Theory of Elastic Scattering of Alpha Particles by Heavy Nuclei*

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A semiclassical model is used to interpret recent experiments of Farwell and Wegner on the elastic scattering of 13–42 Mev alpha particles by heavy nuclei. In this model the outgoing lth partial wave is assumed to vanish if the corresponding classical turning point is less than the radius of the nucleus plus that of the alpha particle; otherwise, it has a phase characteristic of pure Coulomb scattering. The theory predicts that the sum of nuclear and alpha-particle radii is approximately equal to the classical apsidal distance evaluated at the energy for which the experimental cross section is one-quarter of the corresponding Rutherford cross section.

FARWELL and Wegner have measured the differential cross section for the elastic scattering of 13–43 Mev alpha particles as a function of energy for several heavy nuclei at a scattering angle of 60° and for gold at 96°.[1,2] Below a certain "critical energy" E_0, the cross sections have the energy dependence characteristic of pure Coulomb scattering, i.e., they follow the Rutherford formula. Above this critical energy the observed cross sections markedly drop below the pure Coulomb scattering cross section; when the logarithm of the cross section is plotted *versus* the energy, in the region above E_0 the curves are approximately straight lines with slopes which depend chiefly on scattering angle. The energy E_0 is a function of both scattering nucleus and angle. In the present note an attempt is made to explain these experimental results by means of a phase shift analysis which incorporates a crude semiclassical boundary condition at the nuclear surface.

Before discussing this model, however, let us briefly consider the scattering using the hypothesis that the motion of the alpha particle can be described classically and that the alpha particle and nucleus can be pictured as spheres with distinct radii. For classical considerations to have any validity, it is necessary that the reduced wavelength of relative motion, $\lambda\equiv(\hbar/mv)$, be much smaller than nuclear dimensions; in the present experiments, $(\lambda/R)\sim(1/20)$, so that a classical picture should have some qualitative merit. So long as the path of the alpha particle does not allow the nucleus and alpha particle to overlap, a classical alpha particle ollows the well-known hyperbolic path due to pure Coulomb repulsion. For given energy and nuclear charge, the scattered angle in the center-of-mass system, ϕ, is uniquely related to the angular momentum, p, through $p=(ZZ'e^2/v)\cot(\phi/2)$. Further ϕ is uniquely related to the distance of closest approach or apsidal distance, D, by $D=(ZZ'e^2/2E)(1+\csc(\phi/2))$. In these relations, Ze is the nuclear charge, $Z'e=2e$ is the charge of the alpha particle, v is the relative velocity, and E is the energy in the center-of-mass system.

When $D\leqq R$, where $R=R_n+R_\alpha$, the sum of nuclear and alpha-particle radii, deviations from pure Coulomb scattering are expected.[3,4] In particular, if it is assumed that the alpha particles are strongly absorbed or broken up by the nucleus, very few alpha particles should then emerge unscathed from the nucleus. Thus at a given scattering angle, there will be a marked decrease in the scattering cross section as the energy is raised above the energy at which $D=R$. If D_0, the distance of closest approach at E_0, is chosen as a measure of R, radii of a reasonable order of magnitude are obtained although these are slightly larger than the sum of radii obtained by other means;[5] for example, the critical energy for gold at $\phi=96°$ corresponds to $D\cong13(10^{-13})$ cm.[2]

Two explanations seem plausible for the fact that the experimental curves of cross section *versus* energy show a finite slope after the "break." One is that the alpha particle may have a finite mean free path in nuclear matter. The second is that the relative motion is not strictly classical, but rather there will be some quantum-mechanical spread of the trajectory. The second suggestion seems more promising to pursue since it also provides an explanation for the large apparent radii mentioned above. We therefore proceed with a quantum-mechanical discussion of the scattering process.

