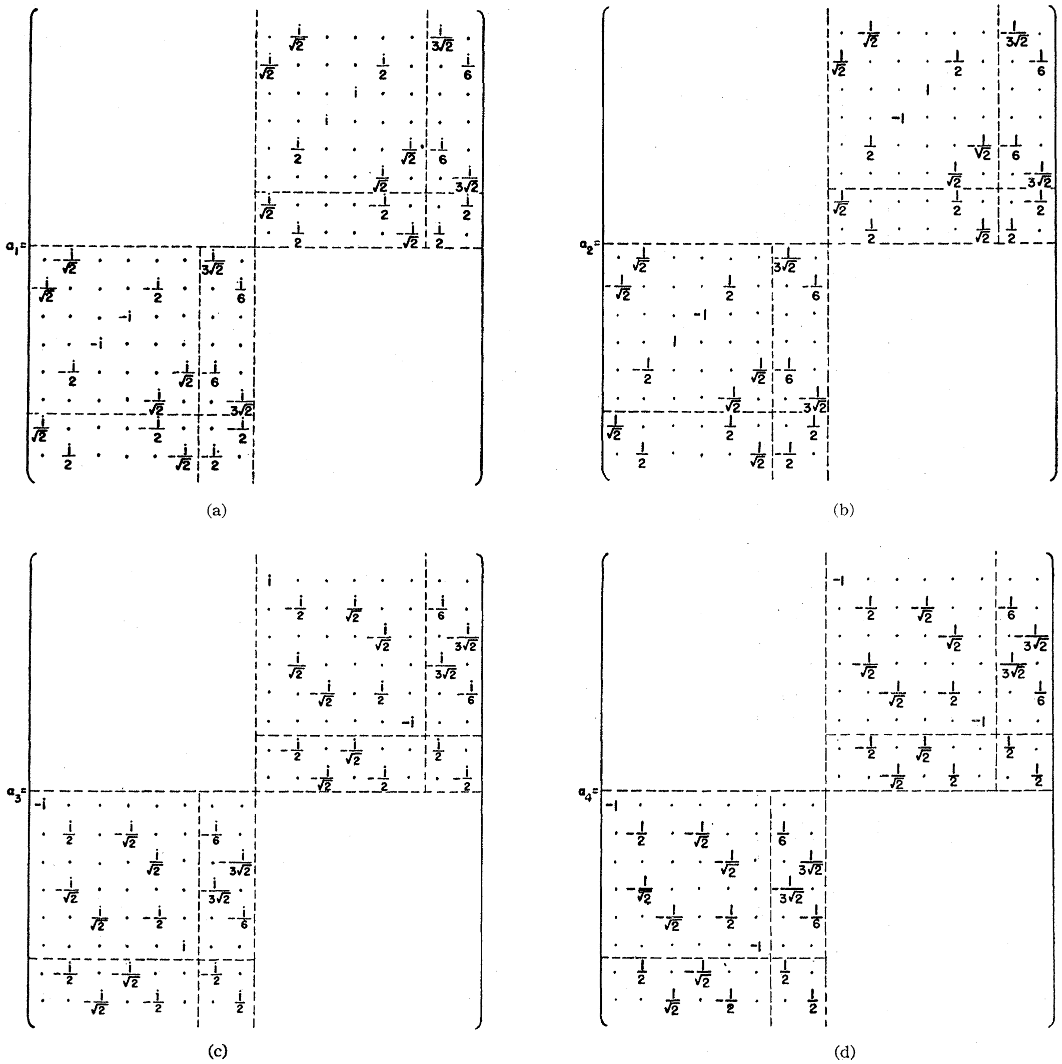

FIG. 2. The matrices α_μ. In these matrices a dot or a blank space indicates a zero or a block of zeros.

Although the field quantity ψ transforms in a rather complicated manner under a Lorentz transformation, we can prove the Lorentz invariance of the above Lagrangian density by showing that (15) is equal to the invariant quantity (2) apart from a constant factor. This also establishes the covariance of all the results obtained from the Lagrangian density (15).

From (15) we obtain in the usual way for the current density four-vector j_μ and the canonical energy-momentum tensor $S_{\mu\nu}$:

$$j_\mu = iec\bar{\psi}\alpha_\mu\psi, \tag{16}$$

$$S_{\mu\nu} = c\hbar\bar{\psi}\alpha_\nu(\partial\psi/\partial x_\mu). \tag{17}$$

