Holladay and R. Capps devoted much of their time to an attempt to formulate a theory based on the notion of free waves outside a nucleon of finite radius. As mentioned in the Introduction, this program did not succeed. Professor Gyo Takeda has been a constant adviser and critic of the work, and he has brought the details of Tomonaga's work to the author's attention. Professor K. M. Watson has made many useful suggestions, some of them quite essential to the completion of the program.

PHYSICAL REVIEW VOLUME 95, NUMBER 4 AUGUST 15, 1954

Application of the Intermediate Coupling Theory to the Scattering of Pseudoscalar Mesons by a Nucleon*

Gyo Takeda†
Department of Physics, University of Wisconsin, Madison, Wisconsin
(Received March 29, 1954; revised manuscript received May 21, 1954)

Tomonaga's intermediate coupling theory is applied to the scattering of pseudoscalar mesons by a nucleon. The scattering cross section depends on the cut-off momentum of the meson field and the coupling strength between the nucleon and the meson field. For reasonable values of the coupling strength, we calculated numerically the wave function of the meson field around a nucleon. By making use of these results the scattering phase shifts are calculated for various values of the cut-off momentum. Agreement with recent experimental results is not very good, although most of the qualitative features of the experiments are reproduced. A modification of the source function of the meson field can be expected to give a better fit with the experimental results.

I. INTRODUCTION

Recently Tomonaga, Maki, and Sato[1] applied the intermediate coupling theory to the problem of meson scattering by a free nucleon. Their calculation is based on the charged longitudinal vector theory for reasons of simplicity. In order to examine the results obtained in a somewhat more realistic theory, we shall apply the same method to the symmetrical pseudoscalar meson theory with pseudovector coupling.

According to the idea of Tomonaga, mesons are classified into bound mesons and unbound mesons. The former are supposed to be in a definite bound state around a core nucleon and the latter in states orthogonal to the bound state. As the function of the bound state we assume that characteristic of the classical field around a nucleon, which is also the same as the wave function of the meson field produced by a nucleon either in the limit of weak coupling or in the limit of strong coupling. This function depends on the cut-off momentum of the meson field or, in other words, on the form of the source function of a nucleon which is treated nonrelativistically.

A real nucleon is a complex system of the core nucleon and the bound mesons around it; its wave function depends on the value of the coupling strength between the core nucleon and the bound field. In case the coupling strength is weak, the probability of those states which have more than one bound meson can be neglected. On the other hand, a large coupling strength would imply that many mesons occupy the bound state around the core nucleon. When the coupling strength is neither weak nor strong, a knowledge of the bound field is obtained by solving the wave equation for the system composed of the core nucleon and the bound field. (Hereafter we shall call the system by the bound system.) In the present case, it is assumed that the probability amplitude of those states containing more than four bound mesons can be neglected. This assumption is justified in Sec. III by a numerical treatment of the wave function obtained for various values of the coupling strength.

The above equation for the bound system has solutions corresponding to excited states of the bound field in addition to the ground state solution. These excited states are unstable for a transition to a state of lower energy by an emission of mesons if we take into account the interaction between the unbound field and the bound system composed of the bound mesons and the core nucleon. Therefore it will not correspond to a real particle.

When we treat the pion-scattering by a nucleon, the unbound mesons have to be taken into account. The scattering process is expressed as a transition from the ground state of the bound system with an incoming unbound meson to a similar state through emissions and absorptions of unbound mesons by the bound system. In this case an excited state of the bound system is occupied during the collision time and when the incident energy of the incoming meson is near to an excitation energy of one of these excited states, we can expect a resonance scattering although its width is

* Supported by the U. S. Atomic Energy Commission
† On leave from Kobe University, Kobe, Japan.
[1] Tomonaga, Maki, and Sato, Progr. Theoret. Phys. (Japan) **9**, 607 (1953).

much larger than the width characteristic of a nuclear reaction.

Another kind of scattering is to be expected due to the fact that the wave function of the unbound meson is distorted by the orthogonality condition of the unbound state on the bound state. We shall call this potential scattering[2] according to Tomonaga's paper.

The resulting phase shift due to the total effect of these two different kinds of scattering is given in Sec. IV. They are calculated numerically for various values of the coupling strength and the cut-off momentum. In the weak-coupling limit these two scattering amplitudes cancel completely with each other, as they should, and as the coupling strength increases the excited states of the bound system start to have a true physical meaning so the phase shift could show a real resonance (90° phase shift) at a certain incident energy.

