PHYSICAL REVIEW VOLUME 95, NUMBER 5 SEPTEMBER 1, 1954

Some Special Examples in Renormalizable Field Theory

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(Received May 4, 1954)

Some special problems of interacting fields that contain removable divergences are treated in detail. Comparisons with the power series renormalization procedures are made. Examination of the closed forms of the solutions before and after renormalization shows that, in one special case solved, coupling-constant renormalization cannot be obtained by any limiting processes that involve only real values of the unrenormalized coupling constant.

I. INTRODUCTION

EVER since the overwhelming success of the applications of renormalization technique in quantum electrodynamics, the problem of understanding this renormalization procedure without the use of perturbation methods has been of great interest. Unfortunately, in all physically realistic cases the Hamiltonians are quite complicated in structure, and so far no solution other than the power series method has been found. Yet the renormalization methods as developed by Dyson,[1] Ward,[2] and others can certainly be applied to a large class of field-theoretical problems that involve removable divergencies. Thus some deeper insight as to the nature of the renormalization procedure may, perhaps, be obtained by examining problems that are similar to, but not as rich as, either the quantum electrodynamics or the relativistic pseudoscalar meson theory. With this motive we shall in this paper treat some simple problems of interacting fields that are both renormalizable and solvable.

We consider first a problem that involves the interactions between two neutral nonrelativistic fermion fields and one relativistic boson field. Although the Hamiltonian of this problem does involve infinities, the Schroedinger equation can still be solved directly. A close examination of the solutions indicates that all the divergent quantities can be removed by both a mass renormalization and a coupling-constant renormalization. The scattering amplitude is then calculated after renormalization and is indeed found to be finite (i.e., not zero). This problem can also be treated by the application of the customary power series renormalization procedure whereby identical conclusions concerning the nature of renormalization quantities are reached. Furthermore, even outside its radius of convergence, the formal sum of the power series can still be used to give the correct closed form. A rather unexpected and quite surprising feature is obtained by comparing the renormalized coupling constant with the unrenormalized coupling constant. In this particular case it can be shown that the result of the renormalization process *cannot* be obtained by any limiting process that involves only *real* values of the unrenormalized coupling constant. This difficulty may, however, be overcome by a modification of the present rules of quantum mechanics.

Next, the well-known soluble problem of neutral scalar mesons with fixed nucleons is studied. The closed forms of the nucleon propagation functions and vertex functions are listed in Appendix I.

In Appendix II, the charged scalar theory together with some interesting identities between the customary matrix elements and the renormalization quantities Z_1, Z_2 are discussed.

[1] F. J. Dyson, Phys. Rev. **75**, 1736 (1949).
[2] J. C. Ward, Proc. Phys. Soc. (London) **A64**, 54 (1951); Phys. Rev. **84**, 897 (1951).

II. HAMILTONIAN

Let us consider the interaction between two neutral fermion fields, V and N, and a neutral scalar boson field, θ. The Hamiltonian for the free fields is

$$H_0 = m_V \int \psi_V^\dagger \psi_V d\tau + m_N \int \psi_N^\dagger \psi_N d\tau + \tfrac{1}{2}\int [\pi^2 + (\nabla\varphi)^2 + \mu^2\varphi^2] d\tau, \quad (1)$$

where $\psi_V^\dagger$, ψ_V and $\psi_N^\dagger$, ψ_N obey the usual anticommutation relation and represent the field variables of V and N, φ and π are the field variable and conjugate momentum of the θ particles. The "observed" masses of V, N, and θ are denoted by m_V, m_N, and μ. Recoil effects of V and N are not included.

It is convenient to write

$$\varphi = A(r) + A^\dagger(r),$$

where

$$A(r) = \sum (2\omega\Omega)^{-\frac{1}{2}} \alpha_k \exp(ik\cdot r),$$
$$A^\dagger(r) = \sum (2\omega\Omega)^{-\frac{1}{2}} \alpha_k^\dagger \exp(-ik\cdot r).$$

α_k and $\alpha_k^\dagger$ are the annihilation and creation operators of the θ particle with wave number k. Ω is the volume of the system and ω is $(k^2+\mu^2)^{\frac{1}{2}}$. The interaction Hamiltonian H_1 that represents the reaction

$$V \rightleftarrows N + \theta \quad (2)$$

can be written as

$$H_1 = g\int [\psi_V^\dagger(r)\psi_N(r)A(r) + \psi_N^\dagger(r)\psi_V(r)A^\dagger(r)] d\tau + \delta m_V \int \psi_V^\dagger \psi_V d\tau, \quad (3)$$

where δm_V is used to cancel any change in mass of the V particle due to the reaction (2).