According to (16) and (17), the charge density and the Hamiltonian density of the field are

$$\rho = j_4/ic = e\bar{\psi}\alpha_4\psi = e\psi^\dagger\psi, \tag{18}$$

$$H = -S_{44} = ic\hbar\bar{\psi}\alpha_4(\partial\psi/\partial x_0) = ic\hbar\psi^\dagger(\partial\psi/\partial x_0), \tag{19}$$

where

$$\psi^\dagger = \bar{\psi}\alpha_4 = \psi^*\eta\alpha_4. \tag{20}$$

4. SOLUTION OF THE FIELD EQUATION

Since ψ satisfies the wave equation (9), a solution of (5) must be of the form

$$\psi = \psi^+(\mathbf{k})\exp[i(\mathbf{k}\cdot\mathbf{x} - \omega x_0)]$$

or

$$\psi = \psi^-(\mathbf{k})\exp[-i(\mathbf{k}\cdot\mathbf{x} - \omega x_0)], \tag{21}$$

where $\mathbf{k}$ is an arbitrary real vector, and

$$\omega = (\mathbf{k}^2 + \kappa^2)^{\frac{1}{2}}. \tag{22}$$

For simplicity, let us first consider the solutions of (5) corresponding to $\mathbf{k} = 0$, which are of the form

$$\psi = \psi^+(0)\exp(-i\kappa x_0) \quad \text{or} \quad \psi = \psi^-(0)\exp(i\kappa x_0). \tag{23}$$

Substituting (23) in (5), we get

$$(\alpha_4 - 1)\psi^+(0) = 0 \quad \text{and} \quad (\alpha_4 + 1)\psi^-(0) = 0. \tag{24}$$

Further, denoting the sixteen components of the one-column matrix $\psi^{+}(0)$ as $c_1, c_2, \cdots, c_{16}$, and making use of the explicit representation for α_4, we can decompose the first relation in (24) into the following four sets involving sixteen algebraic equations:

$$c_1+c_9=0,\quad c_1+c_9=0; \tag{25}$$

$$c_6+c_{14}=0,\quad c_6+c_{14}=0; \tag{26}$$

$$\begin{aligned}
&\sqrt{2}c_3+c_{13}+\tfrac{1}{3}c_{16}=0,\\
&c_5-\tfrac{1}{3}c_8+\sqrt{2}c_{11}=0,\\
&2c_5+\sqrt{2}c_{11}+c_{13}-\tfrac{1}{3}c_{16}=0,\\
&2c_8+\sqrt{2}c_{11}-c_{13}-c_{16}=0,\\
&\sqrt{2}c_3+c_5+\tfrac{1}{3}c_8+2c_{13}=0,\\
&\sqrt{2}c_3-c_5+c_8-2c_{16}=0;
\end{aligned} \tag{27}$$

$$\begin{aligned}
&c_2+\tfrac{1}{3}c_7+\sqrt{2}c_{12}=0,\\
&\sqrt{2}c_4+c_{10}-\tfrac{1}{3}c_{15}=0,\\
&c_2+\sqrt{2}c_4-\tfrac{1}{3}c_7+2c_{10}=0,\\
&c_2-\sqrt{2}c_4+c_7-2c_{15}=0,\\
&2c_2+c_{10}+\sqrt{2}c_{12}+\tfrac{1}{3}c_{15}=0,\\
&2c_7+c_{10}-\sqrt{2}c_{12}-c_{15}=0.
\end{aligned} \tag{28}$$

In order to solve the above simultaneous equations we have to regard one of the c's in each of the four sets as an arbitrary quantity. Thus, taking c_1, c_3, c_6, and c_{12} as an arbitrary quantities, we easily find

$$c_1=\text{arbitrary},\quad c_9=-c_1; \tag{29}$$

$$c_6=\text{arbitrary},\quad c_{14}=-c_6; \tag{30}$$

$$\begin{aligned}
&c_3=\text{arbitrary}, && c_5=\sqrt{2}c_3,\\
&c_{11}=-c_3, && c_{13}=-\sqrt{2}c_3,\\
&c_8=0, && c_{16}=0;
\end{aligned} \tag{31}$$

$$\begin{aligned}
&c_{12}=\text{arbitrary}, && c_{10}=\sqrt{2}c_{12},\\
&c_4=-c_{12}, && c_2=-\sqrt{2}c_{12},\\
&c_{15}=0, && c_7=0.
\end{aligned} \tag{32}$$

Hence, there are four independent solutions of the first relation in (24), which are given in Fig. 4. Similarly, it can be shown that there are four independent solutions of the second relation in (24), which are given in Fig. 5.

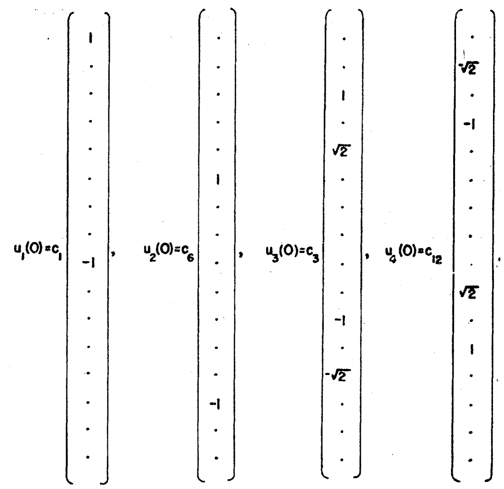

FIG. 4. The four independent solutions for $\psi^{+}(0)$.

