There is only one instance in which the scattering from one element has been observed at two angles; this is the case of gold which has been studied for scattering angles 60° and 96°. It has been previously observed that the one-quarter point recipe provides a good estimate for R when $\phi=90°$. It then seems reasonable to trust the estimate for R obtained from this prescription at the neighboring angle of 96°; Farwell and Wegner find for Au at 96°, $D_{1/4}=(10.45\pm0.25)(10^{-13})$ cm.[2] When $\phi=60°$ it is now interesting to observe that, although $D_{1/4}=(10.05\pm0.16)(10^{-13})$ cm, the experimental curve is straddled by the two theoretical curves for $R=10.3$ and $10.58(10^{-13})$ cm. Thus for this single case, one radius will fit the data within experimental error at two angles.

The agreement between the shape of the experimental and theoretical cross section curves over a range of energy during which the cross section drops by more than a factor 10 suggests not only that the present semiclassical strong absorption model has more merit than its crudity would indicate, but also that it is possible to think of the alpha particle and nucleus as possessing fairly definite collision radii.

The values of R obtained by the "one-quarter point" prescription were given and discussed in the previous paper.[2] In general, these values of R yield theoretical curves whose over-all behavior is in fair agreement with the experimental cross sections. As mentioned before, however, the choice of somewhat larger R gives better agreement for $\phi=60°$ and the deviation between these values of R becomes larger as Z is decreased. It will be noted that, if one assumes a reasonable radius of the order $2(10^{-13})$ cm or less for the alpha particle, then the resulting nuclear collision radii can be fitted moderately well with the usual formula, $R_n=r_0A^{\frac{1}{3}}$, where $r_0\cong(1.5)(10^{-13})$ cm. This is in agreement with other estimates of nuclear collision radii and emphasizes the distinction between the "electromagnetic" and "nuclear force" radii.[14–16]

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Internal Bremsstrahlung*

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Some theorems are given which apply to the beta radiation and internal bremsstrahlung emitted by light nuclei. Use of these theorems simplifies the calculation of approximate spectra and angular correlations. The principal new result is a simple, explicit relation between the spectra and angular correlations of the internal bremsstrahlung of K capture and the spectra and angular correlations of positrons.

1. INTRODUCTION

The internal bremsstrahlung which accompanies beta emission has been studied by many writers, both theoretically[1] and experimentally,[2] and for allowed as well as for certain forbidden transitions. The spectra and angular correlations of this gamma ray for all cases agree quite well with the predictions of the semiclassical theory of Knipp and Uhlenbeck. The spectrum of the internal bremsstrahlung of K capture, which has been given by Morrison and Schiff,[3] is also in agreement with the measured spectra,[4] but the theory of this process, which does not have a classical analog, has been given hitherto only for allowed transitions. The principal object of this study is to examine the properties of the internal bremsstrahlung of K capture for forbidden transitions; in particular, the spectra and the angular correlation with a subsequent (nuclear) gamma ray.

We show that one can obtain as much information about electron-capturing nuclei by studying the gamma rays they emit as one can obtain about beta-emitting

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nuclei by studying their beta radiation. Experiments which make use of the internal bremsstrahlung are extremely difficult, but would provide invaluable information, as they are the only source of information about some of the properties of electron-capturing nuclei. The calculation of the spectra and angular correlations is greatly facilitated by two techniques, illustrated hereafter, which have applications to a much wider class of problems than that which is the chief concern of this paper. In Sec. 2 a way of treating the Coulomb wave functions will be introduced which is extremely convenient in the lowest approximation which does not neglect the large spin-orbit coupling effect. In Sec. 3, using this method, relationships between various radiative and nonradiative beta processes are given which enable one to obtain the spectra and angular correlations of the internal bremsstrahlung from those which have been given for beta decay.[5,6]