Upon examining the total Hamiltonian

$$H=H_0+H_1, \tag{4}$$

one sees that this system possesses two simple conservation laws:

$$\mathfrak{N}_V+\mathfrak{N}_N=\text{constant},$$

and

$$\mathfrak{N}_V+\mathfrak{N}_\theta=\text{constant}, \tag{5}$$

with $\mathfrak{N}_V$, $\mathfrak{N}_N$, $\mathfrak{N}_\theta$ the total number of V, N, and θ particles, respectively. Because of (5) the eigenfunctions of H contain only a finite number of particles and consequently can be solved directly.

III. GROUND STATES

In the following, we denote the state of a "bare" V particle and that of a "bare" N particle by $|V\rangle$ and $|N\rangle$ while the states of the corresponding "physical" particles are indicated by $|\mathbf{V}\rangle$ and $|\mathbf{N}\rangle$. Thus $|\mathbf{V}\rangle$ and $|\mathbf{N}\rangle$ are eigenstates of the Hamiltonian and satisfy

$$\begin{aligned} H|\mathbf{N}\rangle&=m_N|\mathbf{N}\rangle,\\ H|\mathbf{V}\rangle&=m_V|\mathbf{V}\rangle. \end{aligned} \tag{6}$$

In order that V be a stable particle we assume $m_V-m_N<\mu$. Using (5), one see that

$$|\mathbf{N}\rangle=|N\rangle,$$

and

$$|\mathbf{V}\rangle=Z_2^{\frac{1}{2}}[|V\rangle+g\textstyle\sum_k f(k)\alpha_k^\dagger|N\rangle], \tag{7}$$

where $Z_2^{\frac{1}{2}}$ is a normalization constant and $f(k)$ is proportional to the probability amplitude for finding a θ particle with wave number k in a physical V state.

Applying the Hamiltonian (4) on $|\mathbf{V}\rangle$ and requiring that the eigenvalue should be the "observed" mass m_V, one finds

$$\delta m_V=-g^2\textstyle\sum(2\omega\Omega)^{-1}(m_V-m_N-\omega)^{-1}, \tag{8}$$

$$f(k)=(2\omega\Omega)^{-\frac{1}{2}}(m_V-m_N-\omega)^{-1}, \tag{9}$$

and

$$Z_2^{-1}=1+g^2\textstyle\sum(2\omega\Omega)^{-1}(m_V-m_N-\omega)^{-2}. \tag{10}$$

We remark that the divergent quantity δm_V serves as the renormalization in mass such that the eigenvalue m_V of the Hamiltonian is now finite. The divergent quantity Z_2^{-1} is related to the coupling-constant renormalization and will be studied in the next section.

IV. SCATTERING STATE

We consider first the scattering process

$$N+\theta\rightarrow N+\theta.$$

The eigenfunction $|\mathbf{N}+\boldsymbol{\theta}\rangle$ that represents this process can be written in terms of the physical one-particle states as

$$|\mathbf{N}+\boldsymbol{\theta}\rangle=\textstyle\sum_k\chi(k)\alpha_k^\dagger|\mathbf{N}\rangle+c|\mathbf{V}\rangle, \tag{11}$$

and furthermore it satisfies the Schroedinger equation,

$$H|\mathbf{N}+\boldsymbol{\theta}\rangle=(m_N+\omega_0)|\mathbf{N}+\boldsymbol{\theta}\rangle. \tag{12}$$

On using the special form of H together with (6) and (7), the Schroedinger equation can be readily solved and one obtains

$$c=-\langle\mathbf{V}|\textstyle\sum\chi(k)\alpha_k^\dagger|\mathbf{N}\rangle, \tag{13}$$

and

$$(\omega-\omega_0)\chi(k)=g^2Z_2\int K(k,k')\chi(k')d^3k', \tag{14}$$

where

$$K(k,k')=(m_V-m_N-\omega_0)(4\omega\omega')^{-\frac{1}{2}}\times[8\pi^3(m_V-m_N-\omega)(m_V-m_N-\omega')]^{-1}.$$

One observes that although Z_2^{-1} involves a divergent sum, the cross section will not vanish if the renormalized coupling constant, given by

$$g_c^2=g^2Z_2, \tag{15}$$

is chosen to be finite. Equation (14) can then be easily solved, and the phase shift δ for the scattering process is

$$\tan\delta=\frac{g_c^2k_0}{4\pi(m_V-m_N-\omega_0)}\times\left[1-\frac{g_c^2}{16\pi^3}\mathcal{P}\int\frac{d^3k(m_V-m_N-\omega_0)}{\omega(\omega-\omega_0)(m_V-m_N-\omega)^2}\right]^{-1}, \tag{16}$$

where $\mathcal{P}$ indicates that in the integration the principal value is to be taken. Thus if g_c is finite the expression for the phase shift is indeed free from divergent quantities.