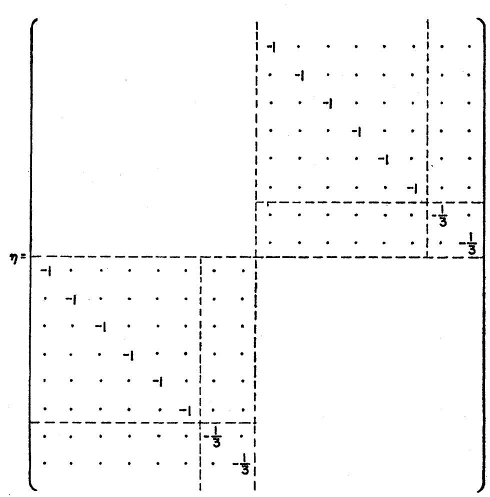

FIG. 3. The matrix η.

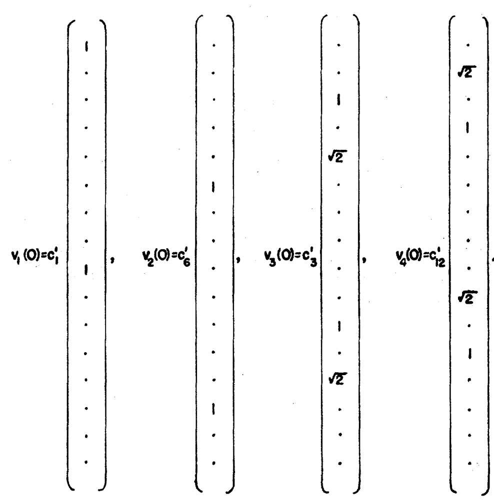

FIG. 5. The four independent solutions for $\psi^{-}(0)$.

We find that there are eight solutions of the field equation for $\mathbf{k}=0$, half of which have a positive frequency while the remaining ones have a negative frequency. But, any solution of the form (21) corresponding to an arbitrary value of $\mathbf{k}$ can be obtained from (23) by means of a suitable proper Lorentz transformation. Therefore, it follows that corresponding to any arbitrary value of $\mathbf{k}$, the field equation has eight solutions, four of which have a positive frequency while the remaining ones have a negative frequency.

It can further be easily verified that each of the solutions, given in Figs. 4 and 5, is an eigenvector of $\eta\alpha_4$ corresponding to the eigenvalue 1. Therefore, if $\psi(0)$ is any linear combination of solutions of the form (23), we have

$$\psi^*(0)\eta\alpha_4\psi(0)=\psi^*(0)\psi(0)=\text{positive definite}. \quad (33)$$

Thus, according to (18), the charge density due to the component $\psi(0)$ of the field is

$$\rho(0)=j_0(0)/c=e\psi^*(0)\eta\alpha_4\psi(0)=\text{positive definite}, \quad (34)$$

where we have taken e as a positive quantity. Moreover, there being no privileged direction for the vector $\mathbf{k}=0$, the current density vector due to the component $\psi(0)$ must vanish, i.e.,

$$j_i(0)=iec\psi^*(0)\eta\alpha_i\psi(0)=0. \quad (35)$$

Since $j_i(0)$ and $ic\rho(0)$ form components of a timelike four vector, it follows that under any proper Lorentz transformation the positive quantity $\rho(0)$ will transform again into a positive quantity, which shows that

$$\psi^*(\mathbf{k})\eta\alpha_4\psi(\mathbf{k})=\text{positive definite}, \quad (36)$$

where $\psi(\mathbf{k})$ denotes any linear combination of solutions of the form (21).

5. FOURIER EXPANSION OF THE FIELD VARIABLES

Let us consider the equation

$$A\psi=\lambda B\psi, \quad (37)$$

where A and B are Hermitian matrices, ψ is a one-column matrix, λ is a number, and $\psi^*B\psi>0$. We can regard (37) as a generalization of the usual eigenvalue equation. Thus, we can call λ an "eigenvalue" of the matrix A corresponding to the "eigenvector" ψ. It can then be easily shown that if $\psi_1, \psi_2, \cdots, \psi_n$ are n independent eigenvectors of A, we can choose these eigenvectors in such a way that they form an "orthonormal" set in the sense

$$\psi_m^*B\psi_n=\delta_{mn}, \quad (38)$$

and we further have

$$\sum_m \psi_{m,\gamma}^*B_{\gamma\alpha}\psi_{m,\beta}=\delta_{\alpha\beta}, \quad (39)$$

where $\psi_{m,\gamma}^*$, $B_{\gamma\alpha}$, and $\psi_{m,\beta}$ denote elements of the matrices ψ_m^*, B, and ψ_m, respectively, and the indices α, β, γ can take the values 1, 2, $\cdots$, n.