The quantum-mechanical expression for the differential elastic scattering cross section is well known[6] in terms of partial waves representing the relative motion of the centers of mass of the nucleus and alpha particle:

$$d\sigma=\left|\frac{\lambda}{2i}\sum_{l=0}^{\infty}(2l+1)(\eta_l-1)P_l(\cos\phi)\right|^2 d\Omega. \qquad (1)$$

In this formula, η_l is the coefficient of the outgoing lth partial wave and is determined by the boundary conditions at the nuclear surface. For pure Coulomb scattering, $\eta_l=\exp(2i\sigma_l)$, where

$$\sigma_l=\sigma_0+\sum_{l''=1}^{l}\arctan(n/l'') \quad\text{and}\quad n=(ZZ'e^2/\hbar v).$$

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[1] G. W. Farwell and H. E. Wegner, Phys. Rev. **93**, 356 (1954).
[2] G. W. Farwell and H. E. Wegner, Phys. Rev. **95**, 1212 (1954).
[3] E. Bieler, Proc. Roy. Soc. (London) **A105**, 434 (1924).
[4] E. Rutherford and J. Chadwick, Phil. Mag. **50**, 889 (1925).
[5] See J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952), pp. 13–16.
[6] See Blatt and Weisskopf (reference 5), p. 323.

Moderate energy proton scattering has been previously investigated with models in which either (a) the interior of the nucleus is represented by a complex potential, thus allowing for both refraction and absorption of the proton wave,[7–9] or (b) the strong absorption of the wave is expressed by the boundary condition that there are no outgoing waves inside the nucleus.[7,9,10] It is difficult to use these models in the present problem, however, since η_l is a function of the Coulomb wave functions evaluated at the nuclear surface; such functions are not tabulated for values of Z and E which are here needed.[11]

As an alternative, the following crude semiclassical model is adopted: First, the nucleus and the alpha particle are again considered to be spherical particles with definite radii. Second, if the potential barrier of the lth wave allows the nucleus and alpha particle to overlap when considered classically, the outgoing lth wave is assumed to vanish. Third, if the barrier is such that the nucleus and alpha particle do not classically overlap the outgoing lth wave has the phase characteristic of pure Coulomb scattering. These assumptions may be more concisely stated:

$$\begin{aligned} \eta_l &= 0 && \text{if} \quad l \leqq l', \\ \eta_l &= \exp(2i\sigma_l) && \text{if} \quad l > l', \end{aligned} \tag{2}$$

where $\hbar l'$ is defined as that angular momentum for which the classical turning point is equal to $R = R_n + R_\alpha$, i.e.,

$$E = (ZZ'e^2/R) + (\hbar^2 l'(l'+1)/2mR^2). \tag{3}$$

The crudity of the model is obvious. The drastic boundary condition when $l > l'$ neglects absorption due to penetration of the barrier. Further, the assumption $\eta_l = 0$ for $l \leqq l'$ neglects reflection due to the sudden change of "refractive index" at the nuclear surface, as well as reflection due to the potential outside the nucleus. In support of the model, however, it should be mentioned that the assumptions of model (b) above lead to $\eta_l \to 0$ when l is so small that the Coulomb wave functions approach their asymptotic amplitudes at the nuclear surface and the change in kinetic energy at the surface is small compared to the energy itself. Correspondingly, it can be demonstrated that model (b) implies that $\eta_l \to \exp(2i\sigma_l)$ when $l \gg l'$. Akhieser and Pomeranchuk[12] and, recently, Clementel and Coen[13] have used this semiclassical model to study the elastic very small angle scattering of high-energy charged particles.

When the η_l are given by Eq. (2), the differential cross section becomes:

$$d\sigma = \tfrac{1}{4}\lambda\!\!\!^{-2} \Big| [-in/\sin^2(\phi/2)] \times \exp[-in \ln \sin^2(\phi/2) + 2i\sigma_0] - \sum_{l=0}^{l'} (2l+1) \exp(2i\sigma_l) P_l(\cos\phi) \Big|^2 d\Omega. \tag{4}$$

According to Eq. (4), the scattered amplitude is simply the amplitude for Coulomb scattering minus the contribution to Coulomb scattering of all outgoing waves up to l'. The ratio of the above cross section to that for pure Coulomb scattering, $(d\sigma/d\sigma_c) \equiv G$, has been computed as a function of l' (l' running from 1 to 30) and a variety of n for $\phi = 90°$ and $60°$. For a given R, Z, and ϕ, these curves can be used to find $d\sigma$ as a function of E since $l' = l'(E,Z,R)$ and $n = n(Z,E)$. (When the ratio G is computed, l' takes on only integer values. In what follows, however, l' is treated as a continuous variable and corresponding values of G are determined by interpolation.)

Before continuing with the detailed comparison between experimental and theoretical curves, let us discuss how the present model relates to the previous classical picture; this digression will be particularly worth while since it leads to a simple prescription for estimating R.