Next we consider the scattering process,

$$V+\theta\rightarrow V+\theta. \tag{17}$$

Again, the corresponding eigenstate that satisfies

$$H|\mathbf{V}+\boldsymbol{\theta}\rangle=(m_V+\omega_0)|\mathbf{V}+\boldsymbol{\theta}\rangle \tag{18}$$

can be written in terms of the physical one-particle states as

$$|\mathbf{V}+\boldsymbol{\theta}\rangle=\sum_k\psi(k)\alpha_k^\dagger|\mathbf{V}\rangle+\sum_{k_1,k_2}\phi(k_1,k_2)\alpha_{k_1}^\dagger\alpha_{k_2}^\dagger|\mathbf{N}\rangle. \tag{19}$$

The Schroedinger equation can also be solved and one finds

$$(\omega-\omega_0)h(\omega)\psi(k)=g^2Z_2\int K'(k,k')\psi(k')d^3k', \tag{20}$$

where

$$K'(k,k')=(16\pi^3)^{-1}(\omega\omega')^{-\frac{1}{2}}(\omega+\omega'-\omega_0-m_V+m_N)^{-1},$$

$$h(\omega)=1-g^2Z_2(16\pi^3)^{-1}\times\int\frac{d^3k'}{(\omega+\omega'-\omega_0-m_V+m_N)(m_V-m_N-\omega')^2\omega'},$$

and

$$(\omega_1+\omega_2-\omega_0-m_V+m_N)\phi(k_1,k_2)$$
$$=-\tfrac{1}{2}(gZ_2^{\frac{1}{2}})[(\omega_1-\omega_0)\psi(k_1)f(k_2)+(\omega_2-\omega_0)\psi(k_2)f(k_1)]. \quad (21)$$

$f(k)$ is given by Eq. (9). By counting the powers of ω and ω' in the kernel K' one sees that the solution $\psi(k)$ and $\phi(k_1,k_2)$ would be free from divergent integrals if the renormalized coupling constant $g_c=gZ_2^{\frac{1}{2}}$ is finite.

V. COMPARISON WITH POWER SERIES METHOD

The above problem can also be renormalized by the power series method. By constructing the S matrix in the interaction representation and commuting all the time ordered operators into S products[1,3] one finds the following rules for the Feynman diagram:

(i) Each V line gives a factor $(p_0-m_V+i\epsilon)^{-1}$, where p_0 is $-i$ times the fourth component of the momentum vector carried by the V particle. ϵ is a positive infinitesimal quantity.

(ii) Each N line gives a factor $(p_0-m_N+i\epsilon)^{-1}$.

(iii) Each θ line gives a factor $(-k_\mu^2-\mu^2+i\epsilon)^{-1}$ and each variable momentum k_μ gives an integration $\int d^4k/(2\pi)^4$.

(iv) Each vertex gives a factor g.

Let S_N', S_V', and Γ denote the sum of all diagrams that contribute to the propagation function of N, the propagation function of V, and the vertex function, respectively. From the nature of the interaction (2), these quantities can be calculated readily and they are

$$\Gamma=1,$$
$$(S_N')^{-1}=p_0-m_N, \quad (22)$$
$$(S_V')^{-1}=p_0-m_V-\delta m_V-\Sigma(p_0),$$

where

$$\Sigma(p_0)=g^2\textstyle\sum_k(2\omega\Omega)^{-1}(p_0-m_N-\omega)^{-1}.$$

Thus, both Γ and S_N' need no renormalization. The renormalization of mass and propagation function of V are given by

$$(S_V')^{-1}=0 \quad \text{at} \quad p_0=m_V,$$

and

$$Z_2^{-1}=d[(S_V')^{-1}]/dp_0 \quad \text{at} \quad p_0=m_V. \quad (23)$$

Hence δm_V and S_V' can be written as

$$\delta m_V=-\textstyle\sum(p_0=m_V)=-g^2\sum(2\omega\Omega)^{-1}(m_V-m_N-\omega)^{-1},$$

and

$$S_V'(p_0)=Z_2S_{Vc}(p_0), \quad (24)$$

where

$$Z_2^{-1}=1+g^2\textstyle\sum(2\omega\Omega)^{-1}(m_V-m_N-\omega)^{-2},$$
$$S_{Vc}^{-1}(p_0)=(p_0-m_V)\left[1-g^2Z_2\sum\frac{p_0-m_V}{(2\omega\Omega)(p_0-m_N-\omega)(m_V-m_N-\omega)^2}\right]. \quad (25)$$