Substituting (21) in (5), we get

$$(ik_1\alpha_1+ik_2\alpha_2+ik_3\alpha_3-\omega\alpha_4+\kappa)\psi^+(\mathbf{k})=0, \quad (40)$$

$$(ik_1\alpha_1+ik_2\alpha_2+ik_3\alpha_3-\omega\alpha_4-\kappa)\psi^-(\mathbf{k})=0, \quad (41)$$

which give, on being multiplied by η,

$$(ik_1\eta\alpha_1+ik_2\eta\alpha_2+ik_3\eta\alpha_3+\kappa\eta)\psi^+(\mathbf{k})=\omega\eta\alpha_4\psi^+(\mathbf{k}), \quad (42)$$

$$(ik_1\eta\alpha_1+ik_2\eta\alpha_2+ik_3\eta\alpha_3+\kappa\eta)\psi^-(-\mathbf{k}) = -\omega\eta\alpha_4\psi^-(-\mathbf{k}). \quad (43)$$

The above equations are of the same form as (37), where the Hermitian operators $ik_1\eta\alpha_1+ik_2\eta\alpha_2+ik_3\eta\alpha_3+\kappa\eta$ and $\eta\alpha_4$ correspond to A and B, respectively, ω and $-\omega$ are the eigenvalues, and $\eta\alpha_4$ satisfies the condition (36). Hence, all possible solutions of (42) and (43) can be chosen in such a way that

$$u_r^\dagger(\mathbf{k})u_s(\mathbf{k})=v_r^\dagger(-\mathbf{k})v_s(-\mathbf{k})=\delta_{rs}, \quad u_r^\dagger(\mathbf{k})v_s(-\mathbf{k})=v_s^\dagger(-\mathbf{k})u_r(\mathbf{k})=0, \quad (44)$$

and

$$\sum_r[u_{r,\alpha}^\dagger(\mathbf{k})u_{r,\beta}(\mathbf{k})+v_{r,\alpha}^\dagger(-\mathbf{k})v_{r,\beta}(-\mathbf{k})]=\delta_{\alpha\beta}, \quad (45)$$

where the $u_r(\mathbf{k})$'s are the independent solutions for $\psi^+(\mathbf{k})$, the $v_r(\mathbf{k})$'s are the independent solutions for $\psi^-(\mathbf{k})$, and

$$u_r^\dagger(\mathbf{k})=u_r^*(\mathbf{k})\eta\alpha_4, \quad v_r^\dagger(\mathbf{k})=v_r^*(\mathbf{k})\eta\alpha_4. \quad (46)$$

Since we have shown in the preceding section that corresponding to any value of $\mathbf{k}$ there are four independent solutions for $\psi^+(\mathbf{k})$ as well as for $\psi^-(\mathbf{k})$, the indices r and s in (44) and (45) can take the values 1, 2, 3, 4. On the other hand, the indices α and β in (45) can take the values 1, 2, $\cdots$, 16.

We can now assume in the usual way that the field is enclosed in a large cubical box of volume V, and carry out a Fourier expansion of the field variable ψ as

$$\psi_\alpha=V^{-\frac{1}{2}}\sum_{\mathbf{k}}\sum_r[a_r(\mathbf{k})u_{r,\alpha}(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{x}-\omega x_0)} +b_r^*(\mathbf{k})v_{r,\alpha}(\mathbf{k})e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega x_0)}], \quad (47)$$

where $a_r(\mathbf{k})$ and $b_r^*(\mathbf{k})$ are arbitrary amplitudes. We then also have

$$\psi_\alpha^\dagger=V^{-\frac{1}{2}}\sum_{\mathbf{k}}\sum_r[a_r^*(\mathbf{k})u_{r,\alpha}^\dagger(\mathbf{k})e^{-(i\mathbf{k}\cdot\mathbf{x}-\omega x_0)} +b_r(\mathbf{k})v_{r,\alpha}^\dagger(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{x}-\omega x_0)}]. \quad (48)$$

6. QUANTIZATION OF THE FIELD

The quantization of the Fierz-Pauli field can now be easily carried out. The canonical conjugate of ψ_α is given by

$$\pi_\alpha=\frac{\partial L}{\partial(\partial\psi_\alpha/\partial t)}=i\hbar\bar{\psi}_\beta(\alpha_4)_{\beta\alpha}=i\hbar\psi_\alpha^\dagger. \quad (49)$$

Since fields of half-integral spin obey Fermi statistics, the appropriate commutation relations in the present case are

$$\{\psi_\alpha(\mathbf{x},t),\psi_\beta(\mathbf{x}',t)\}=0,\quad \{\psi_\alpha^\dagger(\mathbf{x},t),\psi_\beta^\dagger(\mathbf{x}',t)\}=0,\quad (50)$$

$$\{\psi_\alpha(\mathbf{x},t),\psi_\beta^\dagger(\mathbf{x}',t)\}=\delta_{\alpha\beta}\delta(\mathbf{x}-\mathbf{x}'). \quad (51)$$

Substituting (47) and (48) in (50) and (51), and using (44), we get

$$\{a_r(\mathbf{k}),\,a_s^*(\mathbf{k}')\}=\delta_{rs}\delta_{\mathbf{k},\mathbf{k}'},\quad \{b_r(\mathbf{k}),\,b_s^*(\mathbf{k}')\}=\delta_{rs}\delta_{\mathbf{k},\mathbf{k}'}, \quad (52)$$

while any other anticommutator involving a pair of the operators $a_r(\mathbf{k})$, $a_r^*(\mathbf{k})$, $b_r(\mathbf{k})$, or $b_r^*(\mathbf{k})$ vanishes.

