term which is

phase correction

$$=2\pi\{[(\tau/\sigma)_{\Phi=\frac{1}{2}(\Phi_0+\Phi_1)}]^{\frac{1}{2}}-[(\tau/\sigma)_{\Phi=\Phi_0}]^{\frac{1}{2}}\}$$
$$=-\pi\sqrt{2}/16k. \qquad \text{(x)}$$

A similar phase correction in the second sine term will be neglected. Combination of (u), (w), and (x) yields then, for $\Phi=\frac{1}{2}(\Phi_0+\Phi_1)$ and $k^2=q$,

$$\begin{aligned}S=2k^2-8ke^{-\sqrt{2}\pi}\sin[2\pi\sqrt{2}k+(\sqrt{2}\pi+1)/8k]\\+8ke^{-2\pi\sqrt{2}}\sin[4\pi\sqrt{2}k+O(1/k)]\\+O(1/k^2)+O(e^{-\sqrt{2}\pi}/k)\sin(2\pi\sqrt{2}k)\\+O(ke^{-3\pi\sqrt{2}})+O(e^{-3.4k}).\qquad \text{(y)}\end{aligned}$$

Inserting this value into Eq. (38) of Sec. III, we end up with the following value for the ratio of the quantum to the classical capture cross section:

$$\begin{aligned}\frac{\sigma_c}{\sigma_0}=1-\frac{4}{k}e^{-\sqrt{2}\pi}\sin[2\pi\sqrt{2}k+(\sqrt{2}\pi+1)/8k]\\+\frac{4}{k}e^{-2\pi\sqrt{2}}\sin[4\pi\sqrt{2}k+O(1/k)]\\+O\left(\frac{1}{k^4}\right)+O\left(\frac{1}{k^3}e^{-\pi\sqrt{2}}\right)\sin(2\pi\sqrt{2}k)\\+O\left(\frac{1}{k}e^{-3\pi\sqrt{2}}\right)+O\left(\frac{1}{k^2}e^{-3.4k}\right).\qquad \text{(z)}\end{aligned}$$

PHYSICAL REVIEW VOLUME 95, NUMBER 5 SEPTEMBER 1, 1954

Ionizing Collisions of Very Fast Particles and the Dipole Strength of Optical Transitions

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(Received May 21, 1954)

This note discusses how the Bethe theory of ionizing collisions should be adapted to gases other than atomic hydrogen. Experimental data on primary ionization, analyzed by the method of McClure, yield the value of the total dipole strength for ionization, $\int_I^\infty |x_W|^2 dW$, an atomic property which is otherwise poorly known and whose relationship to the diamagnetic susceptibility should be of interest.

THE primary specific ionization of fast charged particles in gases has been calculated long ago by Bethe.[1,2] His calculation applies specifically to atomic hydrogen, and its application to other gases does not seem to have been developed adequately.[3,4] McClure[4] utilized the general structure of the Bethe theory to express his experimental results—specific ionization *vs* energy of incident electrons—in terms of two empirical constants for each gas. It is proposed, in the present note, to review briefly the pertinent elements of the Bethe theory, to specify the theoretical definition of McClure's constants, and to point out that one of these constants represents an atomic property of considerable interest and is a worthwhile target for systematic experimental study.

The majority of the ionizing collisions of fast particles are of the "glancing" (or "optical") type, with large impact parameters. These collisions affect gas molecules through a short electromagnetic pulse whose spectrum has a practically uniform intensity over the frequency range of interest. The portion of this spectrum most effective in producing ionizations lies in the very far ultraviolet, where the optical properties of different substances have not yet been studied systematically. Therefore, data on ionizing collisions may complement conventional optical studies.

This connection with optical properties takes a precise form through an analysis of the *dependence of ionizing collisions on the energy* of the incident particles. When the collisions are classified according to impact parameter—more precisely, according to momentum transfer—the majority of them have a probability inversely proportional to the particle velocity squared. In addition, the maximum impact parameter depends on the velocity. This particular dependence affects only "optical" collisions and thus provides an opportunity to disentangle optical properties of gas molecules from other properties which influence the occurrence of close collisions.