Especially for an excited state of the bound system having total angular momenta 3/2 in both ordinary and isotopic space is the binding energy fairly large, and for one of the values of the coupling constant chosen here we find a real resonance for the scattering via this excited state.

II. GENERAL FORMULATION

We consider the system composed of a single nucleon plus the symmetric pseudoscalar meson field interacting through the pseudovector coupling. The system is treated nonrelativistically and the nucleon's recoil is neglected. Then the total Hamiltonian of the system is

$$H=\int\{\pi^*\pi+\nabla\psi^*\cdot\nabla\psi+\mu^2\psi^*\psi +\tfrac{1}{2}[\pi_0^2+\nabla\psi_0\cdot\nabla\psi_0+\mu^2\psi_0^2]\}dV -(g/\mu)\int\{\tau_+(\boldsymbol{\sigma}\cdot\nabla)\psi+\tau_-(\boldsymbol{\sigma}\cdot\nabla)\psi^* +\tau_0(\boldsymbol{\sigma}\cdot\nabla)\psi_0\}U(x)dV. \quad (1)$$

Here $\psi_0(x)$, $\pi_0(x)$ are the (real) field variables of the neutral field and $\psi(x)$, $\pi(x)$ those of the charged field. $\boldsymbol{\sigma}$ is the Dirac matrix and τ_+, τ_-, τ_0 those matrices in charge space defined by

$$\tau_\pm=(\tau_x\pm i\tau_y)/\sqrt{2},\quad \tau_0=\tau_z; \quad (2)$$

g is the coupling constant measuring the meson-nucleon interaction and μ is the rest mass of a meson. We use the natural units $\hbar=c=1$. The extended source function $U(x)$ is assumed to be spherically symmetric and to satisfy the normalization condition

$$\int U(x)dV=1. \quad (3)$$

Since only the p-wave part of the meson field can interact with a nucleon, the bound field around a nucleon is composed of p waves. Also in the problem of a meson scattering by a nucleon only the p-wave component of the incident meson will be scattered. So we shall consider only the p-wave part of the meson field.

Expanding the field quantities into spherical waves and leaving only the p-wave part of them, we have

$$\begin{aligned}\psi(x)&\sim\sum_{m=\pm1,0}\int\frac{1}{(2k_0)^{\frac{1}{2}}}\{a_m^1(k)+(-1)^m a_{-m}^{-1*}(k)\}\times\varphi_{k,1,m}(x)dk,\\ \pi(x)&\sim\sum_{m=\pm1,0}\int i(k_0/2)^{\frac{1}{2}}\{(-1)^m a_m^{1*}(k)-a_{-m}^{-1}(k)\}\times\varphi_{k,1,-m}(x)dk,\end{aligned} \quad (4)$$

and similar equations for ψ^*, π^*, ψ_0, and π_0.

$a_m^\rho(k)$ and $a_m^{\rho*}(k)$ ($\rho=\pm1$, 0; $m=\pm1$, 0) are the creation and annihilation operators for a meson of angular momentum 1 (P wave), z component of angular momentum m, wave number k, and charge $e\rho$. These satisfy the ordinary commutation relations with each other. $\varphi_{k,1,m}$ and k_0 are defined by

$$\varphi_{k,1,m}(x)=(k/r)^{\frac{1}{2}}J_{\frac{3}{2}}(kr)Y_{1,m}(\theta,\varphi), \quad (5)$$

and

$$k_0=(k^2+\mu^2)^{\frac{1}{2}}. \quad (6)$$

$Y_{1,m}$ is the normalized spherical harmonic of the p-wave function.[3]

According to Tomonaga's idea, we divide the p-wave part of the field quantities into a bound field and an unbound field in the following way:

$$\begin{aligned}a_m^\rho(k)&=a_m^{\rho,0}\psi_0(k)+\sum_{u=1}^{\infty}a_m^{\rho,u}\psi_u(k),\\ a_m^{\rho*}(k)&=a_m^{\rho,0*}\psi_0(k)+\sum_{u=1}^{\infty}a_m^{\rho,u*}\psi_u(k),\end{aligned} \quad (7)$$

where $\psi_0(k)$ and $\psi_u(k)$ ($u=1, 2, \cdots$) compose an orthonormal set of functions in k space. $a_m^{\rho,u*}$ and $a_m^{\rho,u}$ are the creation and annihilation operator of the meson in the p orbit "u" with the z component of the angular momentum m and charge $e\rho$.