The parameter n can be considered a measure of how "classical" is the pure Coulomb scattering since it equals a distance characteristic of the classical solution, namely D evaluated at $\phi = 180°$, divided by twice the reduced wavelength. In the present alpha-particle experiments n is of the order 10, so that for these values of n the quantum-mechanical should approach the classical description of Coulomb scattering. In particular, it is anticipated that the most important contribution to Coulomb scattering at a specific angle comes from those partial waves whose angular momentum is of the order of the classical angular momentum, p, i.e., from partial waves with l of the order $l_{cl} \equiv n \cot(\phi/2)$.

The plot of G *vs* (l'/l_{cl}) when $\phi = 90°$, shown in Fig. 1, verifies this expectation. As was mentioned earlier, $d\sigma$ can be interpreted as the cross section obtained when the contributions of all partial waves with l up to l' are sliced out of the pure Coulomb scattering amplitude. Thus the graph of G *vs* (l'/l_{cl}) shows the relative contributions of the various partial waves to Coulomb scattering. It is seen in Fig. 1 that G is of the order unity for small l', rapidly decreases for l' of the order l_{cl}, and flattens out for l' much larger than l_{cl}. Further the decrease in G is relatively more abrupt with the larger n.

In addition it is observed that all of the curves in Fig. 1 cross at $l' = l_{cl}$ and the value of G at this point is approximately (1/4). This phenomenon is interpreted to mean that the pure Coulomb scattering amplitude is "centered" about l_{cl} in the sense that half of the scatter-

[7] R. E. LeLevier and D. S. Saxon, Phys. Rev. **87**, 40 (1952).
[8] R. Britten, Phys. Rev. **88**, 283 (1952).
[9] D. M. Chase and F. Rohrlich, Phys. Rev. **94**, 81 (1954).
[10] H. Feshbach and V. F. Weisskopf, Phys. Rev. **76**, 1550 (1949).
[11] Bloch, Hull, Broyles, Bouricius, Freeman, and Breit, Revs. Modern Phys. **23**, 147 (1951).
[12] A. Akhieser and I. Pomeranchuk, J. Phys. (U.S.S.R.) **9**, 471 (1945).
[13] E. Clementel and A. Coen, Nuovo cimento **10**, 988 (1953).

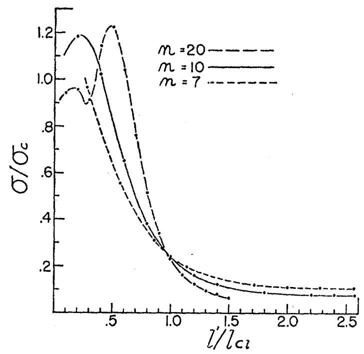

FIG. 1. Ratio of cross sections, σ/σ_c, plotted *versus* the ratio, l'/l_{c1}, for three values of n and $\phi=90°$. The ratio can be computed only for even integer values of l', indicated by the dots, and the smooth curves are interpolated.

ing amplitude is due to partial waves with $l<l_{c1}$; when these waves are removed from the scattered wave the resulting intensity is one-quarter of what it is when no partial waves are subtracted.

The above observation makes it possible to estimate R in a simple manner: The experimental cross section curves are examined to determine the energy at which $d\sigma/d\sigma_c=\frac{1}{4}$. Now it was just seen that the semiclassical strong absorption model predicts that $l'=l_{c1}$ when $d\sigma/d\sigma_c=\frac{1}{4}$ regardless of the value of n. Further, by definition, l_{c1} corresponds to the partial wave with classical turning point equal to D while l' corresponds to the partial wave whose classical turning point is R. Thus at the one-quarter point energy, $E_{1/4}$,

$$R=D_{1/4}\equiv(ZZ'e^2/2E_{1/4})[1+\csc(\phi/2)]. \quad (5)$$

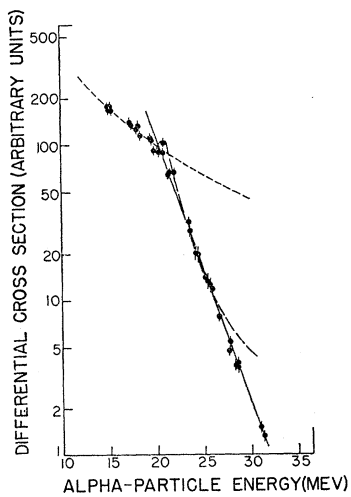

FIG. 2. Differential scattering cross section for gold at $\phi=90°$. The short dashed curve gives the Rutherford cross section; the computed cross section for $R=10.95(10^{-13})$ cm agrees with this for $E<20.5$ Mev but beyond this energy is given by broad dashed curve. The points give the experimental cross sections observed by Farwell and Wegner (reference 1) at the 90° port; the theoretical and experimental cross sections are normalized at 15 Mev. The straight line is the best straight line representing the experimental values beyond the critical energy.