Again, substituting (47) and (48) in (18) and (19), and using (44), we obtain for the total Hamiltonian and the total charge of the field

$$\mathcal{H}=\sum_{\mathbf{k}}\sum_r c\hbar\omega[a_r^*(\mathbf{k})a_r(\mathbf{k})-b_r(\mathbf{k})b_r^*(\mathbf{k})], \quad (53)$$

$$\int\rho dV=\sum_{\mathbf{k}}\sum_r e[a_r^*(\mathbf{k})a_r(\mathbf{k})+b_r(\mathbf{k})b_r^*(\mathbf{k})]. \quad (54)$$

Using the commutation relations (52), and ignoring the zero-point energy and charge, we can write (53) and (54) as

$$\mathcal{H}=\sum_{\mathbf{k}}\sum_r c\hbar\omega[a_r^*(\mathbf{k})a_r(\mathbf{k})+b_r^*(\mathbf{k})b_r(\mathbf{k})], \quad (55)$$

$$\int\rho dV=\sum_{\mathbf{k}}\sum_r e[a_r^*(\mathbf{k})a_r(\mathbf{k})-b_r^*(\mathbf{k})b_r(\mathbf{k})]. \quad (56)$$

It follows in the usual way that the quantities $a_1^*(\mathbf{k})a_1(\mathbf{k})$, $b_1^*(\mathbf{k})b_1(\mathbf{k})$, $a_2^*(\mathbf{k})a_2(\mathbf{k})$, $b_2^*(\mathbf{k})b_2(\mathbf{k})$, . . . can have only the eigenvalues 0 or 1, so that the present field describes particles of positive energy obeying Pauli's exclusion principle. Moreover, the $a_r(\mathbf{k})$ are the absorption operators for particles of energy $c\hbar\omega$ and charge e, while the $b_r(\mathbf{k})$ are the absorption operators for particles of energy $c\hbar w$ and charge $-e$. The Hermitian conjugates of these operators are the corresponding emission operators. Thus, if no particles of charge e are present in a state Ψ, we have

$$\psi^+\Psi=0, \quad (57)$$

and similarly, in the absence of particles of charge $-e$ in a state Ψ,

$$\bar{\psi}^+\Psi=0, \quad (58)$$

where ψ^+ and $\bar{\psi}^+$ denote positive frequency parts of ψ and $\bar{\psi}$, respectively.

7. COVARIANT COMMUTATION RELATIONS

We shall now obtain covariant commutation relations between the field variables at different times. For this purpose it is necessary to use the variables ψ_α and $\bar{\psi}_\alpha$ instead of ψ_α and $\psi_\alpha^\dagger$. It is, of course, evident that

$$\{\psi_\alpha(x),\psi_\beta(x')\}=\{\bar{\psi}_\alpha(x),\bar{\psi}_\beta(x')\}=0. \quad (59)$$

In order to find the required commutation relation between ψ_α and $\bar{\psi}_\beta$ let us put

$$\{\psi_\alpha(x),\bar{\psi}_\beta(x')\}=f_{\alpha\beta}\Delta(x-x'), \quad (60)$$

where the quantity $f_{\alpha\beta}$ has to be determined, and $\Delta(x-x')$ is a well-known singular function with the properties

$$(\Box^2-\kappa^2)\Delta(x-x')=0, \quad (61)$$

$$[\Delta(x-x')]_{x_0'=x_0}=0,\quad [\partial\Delta(x-x')/\partial x_0]_{x_0'=x_0}=-\delta(\mathbf{x}-\mathbf{x}'). \quad (62)$$

Since $\psi_\alpha(x)$ satisfies the field equation (5), we must have

$$[\alpha_\mu(\partial/\partial x_\mu)+\kappa]_{\alpha\beta}f_{\beta\gamma}\Delta(x-x')=0. \quad (63)$$

We also observe that

$$[\alpha_\mu(\partial/\partial x_\mu)+\kappa][\alpha_\nu\alpha_\lambda\alpha_\rho(\partial^3/\partial x_\nu\partial x_\lambda\partial x_\rho)-\kappa\alpha_\lambda\alpha_\rho(\partial^2/\partial x_\lambda\partial x_\rho)]\Delta(x-x')$$
$$=\alpha_\lambda\alpha_\rho(\partial^2/\partial x_\lambda\partial x_\rho)(\Box^2-\kappa^2)\Delta(x-x')=0, \quad (64)$$

which suggests that $f_{\beta\gamma}$ should be of the form

$$f_{\beta\gamma}=C[\alpha_\nu\alpha_\lambda\alpha_\rho(\partial^3/\partial x_\nu\partial x_\lambda\partial x_\rho)-\kappa\alpha_\lambda\alpha_\rho(\partial^2/\partial x_\lambda\partial x_\rho)]_{\beta\gamma}, \quad (65)$$

where C is a constant. We have now to verify that the commutation relation, given by (60) and (65), agrees with (51), and for this we shall derive a relation between $\bar{\psi}$ and $\psi^\dagger$.