$\psi_0(k)$ is called the bound state function (0-orbit) and it is assumed that a real nucleon is a complex system of the core nucleon and bound mesons in "0 orbits" with different m and ρ. Here $\psi_0(k)$ is taken to have the form

$$\psi_0(k)=(1/3N)^{1/2}(k^2U(k)/2\pi k_0^{3/2}), \quad (8)$$

where $U(k)$ is the Fourier transform of the source function $U(x)$ and N is the normalization constant of ψ_0.

This $\psi_0(k)$ is just the orbit associated both with the classical field and with either the weak or strong coupling limit of the quantized theory in the absence of recoil. When the coupling strength is neither weak nor

[2] This kind of scattering is particularly emphasized by R. G. Sachs, Phys. Rev. **95**, 1065 (1954).

[3] E. Condon and G. Shortley, *The Theory of Atomic Spectra* (Cambridge University Press, Cambridge, 1935), p. 76.

strong, the real field around a nucleon has a different shape. But because of its simplicity which leads to some interesting features mentioned later and because of the possibility that we can take into account the deviation of ψ_0 from Eq. (8) by considering the unbound states ψ_u ($u=1, 2, \cdots$) we use ψ_0 defined by Eq. (8).

By using the expansions (4) and (7), the p-wave part of the total Hamiltonian becomes

$$H=H_0+H_u+V, \tag{9}$$

where

$$H_0=\omega_0\{\sum_{\rho,m} a_m{}^{\rho,0*}a_m{}^{\rho,0} -f\sum_{\rho,m}\tau_\rho\sigma_m(a_m{}^{\rho,0}+(-1)^m a_{-m}{}^{\rho,0*})\}, \tag{10}$$

$$H_u=\sum_{\rho,m}\sum_{u,u'}\omega_{u,u'}a_m{}^{\rho,u*}a_m{}^{\rho,u'}, \tag{11}$$

$$V=\sum_{\rho,m}\sum_{u}\{\omega_{0,u}(a_m{}^{\rho,0*}-f\tau_\rho\sigma_m)a_m{}^{\rho,u} +\omega_{0,u}{}^*(a_m{}^{\rho,0}-f(-1)^m\tau_{-\rho}\sigma_{-m})a_m{}^{\rho,u*}\}. \tag{12}$$

ω_0 $(=\omega_{0,0})$, $\omega_{0,u}$, and $\omega_{u,u'}$ are defined by

$$\omega_{u,u'}=\int k_0\psi_u{}^*(k)\psi_{u'}(k)dk \;\; (u \text{ or } u'=0, 1, 2, \cdots), \tag{13}$$

and f is a dimensionless coupling constant defined by

$$f=(gN^{1/2}/\mu). \tag{14}$$

H_0 is the Hamiltonian of the bound system and the lowest eigenstate of H_0 corresponds to an actual nucleon. H_u is the Hamiltonian of the unbound field and V is the interaction between the bound system and the unbound field.

Although a different choice of a set of unbound functions does not change the scattering amplitude or any physical quantity as far as they are orthogonal to $\psi_0(k)$ and to each other, we shall choose $\psi_u(k)$ in such a way that $\omega_{u,u'}$ (u or $u'=1, 2, \cdots$) are diagonal in u and u'.

$$\omega_{uu'}=\omega_u\delta_{u,u'} \;\; (u \text{ or } u'=1, 2, \cdots). \tag{15}$$

Then a state in which we find unbound mesons in u_1, u_2, $\cdots$, u_n orbits is an eigenstate of H_u and the energy of the state is simply a sum of ω_{u1}, ω_{u2}, $\cdots$, ω_{un}. As is already shown by Tomonaga *et al.*,[4] from the orthogonality of ψ_u and the Eq. (15) ψ_u must satisfy the following equation:

$$(k_0-\omega_u)\psi_u(k)=\psi_0(k)\int k_0\psi_0{}^*(k)\psi_u(k)dk. \tag{16}$$

III. THE BOUND FIELD OF MESONS AROUND A NUCLEON

As mentioned in Sec. II, the lowest eigenstate of H_0 corresponds to the state of an actual nucleon. Other eigenstates of H_0 corresponding to excited states of the bound system are produced artificially by restricting our attention to H_0 rather than to H and if we include the interaction V with the unbound field, these states are unstable for an emission of unbound mesons.[5] But in the problem of pion scattering by a nucleon, these states will be present during the collision time due to the forced oscillation of the bound field by an incoming meson and make an important role for the scattering problem. So here we consider the eigenvalue of H_0 and solve it numerically for the lowest eigenstate as well as for some excited states.