A particle of charge ze and velocity $v=\beta c$ produces per cm path in a gas an average number of ionizations ("primary specific ionization") represented by the formula

$$S=(2\pi z^2e^4N/mc^2)\beta^{-2}\int_I^\infty dW\int_{Q_{\min}}^{Q_{\max}}dQ\,Q^{-2}|\eta_W(Q)|^2,\qquad (1)$$

which is an integral of (B50.8).[2,5] Here e and m are the

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[1] H. A. Bethe, Ann. Physik **5**, 325 (1930).

[2] Geiger-Scheel, *Handbuch der Physik* (J. Springer, Berlin, 1933), Vol. 24/1, p. 491 ff., which will be referred to as "B".

[3] See, e.g., B. Rossi, *High Energy Particles* (Prentice Hall, Inc., New York, 1952), p. 45.

[4] G. W. McClure, Phys. Rev. **90**, 796 (1953).

[5] A factor $E'/E=1-W/E$ has been dropped from (B50.8) because it practically equals 1 in most collisions as pointed out in reference 2.

charge and mass of the atomic electrons, N is the number of gas molecules per cm^3, W the energy absorbed by the electron ejected in a collision (which must exceed the ionization threshold I); Q is a function of the momentum Δp transferred in a collision and equals the recoil energy $\Delta p^2/2m$ of an electron assumed initially free and at rest, and $\eta_W(Q)$ is the relativistic form factor (B50.2).[2] ($|\eta_W(Q)|^2$ represents the probability that the atomic electrons absorb the energy W as a consequence of receiving the momentum transfer indicated by Q.)

The form factor η_W is not known in detail for gases other than atomic hydrogen. Therefore this discussion aims at analyzing the dependence of S on the incident particle's velocity by taking advantage of available general information about η_W. As discussed in reference 2 below Eq. (56.13), no appreciable error is incurred for our purpose by taking

$$|\eta_W(Q)|^2=|F_W(Q)|^2-|x_W|^2 2m\hbar^{-2}W^2(1-\beta^2)/2mc^2, \tag{2}$$

where F_W is the nonrelativistic form factor, and x_W is an optical constant, namely the dipole matrix element corresponding to the transition of the ejected electron. Both F_W and x_W are independent of the energy of the incident particle.

The integration limits $Q_{\min}$ and $Q_{\max}$ depend on the incident energy. However, $Q_{\max}$ can be effectively replaced by ∞ in calculations of the *number* of collisions, because large-Q collisions are exceedingly rare (though their energetic import is essential in stopping power calculations). The lower limit of integration $Q_{\min}$ is given by (B50.9),[2]

$$Q_{\min}=W^2(1-\beta^2)/2mc^2\beta^2. \tag{3}$$

The second term of (2), multiplied by Q^{-2} and integrated over Q from the limit (3) to ∞, yields

$$|x_W|^2 2m\hbar^{-2}\beta^2. \tag{4}$$

The first term of (2), $|F_W(Q)|^2$, is not known for most gas molecules. However, for the "optical" collisions, which are characterized by $Q\ll W$, it has the form

$$|F_W(Q)|^2=Q|x_W|^2 2m\hbar^{-2}. \tag{5}$$

To take advantage of this simpler form, we split the integral over Q of $|F_W|^2 dQ\,Q^{-2}$ into two sections. One section is confined to optical collisions and its value depends on the incident particle's velocity through the lower limit $Q_{\min}$. The remaining section is more difficult to calculate but is independent of the incident particle's energy. It is convenient to separate the two sections at $Q=W^2/2mc^2\ll W$, and we write, utilizing (5) and (3),