We note that the relation (6) gives us

$$\alpha_4^2-\alpha_4^4=0 \quad (66)$$

$$\alpha_4^3\alpha_i+\alpha_4^2\alpha_i\alpha_4+\alpha_4\alpha_i\alpha_4^2+\alpha_i\alpha_4^3-\alpha_4\alpha_i-\alpha_i\alpha_4=0, \quad (67)$$

$$\alpha_4^2\alpha_k\alpha_i+\alpha_4\alpha_k\alpha_4\alpha_i+\alpha_4\alpha_k\alpha_i\alpha_4-\delta_{ik}\alpha_4^2+\alpha_k\alpha_i\alpha_4^2$$
$$+\alpha_k\alpha_4\alpha_i\alpha_4+\alpha_k\alpha_4^2\alpha_i-\alpha_k\alpha_i+(i\rightleftarrows k)=0, \quad (68)$$

where $(i\rightleftarrows k)$ on the left-hand side of (68) denotes an expression obtained by interchanging the indices i and k in the preceding expression.

Multiplying (13) by $\alpha_4-\alpha_4^3$, and using (66), we get

$$\kappa\bar{\psi}(\alpha_4-\alpha_4^3)=(\partial\bar{\psi}/\partial x_i)\alpha_i(\alpha_4-\alpha_4^3),$$

which gives, on using (67),

$$\kappa\bar{\psi}\alpha_4(1-\alpha_4^2)=(\partial\bar{\psi}/\partial x_i)\alpha_4(\alpha_4^2\alpha_i+\alpha_4\alpha_i\alpha_4+\alpha_i\alpha_4^2-\alpha_i). \quad (69)$$

Differentiating (69) with respect to x_4, and using (13), we obtain

$$\kappa[\kappa\bar{\psi}-(\partial\bar{\psi}/\partial x_i)\alpha_i](1-\alpha_4^2)$$
$$=(\partial/\partial x_i)[\kappa\bar{\psi}-(\partial\bar{\psi}/\partial x_k)\alpha_k]$$
$$\times(\alpha_4^2\alpha_i+\alpha_4\alpha_i\alpha_4+\alpha_i\alpha_4^2-\alpha_i),$$

or

$$\bar{\psi}=\bar{\psi}\alpha_4^2+\frac{1}{\kappa}\frac{\partial\bar{\psi}}{\partial x_i}(\alpha_4^2\alpha_i+\alpha_4\alpha_i\alpha_4)-\frac{1}{\kappa^2}\frac{\partial^2\bar{\psi}}{\partial x_i\partial x_k}\alpha_k(\alpha_4^2\alpha_i+\alpha_4\alpha_i\alpha_4+\alpha_i\alpha_4^2-\alpha_i).$$

Further, using (66) and (68), we can write the above relation as

$$\bar{\psi}=\bar{\psi}\alpha_4^4+\frac{1}{\kappa}\frac{\partial\bar{\psi}\alpha_4}{\partial x_i}(\alpha_4\alpha_i+\alpha_i\alpha_4)+\frac{1}{\kappa^2}\frac{\partial^2\bar{\psi}\alpha_4}{\partial x_i\partial x_k}(\alpha_4\alpha_k\alpha_i+\alpha_k\alpha_4\alpha_i+\alpha_k\alpha_i\alpha_4-\delta_{ik}\alpha_4^3),$$

or

$$\bar{\psi}=\psi^\dagger\alpha_4^3+\frac{1}{\kappa}\frac{\partial\psi^\dagger}{\partial x_i}(\alpha_4\alpha_i+\alpha_i\alpha_4)+\frac{1}{\kappa^2}\frac{\partial^2\psi^\dagger}{\partial x_i\partial x_k}(\alpha_4\alpha_k\alpha_i+\alpha_k\alpha_4\alpha_i+\alpha_k\alpha_i\alpha_4-\delta_{ik}\alpha_4^3). \quad (70)$$