H_0 allows constants of motion to be specified: the total angular momentum J of the system, its z component J_z and similar quantities T and T_3 in charge space. Here we shall express a nth excited state of H_0 for given J, J_z, T and T_3 in the following way:

$$\Psi(T_3,J_z;T,J,n)=\sum_{S,L,N,\lambda_i} C(T,J,n;S,L,N,\Lambda_i) \cdot\Psi(T_3,J_z;T,J;S,L,N,\Lambda_i), \tag{17}$$

where $\psi(T_3,J_z;T,J;S,L,N,\Lambda_i)$ is a state of the bound system with the total number of bound mesons N, the angular momentum L, and isotopic spin S, of these N mesons, Actually N, L, and S do not specify the state of the bound mesons completely for given values of J, J_z, T, and T_3, so we must introduce some other quantities Λ_1, Λ_2, $\cdots$ which commute with those and with each other in order to remove the degeneracy. We can expect the appearance of the Λ-type degeneracy only for states with $N\geqq 3$ and these quantities are related to the symmetric permutation group.[6]

As H_0 does not commute with N, L, S, and Λ_i the wave equation for the bound system,

$$H_0\Psi(T_3,J_z;T,J,n)=\epsilon_{T,J,n}\Psi(T_3,J_z;T,J,n), \tag{18}$$

becomes an equation for the coefficients $C(T,J,n;S,L,N,\Lambda_i)$:

$$\begin{aligned}(\epsilon_{T,J,n}-N\omega_0)&C(T,J,n;S,L,N,\Lambda_i)\\ =-f\omega_0\sum_{S',L',\Lambda_i'}&\{C(T,J,n;S',L',N+1,\Lambda_i')\\ &\times K(S',L',N+1,\Lambda_i';S,L,N,\Lambda_i)\\ &+C(T,J,n;S',L',N-1,\Lambda_i')\\ &\times K(S',L',N-1,\Lambda_i';S,L,N,\Lambda_i)\},\end{aligned} \tag{19}$$

where $K(S',L',N',\Lambda_i';S,L,N,\Lambda_i)$ is the matrix element of $\Sigma_{\rho,m}\tau_\rho\sigma_m(a_m{}^{\rho,0}+(-1)^m a_{-m}{}^{\rho,0*})$ from a state $\psi(T_3,J_z;T,J;S',L',N',\Lambda_i')$ to a state $\psi(T_3,J_z;T,J;S,L,N,\Lambda_i)$.

[4] See reference 2, Eq. (41).

[5] We shall neglect the possibility of the presence of some stable excited states, for which there appears to be no direct experimental evidence.

[6] Here we used

$$\Lambda=(\Sigma_{\rho,m}a_m{}^{\rho,0*}a_{-m}{}^{-\rho,0*})(\Sigma_{\rho',m'}a_{m'}{}^{\rho',0}a_{-m'}{}^{-\rho',0}).$$

For states with $N\geqq 5$ or $N=4$, $S=L=2$, Λ does not completely remove the degeneracy and other Λ_i would be required. For $N=4$, $S=L=2$ we used an operator $\Lambda'=\Sigma_{\rho,m}(a^*\times a^*)_m{}^\rho(a\times a)_m{}^\rho$ instead of Λ.

TABLE I. Calculated values of $\epsilon_{T,J,n}$, $\Delta\epsilon_{T,J,n}$ and $C(T,J,n;S,L,N,\Lambda)$ for $T=J=1/2$.