2. COULOMB WAVE FUNCTIONS

Those who have studied the effect of the Coulomb field on the processes of beta decay[6] have proceeded by expanding the electron wave functions in angular momentum eigenstates, and then at some point in the calculation have often made the assumption that Z is small and have neglected terms of order αZ, while keeping the dominant terms of order $\alpha Z/mR$ (m is the mass of the electron, R is the radius of the nucleus, and we let $\hbar = c = 1$). This cumbersome procedure can be circumvented easily if one is not attempting to find exact expressions, by first making the above approximation and then summing to find the approximate solutions which asymptotically have the form of plane waves (with ingoing spherical waves). More precisely, we note that it is possible to write

$$\Psi(\mathbf{r};\mathbf{p},\sigma) = A(\mathbf{r},\mathbf{p})u(\mathbf{p},\sigma), \quad (1)$$

where $\Psi(\mathbf{r};\mathbf{p},\sigma)$ is the wave function with asymptotic momentum $\mathbf{p}$ and spin σ, $A(\mathbf{r},\mathbf{p})$ is a matrix function, and $u(\mathbf{p},\sigma)$ is a plane-wave spinor. If the states are restricted to have positive energy, A has the form

$$A = f_1(\mathbf{r},\mathbf{p}) + \beta f_2(\mathbf{r},\mathbf{p}) + i\boldsymbol{\alpha}\cdot\mathbf{r}[f_3(\mathbf{r},\mathbf{p}) + \beta f_4(\mathbf{r},\mathbf{p})], \quad (2)$$

where the f's are ordinary scalar functions. We wish to find an approximate expression for A for $r < R$ and small Z, which may be done by comparing (1) with the expansion for a plane wave traveling along the z axis and then using the fact that A must be invariant in rotations.

One finds

$$A(\mathbf{r},\mathbf{p}) = 1 + i(\mathbf{p}\cdot\mathbf{r}) - \tfrac{1}{2}(\mathbf{p}\cdot\mathbf{r})^2 + i\xi(\boldsymbol{\alpha}\cdot\mathbf{r}) - \tfrac{1}{2}\xi(\boldsymbol{\alpha}\cdot\mathbf{r})(\mathbf{p}\cdot\mathbf{r}) + \cdots, \quad (3)$$

where $\xi = \alpha Z/2R$. To obtain Eq. (3) we have discarded terms strictly of order αZ or smaller, as well as terms which contribute only to third or higher forbidden transitions. We have also supposed that the velocity of the electron is greater than αZ, and that the radial wave function of a component with orbital angular momentum l is adequately represented inside the nucleus by a term of the form ar^l. The wave functions inside and outside of the nucleus have been matched in the naive fashion of equating this term ar^l to the value of the Coulomb wave function at $r = R$, which is equivalent to the usual procedure. We remark that (3) can also be obtained from the Furry wave function.[7] The matrix (3) applies to both spin states, and to states with either ingoing or outgoing scattered waves. It also applies to states with either positive or negative energy; we remember that the negative energy states are related to positive energy states by the operation of charge conjugation and observe that

$$A(\mathbf{r},\mathbf{p},Z) = CA^*(\mathbf{r}, -\mathbf{p}, -Z)C^{-1}. \quad (4)$$

In this equation the asterisk denotes the complex conjugate and C is the charge conjugation matrix. The utility of the matrix $A(\mathbf{r},\mathbf{p})$ arises from the fact that it enables one to perform all spin sums by the standard method of projection operators and spurs and thus exhibits the effect of the spin-orbit coupling in a particularly convenient form.

The interaction Hamiltonian density for beta decay may be written as

$$H_\beta = \sum_{ka} G_k(\bar{\psi}_N\lambda_a{}^k\psi_P)(\bar{\psi}_\nu\lambda_a{}^k\psi_e) + \text{Hermitian conjugate}, \quad (5)$$

where ψ_N, ψ_P, ψ_ν, and ψ_e are the field operators for the neutron, proton, neutrino, and electron fields, and $\lambda_a{}^k$ is the ath matrix of the kth covariant set of matrices in the Dirac algebra. The matrix element for emission of an electron with momentum $\mathbf{p}$, spin σ, and an antineutrino with momentum $\mathbf{q}$, spin ρ, while the nucleus makes a transition between the states with wave functions V and U, is

$$X_\beta = \bar{u}_e(\mathbf{p},\sigma)\sum_{ka} G_k\left(U, \left[\int \bar{\psi}_P\lambda_a{}^k\psi_N e^{-i\mathbf{q}\cdot\mathbf{r}} \times \bar{A}(\mathbf{r},\mathbf{p})d^3x\right]V\right)\lambda_a{}^k v(\mathbf{q},\rho)$$
$$= \bar{u}(\mathbf{p},\sigma)\Lambda v(\mathbf{q},\rho). \quad (6)$$