If, as in (15), g^2Z_2 is set to be g_c^2 and remains finite, then the renormalized propagation function $S_{Vc}(p_0)$ will be free from divergent quantities.[4]

For any other physical processes it can then be shown that one needs only to consider the irreducible diagrams and use S_{Vc}, $S_{Nc}=S_N'$, and g_c for the propagation lines of V, N, and the vertex, respectively. Although the number of irreducible diagrams may still be infinite, the contribution of each diagram will be free from divergent integrals. In this case, using (25), one can compute the scattering processes in powers of g_c and one obtains indeed the same results as discussed in the previous section.

Thus, identical conclusions concerning both the mass renormalization and coupling-constant renormalization are obtained either by the power series method or by directly solving the Schroedinger equation. A comparison between these two methods shows that Z_2, which in the power series method is defined as the residue of S_V' at its pole, is actually the probability[5] of finding a "bare" V particle in the state of a "physical" V particle as shown by (7). It may be of interest to notice that, for example, in (25) the radius of convergence for the power series expansion of $S_{Vc}(p_0)$ in g_c^2 depends on the variable p_0 and it is 0 if $p_0=\infty$. Yet, the power series can always be summed formally and still gives the correct result.

VI. DISCUSSION OF COUPLING CONSTANT RENORMALIZATION

The nature of the coupling constant renormalization may be investigated by expressing g^2 and Z_2 in terms of the renormalized coupling constant g_c. Using (10) and (15) one finds

$$g^2=g_c^2[1-g_c^2\textstyle\sum(2\omega\Omega)^{-1}(m_V-m_N-\omega)^{-2}]^{-1}, \quad (26)$$

and

$$Z_2=1-g_c^2\textstyle\sum(2\omega\Omega)^{-1}(m_V-m_N-\omega)^{-2}. \quad (27)$$

Hence, if g_c does not vanish and remains finite, the unrenormalized coupling constant becomes

$$g=i\infty^{-1},$$

while Z_2 which, being a probability, should be between 0 and 1 is actually

$$Z_2=-\infty.$$

This shows that the result of coupling-constant renormalization in this problem cannot be realized by any limiting process if g is restricted to be on the real axis. Instead, the expressions for cross sections in previous sections may be obtained by allowing in the original Hamiltonian an unrenormalized coupling constant

[3] G. C. Wick, Phys. Rev. 80, 268 (1950).

[4] It is of interest to notice that if g_c^2 is finite, then S_{Vc} has another pole besides $p_0=m_V$. This pole corresponds to another stable state for the V particle. Identical conclusions can be obtained either by studying the bound state solutions for the state $|\mathbf{N}+\boldsymbol{\theta}\rangle$ or by examining the energy of the outgoing θ particle in the scattering state $|\mathbf{V}+\boldsymbol{\theta}\rangle$.

[5] See Appendix II for a general discussion of this property.

which is pure imaginary and which approaches zero while the upper limit of the sum in momentum space approaches infinity as described by (26).

This raises immediately the question as to what changes in the present rules of quantum mechanics we have made by allowing the Hamiltonian to be non-Hermitian. As shown in the preceding sections, the eigenvalues of the Hamiltonian and the matrix elements of the collision matrix are only functions of the renormalized coupling constant g_c, which is always real. Thus, even if g is imaginary, the unitarity of the collision matrix and the reality of the energy spectrum can still be preserved. Consequently, it is possible to transform this non-Hermitian Hamiltonian into a Hermitian matrix by a similarity transformation. As is well known, a similarity transformation preserves all relations between matrices and vectors; the laws of quantum mechanics may still be applied to this non-Hermitian Hamiltonian provided some attention is given to the formal differences between a unitary transformation and a similarity transformation. In particular, the transformation between the bare particle states and the physical particle states is not unitary, which explains why Z_2 as shown by (27) is not confined between 0 and 1.

The author wishes to thank Professors R. Serber and N. Kroll for discussions.