$C(1/2,1/2,n;S,L,N,\Lambda)$ N,S,L,Λ	$f=0.2$ $n=0$	$n=1$	$f=0.4$ $n=0$	$n=1$	$f=0.6$ $n=0$	$n=1$
0.0.0.0	−0.8750	−0.4528	−0.7101	−0.5350	−0.5856	−0.5363
1.1.1.0	0.4636	−0.6854	0.6142	−0.1901	0.6350	0.0878
2.0.0.18	−0.0630	0.2711	−0.1567	0.4161	−0.2277	0.4121
2.1.1.0	−0.1205	0.4558	−0.2749	0.4558	−0.3695	0.2959
3.0.0.0	0.0134	−0.0861	0.0559	−0.1904	0.1023	−0.2001
3.1.1.22	0.0224	−0.1540	0.0968	−0.4124	0.1815	−0.5127
3.1.1.0	0.0160	−0.1007	0.0645	−0.2012	0.1135	−0.1947
4.0.0.44	−0.0021	0.0199	−0.0154	0.0923	−0.0366	0.1432
4.0.0.0	−0.0015	0.0130	−0.0102	0.0450	−0.0229	0.0544
4.1.1.26	−0.0042	0.0399	−0.0309	0.1753	−0.0728	0.2619
4.1.1.0	−0.0032	0.0249	−0.0198	0.0931	0.0456	0.1201
$\epsilon_{1/2,1/2,n}/\omega_0$	−0.3179	0.9081	−1.0379	0.4263	−1.9518	−0.2948
$\Delta\epsilon_{1/2,1/2,n}/\omega_0$	0	1.2260	0	1.4642	0	1.6570

For $T=J=1/2$; $T=J=3/2$; $T=1/2$, $J=3/2$ and $T=3/2$, $J=1/2$, values of K are calculated[7] by using group theory, and as an illustration we listed values of K for $T=J=1/2$ in Fig. 1.

The lowest eigenstate ($n=0$) of Eq. (17) for $T=J=1/2$ corresponds to a state of a proton or a neutron and the eigenvalue $\epsilon_{1/2,1/2,0}$ of it will give the self-energy of a nucleon, which together with the mass of the core nucleon ought to give the observed mass of a nucleon. States with a different value of T, J or n represent excited states of the bound system and the energy difference $\Delta\epsilon_{T,J,n}$ between the energy $\epsilon_{T,J,n}$ of these states and $\epsilon_{1/2,1/2,0}$ gives their excitation energy.

Equation (19) is solved numerically[8] for the ground state ($n=0$, $T=J=1/2$) and for the first excited state ($n=1$) with $T=J=1/2$; $T=J=3/2$; $T=1/2$, $J=3/2$ or $T=3/2$, $J=1/2$ under the normalization condition

$$\sum_{S,L,N,\Lambda_i} |C(T,J,n;S,L,N,\Lambda_i)|^2=1. \qquad (20)$$

Here it is assumed that the coefficients C with $N\geqq 5$ are small and can be neglected, which was justified by the numerical results obtained for various values of f chosen here.

Calculated coefficients $C(T,J,n;S,L,N,\Lambda_i)$ and eigenvalues $\epsilon_{T,J,n}$ are given in Tables I, II, and III for different values of $f=0.2$, $f=0.4$ and $f=0.6$. For a value of f chosen here and for a reasonable source function, the excitation energies are larger than the rest mass of a meson.

IV. SCATTERING OF A PION BY A NUCLEON

Scattering of a meson by a free nucleon can be treated by solving

$$(H_0+H_u+V)\Psi=E\Psi. \qquad (21)$$

In the absence of the interaction term V, a solution of Eq. (21) is a state where the bound system is in the ground state or in one of the excited states and the unbound field is composed of several "u" mesons in orbits u_1, u_2, $\cdots u_n$. In particular, a system composed of an incident pion in the asymptotic region and a free nucleon at rest is supposed to be the ground state of the "bound system" plus a single "u" meson.

The "u" functions already include the scattering caused by the orthogonality condition to ψ_0 as is seen in Eq. (16). Besides this, there is another kind of scattering due to the interaction V through an absorption or an emission of "u" mesons with a simultaneous transition of the bound system from one state to another.

We call these two different types of scattering "potential" scattering and "resonance" scattering, respectively, according to the previous paper.[1]