With $Q_m{}^{k(\alpha)}(\mathbf{r})$ and $P_m{}^{k(\alpha)}(Z,\mathbf{p},\mathbf{q})$ denoting the mth components of the irreducible tensors[8] contained in (6),

$$\Lambda = \sum_{k\alpha m} G_k\left(U, \left[\int \psi_P{}^* Q_m{}^{k(\alpha)}(\mathbf{r})\psi_N d^3x\right]V\right) \times P_m{}^{k(\alpha)}(Z,\mathbf{p},\mathbf{q}). \quad (7)$$

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For first forbidden transitions, for example, the tensor interaction gives

$$\mathbf{Q}^{T(1\alpha)}(\mathbf{r})=\beta\boldsymbol{\alpha},\quad \mathbf{P}^{T(1\alpha)}(Z,\mathbf{p},\mathbf{q})=\boldsymbol{\alpha};\quad Q^{T(0)}(\mathbf{r})=\beta\boldsymbol{\sigma}\cdot\mathbf{r},$$

$$P^{T(0)}(Z)=\tfrac{1}{3}(-i\mathbf{p}-i\mathbf{q}+i\xi\boldsymbol{\alpha})\cdot\boldsymbol{\sigma}$$
$$=P^{T(0)}(Z=0)+i\xi P^{PS}(Z=0),$$

$$\mathbf{Q}^{T(1)}=\beta\boldsymbol{\sigma}\times\mathbf{r},$$

$$\mathbf{P}^{T(1)}(Z)=-\tfrac{1}{2}(-i\mathbf{p}-i\mathbf{q}+i\xi\boldsymbol{\alpha})\times\boldsymbol{\sigma}$$
$$=\mathbf{P}^{T(1)}(Z=0)+\xi\mathbf{P}^{T(1\alpha)}(Z=0),\qquad (8a)$$

$$Q_{ij}{}^{T(2)}=\beta(\tfrac{1}{2}\sigma_i r_j+\tfrac{1}{2}\sigma_j r_i-\tfrac{1}{3}\boldsymbol{\sigma}\cdot\mathbf{r}\delta_{ij}),$$

$$P_{ij}{}^{T(2)}=\tfrac{1}{2}(-ip_i-iq_i+i\xi\alpha_i)\sigma_j$$
$$+\tfrac{1}{2}(-ip_j-iq_j+i\xi\alpha_j)\sigma_i-P^{T(0)}(Z)\delta_{ij}$$
$$=P_{ij}{}^{T(2)}(Z=0);$$

the scalar interaction gives

$$\mathbf{Q}^{S}(\mathbf{r})=\beta\mathbf{r},$$

$$\mathbf{P}^{S}(Z,\mathbf{p},\mathbf{q})=-i\mathbf{p}-i\mathbf{q}+i\xi\boldsymbol{\alpha}$$
$$=\mathbf{P}^{S}(Z=0)+i\xi\mathbf{P}^{T(1\alpha)}(Z=0).\qquad (8b)$$

For all first forbidden transitions, the terms in the P's which are proportioned to ξ are the same as other first forbidden P's for $Z=0$. The transition probabilities for all processes, which involve the P's quadratically, can therefore be obtained easily as linear combinations of corresponding transition probabilities for $Z=0$, in this approximation. Note that (8) and the similar relations for the other interactions give directly the well-known results that $\Delta I=0$, ±1 (yes) transitions will almost always have allowed spectra and allowed angular correlations, and that the $\Delta I=\pm2$ (yes) transitions are not so greatly affected by the Coulomb field. Second forbidden transitions are not quite as simple. The transition probabilities for small Z cannot always be expressed in terms of the transition probabilities for the same process in the limit $Z=0$. We easily find, however, that the ξ^2 terms in $|X_\beta|^2$ are always the same as the terms associated with the matrix elements A_{ij} and $A_{ij}{}^{\beta}$, with the neutrino momentum doubled, and the fact that there is no large Coulomb effect in $\Delta I=\pm3$ (no) transitions appears as a consequence of the anticommutativity of the Dirac matrices. Similar relations hold in nth forbidden transitions.

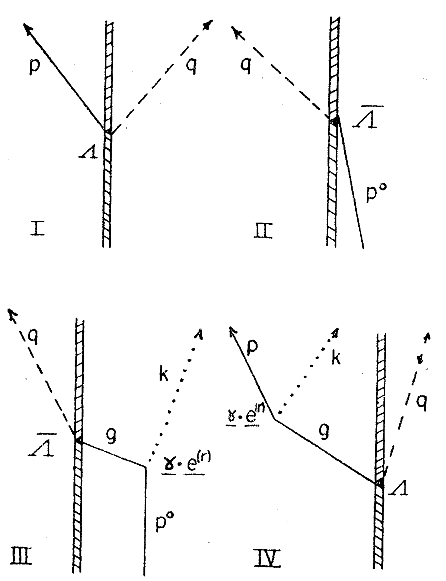

FIG. 1. Feynman diagrams for several beta-decay processes. The nucleus is supposed to be infinitely heavy and at rest.