APPENDIX I[6]

Another soluble problem is that of the neutral scalar meson with fixed nucleon. The Hamiltonian is H_0+H_1, where

$$H_0=\int m\psi_N{}^\dagger\psi_N d\tau+\tfrac{1}{2}\int[\pi^2+(\nabla\varphi)^2+\mu^2\varphi^2]d\tau,$$

and

$$H_1=g\int\psi_N{}^\dagger\psi_N\varphi d\tau+\delta m\int\psi_N{}^\dagger\psi_N d\tau,$$

where $\psi_N{}^\dagger$, ψ_N describe the nucleon field and π, φ the meson field, g the coupling constant, and m the physical mass of the nucleon. While the Schroedinger equation of the above Hamiltonian is well known to be solvable, it is of some interest to examine the nucleon propagation function S', the vertex function Γ, and the renormalization quantities δm and Z_1, Z_2.

The mass renormalization is

$$\delta m=-g^2\sum(2\omega^2\Omega)^{-1}, \tag{A1}$$

where Ω is the volume of the system. The nucleon propagation function can be expressed as[7]

$$\mathbf{S}'(x-x')=T\langle\text{vac}|\,\boldsymbol{\psi}_N(x)\boldsymbol{\psi}_N{}^\dagger(x')\,|\text{vac}\rangle, \tag{A2}$$

where $\boldsymbol{\psi}_N$ and $\boldsymbol{\psi}_N{}^\dagger$ are operators in the Heisenberg representations and T is the T product as defined by Wick.[3]

[6] Some of the results obtained in the appendices have been obtained by other authors: M. Gell-Mann and F. Low (private communication); S. Edwards and R. Peierls (to be published).

[7] M. Gell-Mann and F. Low, Phys. Rev. **84**, 350 (1951).

Using the known solutions in the Heisenberg representation, one finds

$$\mathbf{S}'(x-x')=Z_2\mathbf{S}_c(x-x'), \tag{A3}$$

where

$$Z_2=\exp[-g^2\sum(2\omega^3\Omega)^{-1}] \tag{A4}$$

and

$$\begin{aligned}\mathbf{S}_c(x-x')=&\,\delta^3(r-r')\exp[-im(t-t')]\\&\times\exp[g^2\sum(2\omega^3\Omega)^{-1}e^{-i\omega(t-t')}]\text{ if }t>t';\\=&\,0\text{ if }t<t'.\end{aligned} \tag{A5}$$

It is of interest to notice that Z_2 is again the probability of finding the bare nucleon in the state of a physical nucleon.[5] If $t\neq t'$, $\mathbf{S}_c(x-x')$ is always free from divergent quantities; furthermore it can always be expanded into convergent power series in g.

For convenience we shall in the following omit the trivial spatial dependence of $\mathbf{S}'(x)$ and denote its Fourier transform by $S'(p_0)$. The vertex function $\Gamma(p_0,p_0')$ is related to $S'(p_0)$ by[8]

$$\Gamma(p_0,p_0')(p_0-p_0')=[S'(p_0)]^{-1}-[S'(p_0')]^{-1}. \tag{A6}$$

Following Dyson's notation,[1] we write

$$\Gamma=Z_1{}^{-1}\Gamma_c. \tag{A7}$$

If we use (A3), Γ_c would be free from a divergent sum if

$$Z_1=Z_2. \tag{A8}$$

Thus, no coupling-constant renormalization is needed in this case and Z_2 is actually ∞^{-1} for any finite value of g.

APPENDIX II

There exist some interesting relations between the renormalization quantities Z_1, Z_2, etc., and the ordinary matrix elements. In this section we shall illustrate the proof of these relations by considering the problem of the charged scalar meson field with a fixed nucleon. For simplicity, we shall set the nucleon at the origin. The Hamiltonian is

$$H=(m+\delta m)\psi^\dagger\psi+\tfrac{1}{2}\int\sum_{i=1}^{2}[\pi_i{}^2+(\nabla\varphi_i)^2+\mu^2\varphi_i{}^2]d\tau+(g/\sqrt{2})\sum_{i=1}^{2}\psi^\dagger\tau_i\psi\varphi_i(0), \tag{A9}$$

where

$$\psi=\begin{pmatrix}\psi_P(t)\\\psi_N(t)\end{pmatrix}$$

describes the nucleon field and τ_1, τ_2 are the Pauli matrices.