The "one-quarter point" phenomenon can be described perhaps more pictorially as follows: The outgoing wave describing pure Coulomb scattering is centered about the classical trajectory in the sense that half of the contributions to the outgoing wave come from partial waves which classically penetrate closer to the origin than does the classical trajectory. When in addition to the Coulomb potential, there is present strongly absorbing nuclear matter extending to a distance $R=D$, the amplitude of the outgoing component of these waves is damped so that only half of the scattering amplitude remains, namely that contribution due to partial waves which do not classically penetrate beyond D.

For this "one-quarter point" recipe to have any meaning, the theory must also give the correct shape of the curve of cross section *vs* E, and predict the correct location of the critical energy, E_0; as will be seen later, these features are fairly well duplicated. It is also only fair to point out two inaccuracies of this one-quarter point recipe: (1) There is the familiar uncertainty of order one in l simply because quantum-mechanically the square of the angular momentum is not $(\hbar l)^2$ but $\hbar^2 l(l+1)$. Strict application of the definition of l', Eq. (3), leads to values of R which are of the order $0.2(10^{-13})$ cm larger than that given by Eq. (5). (2) For ϕ different from 90°, the plots of G *vs* (l'/n) are not so easily interpolated and thus make the analysis less certain. However, the best smooth curves at $\phi=60°$, for example, do not appear to cross exactly at $l'=l_{c1}$, and indicate that G is slightly higher than 0.25 at $l'=l_{c1}$. Thus the best over-all fits to the experimental curves at 60° are obtained with choices for R which are slightly larger than those predicted by the one-quarter point recipe. In the case of Pb, for example, the best fit is obtained for the choice $R\cong 10.6(10^{-13})$ cm while $D_{1/4}=10.26(10^{-13})$ cm.[2] The worst discrepancy between $D_{1/4}$ and the R giving the best fit occurs for the lightest element studied, Ag; in this case a fit to the data is obtained for R as large as $9.5(10^{-13})$ cm although $D_{1/4}=8.3(10^{-13})$ cm.[2]

We shall now continue with the detailed comparison of experimental and theoretical curves. In Fig. 2 the experimental cross section for scattering alpha particles from the 90° port is compared with the cross section computed for $R=10.95(10^{-13})$ cm and $\phi=90°$. The curves are normalized at 15 Mev, an energy well below the Coulomb barrier for "head-on" collisions; at such an energy, the scattering is anticipated to be purely Coulomb. The two curves agree fairly well except for large values of E. The change of slope at these energies can be traced to the leveling off of G for $l'>l_{c1}$. It is believed that this leveling off effect is due to the sharp transition from pure Coulomb phase shifts to complete absorption at $l=l'$, and is analogous to the "ringing" effect observed when a Fourier series is abruptly terminated. A more realistic boundary condition at R is expected to give a theoretical cross section which decreases more rapidly at large E than does that computed from the present model.

Some caution should be exercised when comparing the theoretical and experimental curves of Fig. 2. For one thing, it is estimated that the theoretical curve may be uncertain by as much as 5 percent due to the graphical interpolation which was necessary to obtain G as a function of E. More important, it should be pointed out that the experimental and theoretical curves of Fig. 2 correspond to somewhat different situations: (a) The theoretical curve was computed for $\phi=90°$ in the center-of-mass system while it was later found that the average experimental ϕ corresponding to the 90° port was 97° in the center-of-mass system.[1,2] (b) Moreover, the experimental ϕ changes by perhaps 3° over the energy range here considered.

The points determining G do not lie on such smooth curves when ϕ is different from 90°. This is illustrated in Fig. 3 where the points give the value of G corresponding to the various integers l' when $n=10$ and $\phi=60°$. For purposes of graphical interpolation, a smooth curve is sketched which appears to give the best fit of the scattered points. It is felt that the uncertainty introduced when this smooth curve is used for interpolation is no worse than the uncertainty of any curve designed to pass through the limited number of points. Also, it is believed that a theory which incorporates a more realistic and less sharp boundary condition will predict that these oscillations are damped. Nonetheless, one should be careful in interpreting the cross sections resulting from use of a smooth curve of G *vs* l', since the points may deviate 10 percent from the smooth curve when G is of the order unity and may deviate as much as 50 percent when G is small.