$$\begin{aligned}\int_{Q_{\min}}^{\infty}&|F_W(Q)|^2dQQ^{-2}\\&=|x_W|^2 2m\hbar^{-2}\int_{Q_{\min}}^{W^2/2mc^2}dQ\,Q^{-1}+\int_{W^2/2mc^2}^{\infty}|F_W(Q)|^2dQ\,Q^{-2}\\&=|x_W|^2 2m\hbar^{-2}\ln[\beta^2/(1-\beta^2)]+G(W),\end{aligned} \tag{6}$$

where

$$G(W)=\int_{W^2/2mc^2}^{\infty}|F_W(Q)|^2dQ/Q^2. \tag{7}$$

Equations (6), (4), and (2) together yield

$$\begin{aligned}\int_{Q_{\min}}^{\infty}dQ\,Q^{-2}|\eta_W(Q)|^2=|x_W|^2 2m\hbar^{-2}&\\\times\{\ln[\beta^2/(1-\beta^2)]-\beta^2\}+G(W).&\end{aligned} \tag{8}$$

It is convenient to express $2m/\hbar^2$ in (8) in terms of the Bohr radius $a_0=0.53\times10^{-8}$ cm and of the ionization potential of atomic hydrogen, $Ry=\hbar^2/2ma_0^2$. Accordingly we write

$$|x_W|^2 2m\hbar^{-2}=(|x_W|^2/a_0^2)Ry^{-1}, \tag{9}$$

so that the "dipole strength" $|x_W|^2$ is expressed in atomic units. Similarly, Ry plays the role of an atomic unit and should *not* be regarded as the ionization potential of the gas molecules, contrary to previous surmises.[3,4] The factor Ry^{-1} in (9) combines with the factor in parentheses in (1) to yield

$$z^2 2\pi e^4N/mc^2Ry=0.50\ z^2\ \text{cm}^{-1},\quad \text{at NTP}. \tag{10}$$

Substitution of (8) into (1), taking into account (9) and (10), yields the simplified expression of the specific ionization:

$$S=0.50z^2\beta^{-2}\{\bar{A}[\ln(\beta^2/(1-\beta^2))-\beta^2]+\bar{C}\}\ \text{cm}^{-1},\quad \text{at NTP}, \tag{11}$$

where

$$\bar{A}=\int_I^{\infty}dW(|x_W|^2/a_0^2) \tag{12}$$

is the total dipole strength of ionizing transitions in the gas molecules (expressed in atomic units) and

$$\bar{C}=Ry\int_I^{\infty}dW\,G(W) \tag{13}$$

is another molecular constant.[6,7]

The constants $\bar{A}$ and $\bar{C}$ ought to be determined conveniently from the experimental values of S at different velocities of the incident particle by plotting $2.0\beta^2z^{-2}S$ against $\{\ln[\beta^2/(1-\beta^2)]-\beta^2\}$. The plot should be a straight line with slope $\bar{A}$ and intercept $\bar{C}$.

As indicated above, $\bar{A}$ represents a property of each atom or molecule which is of considerable general interest and whose value is hardly known, experimentally or theoretically, except for atomic hydrogen.

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[6] The bar over the symbols $\bar{A}$ and $\bar{C}$ distinguishes them from the corresponding symbols of reference 4 which equal, respectively, 0.50 A cm^{-1} and 0.50 C cm^{-1}.

[7] The constant $\bar{A}$ corresponds to the quantity indicated by Za in reference 4. Factoring out the atomic number might be suggested by the related theory of stopping power but, in fact, is not suited to the theory of collision probability, because this probability should tend to increase, on the whole, in proportion to $Z^{\frac{1}{3}}$, rather than in proportion to Z.

TABLE I. Values of the parameters $\bar{A}$, $\bar{A}_{\text{tot}}$, $\bar{C}$, and $\bar{C}/\bar{A}$.