By applying the power-series renormalization method, one can prove that this problem can be renormalized by both a mass renormalization and a coupling-constant

[8] This can be obtained by generalizing the arguments used in proving Ward's identity. The author wishes to thank Dr. C. N. Yang for informing him of this generalization.

renormalization. The nucleon propagation function $S'(p_0)$ and the vertex function $\Gamma(p_0,p_0')$ can be written as

$$S'(p_0)=Z_2S_c(p_0), \tag{A10}$$

$$\Gamma(p_0,p_0')=Z_1^{-1}\Gamma_c(p_0,p_0'). \tag{A11}$$

The quantities Z_2 and Z_1 are determined by the conditions

$$(p_0-m)S_c(p_0)=1 \quad \text{when} \quad p_0=m, \tag{A12}$$

and

$$\Gamma_c(m,m)=1, \tag{A13}$$

where m is the physical mass of the nucleon. The renormalized coupling constant g_c is connected to the unrenormalized coupling constant g by

$$gZ_2Z_1^{-1}=g_c. \tag{A14}$$

S_c and Γ_c are then finite functions of g_c and p_0.

Let $|\mathbf{P}\rangle$ (or $|\mathbf{N}\rangle$) and $|P\rangle$ (or $|N\rangle$) represent one physical proton (neutron) state and one bare proton (neutron) state, respectively. Z_2 and Z_1 are related to these state vectors through the following theorems.

Theorem 1.

$$Z_2=|\langle P|\mathbf{P}\rangle|^2=|\langle N|\mathbf{N}\rangle|^2, \tag{A15}$$

Theorem 2.

$$Z_2Z_1^{-1}=\langle\mathbf{N}|\tau_-|\mathbf{P}\rangle, \tag{A16}$$

where τ_- is the operator $\frac{1}{2}(\tau_1-i\tau_2)$ which transforms a bare proton into a bare neutron.

(i) Proof of Theorem 1. The propagation function of a proton (or neutron) in time can be written as[7]

$$\mathbf{S}'(t_1-t_2)=T\langle\text{vac}|\boldsymbol{\psi}_P(t_1)\boldsymbol{\psi}_P^\dagger(t_2)|\text{vac}\rangle$$

in the Heisenberg representation. For convenience one chooses $t=0$ as the time when the Heisenberg representation coincides with the interaction representation; i.e.,

$$\boldsymbol{\psi}_P(t=0)=\psi_P(t=0), \tag{A17}$$

where $\boldsymbol{\psi}_P$ is in the Heisenberg representation and ψ_P in the interation representation.

$\mathbf{S}'(t_1-t_2)$ can be rewritten as

$$\mathbf{S}'(t_1-t_2)=\textstyle\sum_n\langle\text{vac}|\boldsymbol{\psi}_P(0)|n\rangle\langle n|\boldsymbol{\psi}_P^\dagger(0)|\text{vac}\rangle$$
$$\times\exp[-iE_n(t_1-t_2)] \text{ if } t_1>t_2, \tag{A18}$$

where $|n\rangle$ is eigenstate of the total Hamiltonian with eigenvalue E_n ($E_{\text{vac}}=0$). Thus,[9]

$$\underset{\substack{t_2\to-\infty\\ t_1\to+\infty}}{\mathcal{L}} \exp[im(t_1-t_2)]\mathbf{S}'(t_1-t_2)=|\langle\text{vac}|\psi_P(0)|\mathbf{P}\rangle|^2.$$

By using (A12) and noticing that $\mathbf{S}'(t)$ is the Fourier transform of $S'(p_0)$, one has

$$Z_2=|\langle\mathbf{P}|P\rangle|^2.$$

[9] $\underset{t\to\pm\infty}{\mathcal{L}}$ represents an operator of the type $\underset{t\to\pm\infty}{\text{Lim}}\ t^{-1}\int_t^{2t}dt$ such that all oscillating terms can be taken to be zero.

(ii) To prove Theorem 2, we consider the quantity A_k defined by

$$A_k=T\langle\text{vac}|\boldsymbol{\psi}_N(t_1)\boldsymbol{\alpha}_k(t_3)\boldsymbol{\psi}_P^\dagger(t_2)|\text{vac}\rangle, \tag{A19}$$

where all operators are in the Heisenberg representation. We denote by $\alpha_k(t)$ the annihilation operator of a positive meson in the interaction representation and by $\boldsymbol{\alpha}_k(t)$ the corresponding operator in the Heisenberg representation. Similarly to (A17), we have

$$\boldsymbol{\alpha}_k(t=0)=\alpha_k(t=0). \tag{A17$'$}$$

By making similar arguments to those used by Gell-Mann and Low[7] in proving (A2), one can prove that

$$A_k=g\int\mathbf{S}'(t_1-t)\mathbf{\Gamma}(t-t'',\,t'-t'')$$
$$\times\mathbf{S}'(t'-t_2)f(t''-t_3)dtdt'dt'', \tag{A20}$$