3. INTERNAL BREMSSTRAHLUNG FOR LIGHT ELEMENTS

From Eq. (6), the beta-emission transition probability is proportional to

$$\sum|X_\beta|^2=\tfrac{1}{4}\operatorname{Tr}\left\{\int\Lambda\left(i\gamma_0-\frac{i\boldsymbol{\gamma}\cdot\mathbf{q}}{q}\right)\times\bar{\Lambda}d\Omega_q\frac{m+i\gamma_0E-i\boldsymbol{\gamma}\cdot\mathbf{p}}{E}\right\}\qquad(9)$$

$$=\tfrac{1}{4}\operatorname{Tr}\left\{N(\mathbf{p},q,\xi)\frac{m+i\gamma_0E-i\boldsymbol{\gamma}\cdot\mathbf{p}}{E}\right\}.$$

The Dirac matrix N which is thus introduced is a function of the nuclear quantum numbers and the coupling constants, as well as of $\mathbf{p}$, q, and ξ. Note that the matrix element (6) may be defined by diagram I of Fig. 1; through the use of the matrix A we may, in this approximation, consider the effect of the Coulomb field to be included in the interaction term Λ rather than in the propagation function of the electron, thus we consider the electron and neutrino to be both in plane wave states.

In the processes of radiative and nonradiative K capture we assume, in addition to the assumptions used in deriving (3), that the binding energy as well as the spread in momentum space and the small components of the K-electron wave function can be neglected, and that the gamma-emission matrix element is given by its value when $Z=0$. The errors thus introduced are all of order αZ or smaller compared to the terms which are kept, if in all continuum states the electron has a velocity $v\gg\alpha Z$.[9] If $p_\mu{}^0$ denotes the 4-momentum of an electron at rest and $\phi(0)$ the Schrödinger wave function of the K electron evaluated at the nucleus, the matrix element for the ordinary K capture, which is described

[9] The Coulomb field in intermediate states depends on whether there is an electron or a positron present. This also gives a small effect, which we disregard.

by diagram II, is

$$X_K=\bar{u}_\nu(\mathbf{q},\rho)\bar{\Lambda}u_e(0,\sigma)\phi(0), \quad (10)$$

where $\Lambda=\Lambda(0,-\mathbf{q},\xi)$; so that the transition probability is

$$P_K=(2\pi)^{-2}W^2\tfrac{1}{4}\,\mathrm{Tr}\{(1+\beta)N(0,-W,\xi)\}[\phi(0)]^2, \quad (11)$$

where W is the total energy liberated in the transition.

The radiative capture, described by diagram III, has the matrix element

$$X_{K\gamma}=ie(2\omega)^{-\frac{1}{2}}\phi(0)\bar{u}_\nu(\mathbf{q},\rho)\bar{\Lambda}\frac{i\gamma_\mu g_\mu-m}{g_\mu{}^2+m^2}\mathbf{e}^{(r)}\cdot\boldsymbol{\gamma}u_e(0,\sigma), \quad (12)$$

where $g_\mu=p_\mu{}^0-k_\mu$, $\omega\equiv k_0$, and $\mathbf{e}^{(r)}$ is the polarization vector of the emitted photon. Squaring and summing gives

$$\sum|X_{K\gamma}|^2=-\frac{2\pi\alpha}{\omega}[\phi(0)]^2\tfrac{1}{4}\,\mathrm{Tr}\left\{\left[\sum_r\frac{i\gamma_\mu g_\mu-m}{g^2+m^2}\boldsymbol{\gamma}\cdot\mathbf{e}^{(r)}\right.\right.$$

$$\left.\left.\times(1+i\gamma_0)\boldsymbol{\gamma}\cdot\mathbf{e}^{(r)}\frac{i\gamma_\mu g_\mu-m}{g^2+m^2}N(-\mathbf{k},-q,\xi)\right]\right\} \quad (13)$$