	H (theory)	H_2	He	Ne	A
$\bar{A}$	0.285	0.71±0.06	0.84±0.08	2.3±0.2	4.7±0.5
$\bar{A}_{\text{tot}}$	1	1.65	0.79	2.8	7.6
$\bar{C}$	4.07	8.70±0.04	7.96±0.04	19.3±0.2	43.6±0.4
$\bar{C}/\bar{A}$	14.3	12.2	9.45	8.53	9.20

Of considerable interest is also the comparison of $\bar{A}$ with the total dipole strength for all transitions,[8]

$$\bar{A}_{\text{tot}}=\bar{A}+\sum_n |x_n|^2/a_0^2=\langle x^2\rangle/a_0^2 =0.42\times 10^{-6}(-\chi_{\text{dia}})M. \quad (14)$$

Here x_n is the dipole matrix element for transitions from the ground state to the nth level of the discrete spectrum, the $\sum_n$ extends over the discrete spectrum, $\langle x^2\rangle$ is the mean square value for the ground state of the sum of the x coordinates of all electrons in a molecule, χ_{dia} is the diamagnetic susceptibility per unit mass of the gas, and M is the molecular weight.

The ratio $\bar{A}/\bar{A}_{\text{tot}}$ should be of the order of $\frac{1}{2}$, larger for substances with high ionization potentials and lower for substances with loosely bound electrons,[9] but there is little real information on this subject. Table I compares the values of $\bar{A}$ measured by McClure[4] with the values of $\bar{A}_{\text{tot}}$ derived from a table of susceptibilities;[10] the values of $\bar{C}$ are also listed, as well as those of $\bar{C}/\bar{A}$ which may vary less sharply from one substance to another, as pointed out by Rossi.[3] (For atomic hydrogen $\bar{C}/\bar{A}$ equals the Bethe constant 3.04 plus 2ln274.)

The values of $\bar{A}$ and $\bar{A}_{\text{tot}}$ for He coincide within the experimental error, whereas (14) implies that $\bar{A}_{\text{tot}}>\bar{A}$. However, the addition of some H_2 to He in McClure's experiment has probably caused *the excitations* produced in He to be *converted into ionizations* of H_2 and thus to be scored as ionizations. Accordingly, the experiment would yield a measurement of $\bar{A}_{\text{tot}}$ itself, rather than of $\bar{A}$. This interpretation stems from the observation of Jesse and Sadauskis[11] that the ionization yield in He (and in other gases with high excitation potentials) increases sharply in the presence even of traces of other gases. The same mechanism should presumably operate in Ne as well as in He, but here McClure's value of $\bar{A}$ amounts to only about 80 percent of $\bar{A}_{\text{tot}}$ and the interpretation of this value of $\bar{A}$ remains unclear. Argon has a lower ionization potential than He or Ne, comparable to that of H_2, but still rather high, so that the measured ratio $\bar{A}/\bar{A}_{\text{tot}}\sim 0.6$ does not appear surprising on first inspection. Similarly, the ratio $\bar{A}/\bar{A}_{\text{tot}}$ for H_2 being higher than for H seems to conform to qualitative expectation since electrons are more tightly bound in H_2 than in H.

In conclusion, however, the present brief discussion of values of $\bar{A}$ and $\bar{A}_{\text{tot}}$ should only indicate the existence of an area for further experimental and theoretical study.

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[8] See, e.g., J. H. Van Vleck, *Electric and Magnetic Susceptibilities* (Oxford University Press, London, 1932), p. 90.

[9] See, e.g., U. Fano, Phys. Rev. **70**, 44 (1946).

[10] *Handbook of Chemistry and Physics* (Chemical Rubber Publishing Company, Cleveland, 1947), thirtieth edition.

[11] W. P. Jesse and J. Sadauskis, Phys. Rev. **88**, 417 (1952). I wish to thank Professor R. L. Platzman for a discussion of his unpublished analysis of the Jesse-Sadauskis results and of its implications regarding the McClure experiments.