where $\mathbf{\Gamma}$, $\mathbf{S}'$ are Fourier transforms of Γ and S'. The function $f(t''-t_3)$ represents the retarded part of the propagation of a positive meson with wave number k from time t'' to time t_3. It can be written as

$$f(t''-t_3)=\frac{1}{2\pi}\int\frac{\exp[ip_0(t''-t_3)]}{(p_0-\omega+i\epsilon)(2\omega\Omega)^{\frac{1}{2}}}dp_0,$$

where Ω is the volume of the system. The factor $(2\omega\Omega)^{\frac{1}{2}}$ arises because we use α_k in the definition for A_k. Equation (A20) thus represents the totality of all diagrams in which a nucleon propagates from t_2 to t' and then a vertex part from t' to t with an emission of a positive meson at t''. The positive meson, then, propagates from t'' to t_3 and the nucleon from t to t_1.

As in the previous proof, we wish to examine the part of A_k that oscillates like $\exp[-im(t_1-t_2)]$ as $t_1\to+\infty$ and $t_2\to-\infty$. This can be achieved by two different ways. On the one hand, we can express the definition of A_k (A19) in terms of the eigenstates $|n\rangle$ of the total Hamiltonian. A_k can then be written as

$$A_k=\sum_{n,m}\langle\text{vac}|\boldsymbol{\psi}_N(0)|n\rangle\langle n|\boldsymbol{\alpha}_k(0)|m\rangle\langle m|\boldsymbol{\psi}_P^\dagger(0)|\text{vac}\rangle$$
$$\times\exp[-iE_n(t_1-t_3)]\exp[-iE_m(t_3-t_2)]; \text{ if } t_1>t_3>t_2.$$

Using (A17) and (A15), we have

$$\underset{\substack{t_1\to+\infty\\ t_2\to-\infty}}{\mathcal{L}} \exp[im(t_1-t_2)]A_k=Z_2\langle\mathbf{N}|\alpha_k(0)|\mathbf{P}\rangle. \tag{A21}$$

On the other hand, by (A18) and the definitions of $\mathbf{S}'$ and $\mathbf{\Gamma}$, one has

$$\underset{t_1\to+\infty}{\mathcal{L}} \exp(imt_1)\mathbf{S}'(t_1-t)=Z_2\exp(imt),$$

$$\underset{t_2\to-\infty}{\mathcal{L}} \exp(-imt_2)\mathbf{S}'(t'-t_2)=Z_2\exp(-imt'),$$

and

$$\Gamma(m,m)=\int \mathbf{\Gamma}(t,t')\exp[im(t-t')]dtdt'.$$

By using (A20), the same limit becomes

$$\mathop{\mathcal{L}}_{\substack{t_1\to+\infty\\ t_2\to-\infty}} \exp[im(t_1-t_2)]A_k=gZ_2{}^2\Gamma(m,m)(-\omega)^{-1}(2\omega\Omega)^{-\frac{1}{2}}, \tag{A22}$$

which, together with (A21), gives

$$Z_2\Gamma(m,m)=g^{-1}(-\omega)(2\omega\Omega)^{\frac{1}{2}}\langle\mathbf{N}|\alpha_k(0)|\mathbf{P}\rangle. \tag{A23}$$

The left-hand side of (A23) is the ratio of the renormalization constants, Z_2/Z_1. The k dependence of the right-hand side can be eliminated by using the identity:

$$\langle\mathbf{N}|[\alpha_k,H]|\mathbf{P}\rangle=0.$$

By commuting α_k with the total Hamiltonian, we have

$$\langle\mathbf{N}|\alpha_k|\mathbf{P}\rangle=-g\omega^{-1}(2\omega\Omega)^{-\frac{1}{2}}\langle\mathbf{N}|\tau_-|\mathbf{P}\rangle. \tag{A24}$$

By comparing (A24) with (A23), we can express the coupling constant renormalization (A14) as

$$g_c/g=Z_2/Z_1=\langle\mathbf{N}|\tau_-|\mathbf{P}\rangle.$$

Thus g_c/g, if real, must be less than unity.