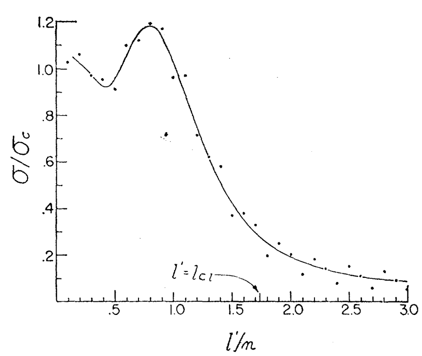

FIG. 3. Ratio of cross sections, σ/σ_c, plotted *versus* l'/n for $n=10$ and $\phi=60°$. The ratio can be computed only for integer values of l', indicated by the dots. The curve is judged to be the best smooth curve representing these points, and is the curve used for interpolation.

The comparisons of experimental and theoretical cross sections at the 60° port are shown in Fig. 4 and Fig. 5. One source of error present in the comparison at the 90° port is less important at 60°, namely, the average experimental scattering angle in the center-of-mass system is 61° which is only 1° larger than value chosen in the computations.

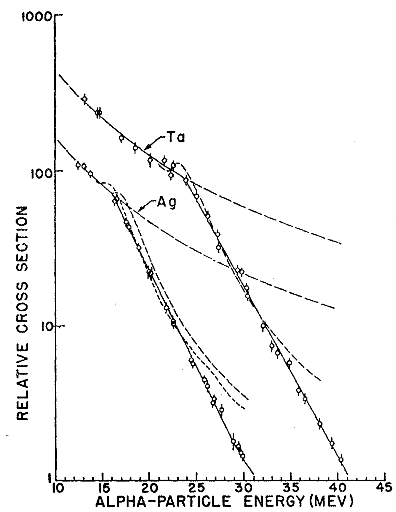

FIG. 4. Differential cross sections for Ag and Ta at $\phi=60°$ as a function of alpha-particle energy. In both cases the broad dashed curve gives the Rutherford cross section; the points and solid curves represent the experimental cross sections observed by Farwell and Wegner (see reference 2). In the case of Ta, the small dashed curve gives the theoretical cross section for $R=10.54(10^{-13})$ cm. In the case of Ag, two theoretical curves are shown; the finer dashed curve is for $R=9.67(10^{-13})$ cm and the larger dashed curve is for $R=8.84(10^{-13})$ cm. (One should not compare the relative magnitude of the Ta and Ag curves; their relative orientation is purely a matter of convenience.)

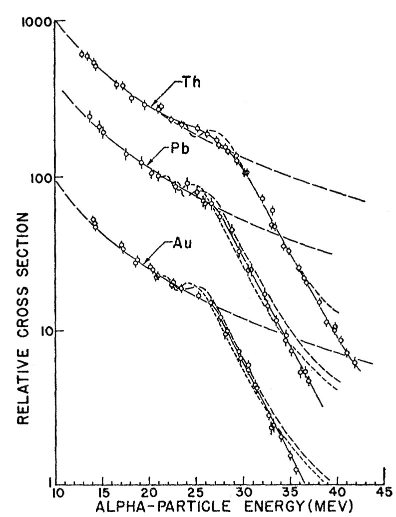

FIG. 5. Differential cross sections for Au, Pb, and Th at $\phi=60°$ as a function of alpha-particle energy. In all cases the broad dashed curves give the Rutherford cross section; the points and solid curves represent the experimental cross sections observed by Farwell and Wegner (see references 1 and 2). Two theoretical curves are shown for Au: the finer dashed curve corresponds to $R=10.58(10^{-13})$ cm while the larger dashed curve corresponds to $R=10.3(10^{-13})$ cm. Two theoretical curves are shown for Pb: the finer dashed curve corresponds to $R=10.87(10^{-13})$ cm while the larger dashed curve corresponds to $R=10.42(10^{-13})$ cm. The dashed theoretical curve in the case of Th corresponds to $R=11.01(10^{-13})$ cm.