From (70) and (51) we get

$$\{\psi_\alpha(\mathbf{x},t),\bar{\psi}_\beta(\mathbf{x}',t)\}=\left[\alpha_4^3+\frac{1}{\kappa}\frac{\partial}{\partial x_i'}(\alpha_4\alpha_i+\alpha_i\alpha_4)+\frac{1}{\kappa^2}\frac{\partial^2}{\partial x_i'\partial x_k'}\times(\alpha_4\alpha_k\alpha_i+\alpha_k\alpha_4\alpha_i+\alpha_k\alpha_i\alpha_4-\delta_{ik}\alpha_4^3)\right]_{\alpha\beta}\delta(\mathbf{x}-\mathbf{x}')$$
$$=\left[\alpha_4^3-\frac{1}{\kappa}\frac{\partial}{\partial x_i}(\alpha_4\alpha_i+\alpha_i\alpha_4)+\frac{1}{\kappa^2}\frac{\partial^2}{\partial x_i\partial x_k}\times(\alpha_4\alpha_k\alpha_i+\alpha_k\alpha_4\alpha_i+\alpha_k\alpha_i\alpha_4-\delta_{ik}\alpha_4^3)\right]_{\alpha\beta}\delta(\mathbf{x}-\mathbf{x}'). \quad (71)$$

On the other hand, substituting (65) in (60), putting $x_0'=x_0$, and using (61) and (62), we obtain

$$\{\psi_\alpha(\mathbf{x},t),\bar{\psi}_\beta(\mathbf{x}',t)\}=iC\left[\kappa^2\alpha_4^3-\kappa\frac{\partial}{\partial x_i}(\alpha_4\alpha_i+\alpha_i\alpha_4)+\frac{\partial^2}{\partial x_i\partial x_k}(\alpha_4\alpha_k\alpha_i+\alpha_k\alpha_4\alpha_i+\alpha_k\alpha_i\alpha_4)-\frac{\partial^2}{\partial x_i^2}\alpha_4^3\right]_{\alpha\beta}\delta(\mathbf{x}-\mathbf{x}'). \quad (72)$$

Comparing (71) and (72), we find that the commutation relation (60) agrees with (51), provided that we choose the constant C in (65) as

$$C=-i/\kappa^2. \quad (73)$$

Thus, from (60), (65), and (73) we obtain the commutation relation:

$$\{\psi_\alpha(x),\bar{\psi}_\beta(x')\}=-\frac{i}{\kappa^2}\left(\alpha_\nu\alpha_\lambda\alpha_\rho\frac{\partial^3}{\partial x_\nu\partial x_\lambda\partial x_\rho}-\kappa\alpha_\lambda\alpha_\rho\frac{\partial^2}{\partial x_\lambda\partial x_\rho}\right)_{\alpha\beta}\times\Delta(x-x'). \quad (74)$$

8. ELECTROMAGNETIC INTERACTION OF PARTICLES OF SPIN 3/2

Following the usual procedure, we can write the Lagrangian density for the Fierz-Pauli field interacting with the electromagnetic field as

$$L=-\frac{1}{4}\left(\frac{\partial A_\nu}{\partial x_\mu}-\frac{\partial A_\mu}{\partial x_\nu}\right)^2-\frac{1}{2}\left(\frac{\partial A_\mu}{\partial x_\mu}\right)^2-c\hbar\left[\bar{\psi}\alpha_\mu\left(\frac{\partial}{\partial x_\mu}-\frac{ie}{c\hbar}A_\mu\right)\psi+\kappa\bar{\psi}\psi\right], \quad (75)$$

which gives us the field equations:

$$\alpha_\mu(\partial\psi/\partial x_\mu)+\kappa\psi=(ie/c\hbar)A_\mu\alpha_\mu\psi,$$
$$(\partial\bar{\psi}/\partial x_\mu)\alpha_\mu-\kappa\bar{\psi}=(-ie/c\hbar)A_\mu\bar{\psi}\alpha_\mu, \quad (76)$$
$$\Box^2A_\mu=-ie\bar{\psi}\alpha_\mu\psi.$$

In order to obtain the rules for writing down the elements of the S matrix in the present case, it will be very convenient to follow the treatment of Yang and Feldman.[7] We first observe that the relation (6) gives us

$$\left(\alpha_\mu\frac{\partial}{\partial x_\mu}+\kappa\right)\left(\alpha_\nu\frac{\partial}{\partial x_\nu}-\kappa\right)\times\left(\frac{1}{\kappa^2}\alpha_\lambda\alpha_\rho\frac{\partial^2}{\partial x_\lambda\partial x_\rho}-\frac{1}{\kappa^2}\Box^2+1\right)=\Box^2-\kappa^2, \quad (77)$$

so that, defining $R^{\text{ret}}(x-x')$ by

$$R^{\text{ret}}(x-x')=\left(\alpha_\nu\frac{\partial}{\partial x_\nu}-\kappa\right)\times\left(\frac{1}{\kappa^2}\alpha_\lambda\alpha_\rho\frac{\partial^2}{\partial x_\lambda\partial x_\rho}-\frac{1}{\kappa^2}\Box^2+1\right)\Delta^{\text{ret}}(x-x'), \quad (78)$$

we get

$$[\alpha_\nu(\partial/\partial x_\nu)+\kappa]R^{\text{ret}}(x-x')=-\delta(x-x'). \quad (79)$$

Similarly, we define a function R^{adv} by replacing Δ^{ret} in (78) by Δ^{adv}, so that R^{adv} also satisfies the relation (79).