These proofs can be obviously generalized to other renormalizable field theoretical problems. In the charged scalar theory these identities can be applied to calculate formally the values of Z_1 and Z_2 by using both the weak-coupling and strong-coupling solutions.[10] They are:

(i) weak-coupling solution:

$$Z_2=1-g_c{}^2\sum(2\omega^3\Omega)^{-1}+\cdots,$$
$$Z_2/Z_1=1-g_c{}^2\sum(2\omega^3\Omega)^{-1}+\cdots;$$

(ii) strong-coupling solution:

$$Z_2=\tfrac{1}{2}\exp[-g_c{}^2\sum(2\omega^3\Omega)^{-1}]+\cdots,\quad Z_2/Z_1=\tfrac{1}{2}.$$

[10] G. Wentzel, Helv. Phys. Acta **13**, 269 (1940); **14**, 633 (1941); R. Serber and S. Dancoff, Phys. Rev. **63**, 143 (1943); S. Tomonaga, Progr. Theoret. Phys. (Japan) **1**, 109 (1946).

PHYSICAL REVIEW VOLUME 95, NUMBER 5 SEPTEMBER 1, 1954

Fierz-Pauli Theory of Particles of Spin 3/2

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(Received April 14, 1954)

The Fierz-Pauli field corresponding to particles of spin 3/2 is quantized, and its interaction with the electromagnetic field is investigated. It is also shown how the elements of the S matrix for collision processes, involving photons and charged particles of spin 3/2, can be obtained in a simple way.

1. INTRODUCTION

A THEORY of particles of arbitrary spin was first developed by Dirac,[1] Fierz and Pauli,[2] and since then several other theories have also been proposed.[3] Such theories are of special interest at the present time, because a number of new particles have been observed in recent years, and some of them may have a spin higher than one. However, except in the case of the gravitational field,[4] the interaction of a quantized field of spin higher than one with other fields has never been investigated.

We shall, therefore, discuss in some detail the Fierz-Pauli theory of particles of spin 3/2. We shall first carry out the quantization of the Fierz-Pauli field, and consider its interaction with the electromagnetic field. It will then be shown that in the present case, too, the contribution of any collision process can be obtained from the S matrix by means of simple rules, which are similar to the Feynman-Dyson[5] rules of quantum electrodynamics.

[1] P. A. M. Dirac, Proc. Roy. Soc. (London) **A155**, 447 (1936).
[2] M. Fierz and W. Pauli, Proc. Roy. Soc. (London) **A173**, 211 (1939).
[3] A specially interesting field equation for particles with two different mass states has recently been given by H. J. Bhabha, Phil. Mag. **43**, 33 (1952).
[4] S. N. Gupta, Proc. Phys. Soc. (London) **A65**, 161, 608 (1952).

2. FIELD EQUATION FOR PARTICLES OF SPIN 3/2

According to Fierz and Pauli,[2] a field corresponding to particles of spin 3/2 is described by the symmetrical spinors

$$a^{\dot\alpha}{}_{\beta\nu}=a^{\dot\alpha}{}_{\nu\beta}\quad\text{and}\quad b_\nu{}^{\dot\alpha\dot\beta}=b_\nu{}^{\dot\beta\dot\alpha}, \tag{1}$$

and the auxiliary spinors c_α and $d^{\dot\alpha}$. The Lagrangian density for the field is

$$\begin{aligned}L=&-(a^{*\nu}{}_{\dot\alpha\dot\beta}\,p^{\dot\beta\rho}\,a^{\dot\alpha}{}_{\nu\rho}+b^{*}{}_{\dot\nu}{}^{\alpha\beta}\,p_{\alpha\dot\rho}\,b_\beta{}^{\dot\nu\dot\rho})\\&+\kappa(a^{*\nu}{}_{\dot\alpha\dot\beta}\,b_\nu{}^{\dot\alpha\dot\beta}+b^{*}{}_{\dot\nu}{}^{\alpha\beta}\,a^{\dot\nu}{}_{\alpha\beta})+(p_{\dot\nu}{}^{\beta}\,d^{*\alpha}\,a^{\dot\nu}{}_{\alpha\beta}\\&+p_{\dot\beta}{}^{\nu}\,c^{*}{}_{\dot\alpha}\,b_\nu{}^{\dot\alpha\dot\beta}-a^{*\nu}{}_{\dot\alpha\dot\beta}\,p_\nu{}^{\dot\beta}\,d^{\dot\alpha}-b^{*}{}_{\dot\nu}{}^{\alpha\beta}\,p_\beta{}^{\dot\nu}\,c_\alpha)\\&+3(d^{*\alpha}\,p_{\alpha\dot\beta}\,d^{\dot\beta}+c^{*}{}_{\dot\alpha}\,p^{\dot\alpha\beta}\,c_\beta)+6\kappa(d^{*\alpha}\,c_\alpha+c^{*}{}_{\dot\alpha}\,d^{\dot\alpha}),\end{aligned} \tag{2}$$

[5] F. J. Dyson, Phys. Rev. **75**, 486, 1736 (1949).