There is only one instance in which the scattering from one element has been observed at two angles; this is the case of gold which has been studied for scattering angles 60° and 96°. It has been previously observed that the one-quarter point recipe provides a good estimate for R when $\phi = 90°$. It then seems reasonable to trust the estimate for R obtained from this prescription at the neighboring angle of 96°; Farwell and Wegner find for Au at 96°, $D_{1/4} = (10.45 \pm 0.25)(10^{-13})$ cm.[2] When $\phi = 60°$ it is now interesting to observe that, although $D_{1/4} = (10.05 \pm 0.16)(10^{-13})$ cm, the experimental curve is straddled by the two theoretical curves for $R = 10.3$ and $10.58(10^{-13})$ cm. Thus for this single case, one radius will fit the data within experimental error at two angles.

The agreement between the shape of the experimental and theoretical cross section curves over a range of energy during which the cross section drops by more than a factor 10 suggests not only that the present semiclassical strong absorption model has more merit than its crudity would indicate, but also that it is possible to think of the alpha particle and nucleus as possessing fairly definite collision radii.

The values of R obtained by the "one-quarter point" prescription were given and discussed in the previous paper.[2] In general, these values of R yield theoretical curves whose over-all behavior is in fair agreement with the experimental cross sections. As mentioned before, however, the choice of somewhat larger R gives better agreement for $\phi = 60°$ and the deviation between these values of R becomes larger as Z is decreased. It will be noted that, if one assumes a reasonable radius of the order $2(10^{-13})$ cm or less for the alpha particle, then the resulting nuclear collision radii can be fitted moderately well with the usual formula, $R_n = r_0 A^{\frac{1}{3}}$, where $r_0 \cong (1.5)(10^{-13})$ cm. This is in agreement with other estimates of nuclear collision radii and emphasizes the distinction between the "electromagnetic" and "nuclear force" radii.[14–16]

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[14] L. N. Cooper and E. M. Henley, Phys. Rev. 92, 801 (1953).
[15] V. L. Fitch and J. Rainwater, Phys. Rev. 92, 789 (1953).
[16] F. Bitter and H. Feshbach, Phys. Rev. 92, 837 (1953).

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Internal Bremsstrahlung*

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Some theorems are given which apply to the beta radiation and internal bremsstrahlung emitted by light nuclei. Use of these theorems simplifies the calculation of approximate spectra and angular correlations. The principal new result is a simple, explicit relation between the spectra and angular correlations of the internal bremsstrahlung of K capture and the spectra and angular correlations of positrons.

1. INTRODUCTION

THE internal bremsstrahlung which accompanies beta emission has been studied by many writers, both theoretically[1] and experimentally,[2] and for allowed as well as for certain forbidden transitions. The spectra and angular correlations of this gamma ray for all cases agree quite well with the predictions of the semiclassical theory of Knipp and Uhlenbeck. The spectrum of the internal bremsstrahlung of K capture, which has been given by Morrison and Schiff,[3] is also in agreement with the measured spectra,[4] but the theory of this process, which does not have a classical analog, has been given hitherto only for allowed transitions. The principal object of this study is to examine the properties of the internal bremsstrahlung of K capture for forbidden transitions; in particular, the spectra and the angular correlation with a subsequent (nuclear) gamma ray.

We show that one can obtain as much information about electron-capturing nuclei by studying the gamma rays they emit as one can obtain about beta-emitting

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[1] J. K. Knipp and G. E. Uhlenbeck, Physica 3, 425 (1936); F. Bloch, Phys. Rev. 50, 272 (1936); C. S. Wang Chang and D. L. Falkoff, Phys. Rev. 76, 365 (1949); J. M. Jauch, Oak Ridge National Laboratory Report ORNL 1102, 1951 (unpublished). Madansky, Lipps, Bolgiano, and Berlin, Phys. Rev. 84, 596 (1951).
[2] T. B. Novey, Phys. Rev. 89, 672 (1953); Bolgiano, Madansky, and Rasetti, Phys. Rev. 89, 679 (1953).
[3] P. Morrison and L. I. Schiff, Phys. Rev. 58, 24 (1940).
[4] Bradt *et al.*, Helv. Phys. Acta 19, 222 (1946); D. Maeder and P. Preiswerk, Phys. Rev. 84, 595 (1951); Anderson, Wheeler, and Watson, Phys. Rev. 87, 608 (1952), and Phys. Rev. 90, 606 (1953).