[7] C. N. Yang and D. Feldman, Phys. Rev. **79**, 972 (1950). We also refer to this paper for the definitions of the functions Δ^{ret}, Δ^{adv}, D^{ret}, and D^{adv}, which we shall be using here.

Further, we define "incoming field operators" ψ^{in}, $\bar{\psi}^{\text{in}}$, and A_μ^{in} by the set of integral equations

$$\psi(x)=\psi^{\text{in}}(x)-(ie/c\hbar)\times\int R^{\text{ret}}(x-x')dx'A_\nu(x')\alpha_\nu\psi(x'),$$

$$\bar{\psi}(x)=\bar{\psi}^{\text{in}}(x)-(ie/c\hbar)\times\int\bar{\psi}(x')\alpha_\nu A_\nu(x')dx'R^{\text{adv}}(x'-x), \quad (80)$$

$$A_\nu(x)=A_\nu^{\text{in}}(x)+ie\times\int D^{\text{ret}}(x-x')dx'\bar{\psi}(x')\alpha_\mu\psi(x'),$$

and the "outgoing field operators" ψ^{out}, $\bar{\psi}^{\text{out}}$, and A_μ^{out} by the set of integral equations:

$$\psi(x)=\psi^{\text{out}}(x)-(ie/c\hbar)\times\int R^{\text{adv}}(x-x')dx'A_\nu(x')\alpha_\nu\psi(x'),$$

$$\bar{\psi}(x)=\bar{\psi}^{\text{out}}(x)-(ie/c\hbar)\times\int\bar{\psi}(x')\alpha_\nu A_\nu(x')dx'R^{\text{ret}}(x'-x), \quad (81)$$

$$A_\nu(x)=A_\nu^{\text{out}}(x)+ie\times\int D^{\text{adv}}(x-x')dx'\bar{\psi}(x')\alpha_\mu\psi(x').$$

It can then be easily shown that both the incoming as well as the outgoing field operators satisfy the free-field equations and the free-field commutation relations.

We can now define the S matrix for the interaction of photons and particles of spin 3/2 as the unitary operator, given by

$$\psi^{\text{out}}(x)=S^{-1}\psi^{\text{in}}(x)S,$$
$$\bar{\psi}^{\text{out}}(x)=S^{-1}\bar{\psi}^{\text{in}}(x)S, \quad (82)$$
$$A_\nu^{\text{out}}(x)=S^{-1}A_\nu^{\text{in}}(x)S.$$

One can obtain the matrix elements of the S matrix by solving (80) by successive approximations in powers of e, and then using (81) and (82). However, as pointed out by Yang and Feldman,[7] in practice one need not take this trouble. For, if we compare the set of Eqs. (80), (81), and (82) with similar equations in quantum electrodynamics, it is evident that the rules for obtaining the elements of the S matrix in the present case will be exactly analogous to the Feynman-Dyson rules of quantum electrodynamics,[5] except that the γ_μ matrices of Dirac have to be replaced by the α_μ matrices of Fig. 2, and the function

$$S_F(x-x')=[\gamma_\mu(\partial/\partial x_\mu)-\kappa]\Delta_F(x-x') \quad (83)$$

has to be replaced by

$$R_F(x-x')=\left(\alpha_\nu\frac{\partial}{\partial x_\nu}-\kappa\right)\times\left(\frac{1}{\kappa^2}\alpha_\lambda\alpha_\rho\frac{\partial^2}{\partial x_\lambda\partial x_\rho}-\frac{1}{\kappa^2}\square^2+1\right)\Delta_F(x-x'), \quad (84)$$

where $\Delta_F(x-x')$ is Feynman's singular function, as defined by Dyson.[5] It is interesting to note that if we replace α_μ by γ_μ in (84), $R_F(x-x')$ reduces to $S_F(x-x')$.

Since $R_F(x-x')$ involves third space and time derivatives of $\Delta_F(x-x')$, it seems at first sight that the divergencies in the present case are even stronger than those in the case of charged particles of spin 1. However, in order to see whether the renormalization theory is really unsuccessful in the case of charged particles of spin 3/2, it would be necessary to carry out actual calculations of the various possibly divergent matrix elements.

We have seen that the quantization of the Fierz-Pauli field does not present any special difficulty. Therefore, it seems to us by no means certain that particles of spin higher than one do not exist in nature, and it would be interesting to carry out further investigations of the properties of such particles. In this connection it should be noted that the intrinsic magnetic moment of charged particles of spin 3/2 has recently been calculated by Belinfante,[8] and shown to be equal to $e\hbar/2mc$, where m is the rest mass of the particles. As pointed out by Belinfante, this seems to suggest that the intrinsic magnetic moment of every elementary particle of nonzero spin is given by the same expression $e\hbar/2mc$, which depends only on the charge and the rest mass of the particle.

[8] F. J. Belinfante, Phys. Rev. **92**, 997 (1953).