$$=2\pi\alpha\omega^{-1}m^{-2}[\phi(0)]^2\tfrac{1}{4}\,\mathrm{Tr}\{(i\gamma_0-i\boldsymbol{\gamma}\cdot\boldsymbol{\kappa})N\},$$

where $\boldsymbol{\kappa}=\omega^{-1}\mathbf{k}$. Referring to (9), we see that this is just equal to $2\pi\alpha\omega^{-1}m^{-2}[\phi(0)]^2$ times the corresponding expression for emission of a positron with momentum $\mathbf{k}$ and zero rest mass. The density of final states is also the same for a γ ray and such a positron, so the above calculation can be summarized by the following rule: Let P_γ be the probability for emission of a γ ray with momentum $\mathbf{k}$ in K capture, and let $P_{\beta+}$: be the probability for emission of a positron with the same momentum [and energy $E=(m^2+k^2)^{\frac{1}{2}}$] when the nucleus makes the same transition. The expression for $P_{\beta+}$ will of course involve the mass of the positron explicitly. *To obtain P_γ, set the rest mass equal to zero in $P_{\beta+}$, and multiply by $2\pi\alpha\omega^{-1}m^{-2}[\phi(0)]^2$.* For example, consider an allowed transition. Provided there is no Fierz interference,

$$N=4\pi i\gamma_0(\sum G^2|M_{VU}|^2), \quad (14)$$

which with (13) gives the result of Morrison and Schiff,[3] and which with (9) gives the well-known allowed beta spectrum.

All radiative corrections to beta decay can be treated in a similar fashion by this method, when the same approximations are made. Consider for example the radiative beta emission, which is given by diagram IV. The matrix element is

$$X_{\beta\gamma}=ie(2\omega)^{-\frac{1}{2}}\bar{u}_e(\mathbf{p},\sigma)\boldsymbol{\gamma}\cdot\mathbf{e}^{(r)}\frac{i\gamma_\mu g_\mu-m}{g^2+m^2}\Lambda v_\nu(\mathbf{q},\rho). \quad (15)$$

The summed square has the form

$$\sum|X_{\beta\gamma}|^2=2\pi\alpha\omega^{-1}\tfrac{1}{4}\,\mathrm{Tr}\{BN(\mathbf{p}+\mathbf{k},q,\xi)\}, \quad (16)$$

where, after some manipulation, one finds, with the notion $\mathbf{p}\cdot\mathbf{k}=pk\cos\theta$, $\Theta=1-pE^{-1}\cos\theta$:

$$B=E^{-3}\omega^{-2}\Theta^{-2}\{mp^2\sin^2\theta+i\gamma_0[(E+\omega)p^2\sin^2\theta$$
$$+E\omega^2\Theta]-i\boldsymbol{\gamma}\cdot\mathbf{p}[p^2\sin^2\theta+E\omega\Theta]$$
$$-i\boldsymbol{\gamma}\cdot\boldsymbol{\kappa}[(E+\omega)E\omega\Theta-m^2\omega]\}. \quad (17)$$

The dominant term is (with $v=p/E$):

$$B_0=\frac{v^2\sin^2\theta}{\omega^2(1-v\cos\theta)^2}\left(\frac{m+i\gamma_0E-i\boldsymbol{\gamma}\cdot\mathbf{p}}{E}\right), \quad (18)$$

which should be compared with (9). The probability for emission of a low-energy photon, in this approximation, is thus

$$dP_{\beta\gamma}=\frac{\alpha}{4\pi^2}\frac{v^2\sin^2\theta}{\omega(1-v\cos\theta)^2}d\omega d\Omega_\gamma dP_\beta, \quad (19)$$

as has been shown for some special cases.[1] The calculation leading to (19) gives a quantum-mechanical verification that the classical theory gives a good description of this process. The corrections to (19) can be determined easily from (17).

The method explained above has been used to calculate the spectra and angular correlations of the K-capture bremsstrahlung for first and second forbidden transitions and for a mixture of the scalar, tensor, and pseudoscalar interactions. These results are not given here because their tabulation would require a great deal of space and because most of them can be obtained easily from the results of others[6] as indicated above. Of more importance than these particular results is the method used, which was found to be simpler than that of Fuchs,[6] whose results have been checked. The approach of Sec. 2 enables one to see easily how the spin-orbit coupling alters the character of many forbidden transitions, and the theorems of Sec. 3 provide a complete description of the internal bremsstrahlung of K capture and beta emission for light nuclei.

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