If we construct an effective potential for the scattering of an unbound meson by the bound system according to the method of Watson and Brueckner,[9] the wave equation for the pion scattering becomes

$$(H_0+H_u+U)\Psi=E\Psi, \qquad (22)$$

where U is the part of the following operator which is

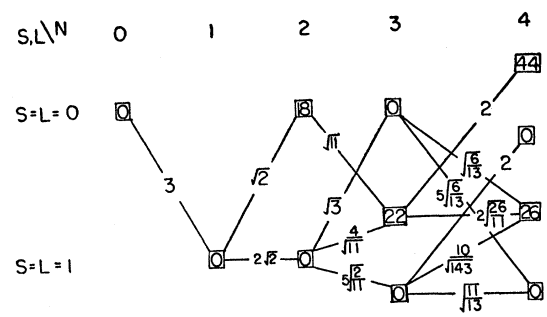

FIG. 1. Values of $K(S',L',N',\Lambda';S,L,N,\Lambda)$ for $T=J=1/2$. Each square in this figure represents a state with definite S, L, N, Λ and the value of Λ is written inside the square. A number attached to each line denotes the value of $K(S',L',N',\Lambda';S,L,N,\Lambda)$ for the corresponding transition and only nonvanishing $K(S',L',N',\Lambda';S,L,N,\Lambda)$ are written in this figure.

[7] K has the following selection rules: $\Delta N=\pm1$; $\Delta L=\pm1, 0$; $\Delta S=\pm1, 0$. (For a state $L=0$ or $S=0$, $\Delta L=0$ or $\Delta S=0$ is prohibited, respectively.) Also the values of K do not depend on T_3 and J_z.

[8] Numerical works were done by Dr. J. L. Gammel at Los Alamos Scientific Laboratory.

[9] K. A. Brueckner and K. M. Watson, Phys. Rev. **90**, 699 (1953).

TABLE II. Calculated values of $\epsilon_{T,J,n}$, $\Delta\epsilon_{T,J,n}$ and $C(T,J,n;S,L,N,\Lambda)$ for $T=1/2$, $J=3/2$ (or $T=3/2$, $J=1/2$) and $n=1$.

$C(1/2,3/2,n;S,L,N,\Lambda)$ N, S, L, Λ	$f=0.2$ $n=1$	$f=0.4$ $n=1$	$f=0.6$ $n=1$
1 1 1 0	−0.8843	−0.6970	−0.5702
2 1 1 0	−0.2482	−0.3656	−0.3966
2 0 2 0	0.3683	0.4985	0.5170
3 1 1 22	−0.0489	−0.1246	−0.1732
3 1 1 0	0.0461	0.1071	0.1410
3 1 2 0	0.1237	0.3002	0.4000
4 1 1 26	−0.0116	−0.0477	−0.0794
4 1 1 0	0.0135	0.0541	0.0886
4 0 2 26	0.0147	0.0601	0.1005
4 0 2 0	0.0104	0.0426	0.0701
4 1 2 0	−0.0167	−0.0653	−0.1064
$\epsilon_{1/2,\,3/2,\,n}/\omega_0$	0.7356	0.06374	−0.8065
$\Delta\epsilon_{1/2,\,3/2,\,n}/\omega_0$	1.0535	1.1016	1.1453

($C(3/2,1/2,n;S,L,N,\Lambda)$ is obtained from $C(1/2,3/2,n;S,L,N,\Lambda)$ by interchanging values of S with those of L.)

diagonal in the occupation numbers of "u" mesons:

$$V\frac{1}{E-H_0-H_u}V. \tag{23}$$

In Eq. (22) we neglected those parts of wave function Ψ which have more than two "u" mesons.

If we introduce the following quantities:[10]

$$\begin{aligned} A_m{}^{\rho*} &= a_m{}^{\rho,0*}-f\tau\sigma_{\rho m}, \\ A_m{}^{\rho} &= a_m{}^{\rho,0}-f\tau_{-\rho}\sigma_{-m}, \end{aligned} \tag{24}$$

H_0 and V can be written as follows:

$$H_0=\omega_0\sum_{\rho,m}A_m{}^{\rho*}A_m{}^{\rho}-9\omega_0 f^2, \tag{25}$$

$$V=\sum_{u=1,2,\cdots}\sum_{\rho,m}(\omega_{0,\,u}A_m{}^{\rho*}a_m{}^{\rho,u}+\omega_{0,\,u}{}^*A_m{}^{\rho}a_m{}^{\rho,u*}). \tag{26}$$

From the commutation relations between H_0 and $A_m{}^{\rho*}$ or $A_m{}^{\rho}$, the transition matrix of $A_m{}^{\rho*}$ or $A_m{}^{\rho}$ from an eigenstate ν of H_0 to another ν' is expressed by

$$\begin{aligned} (\nu'|A_m{}^{\rho*}|\nu) &= f\frac{\epsilon_{\nu'}-\epsilon_\nu}{\omega_0-(\epsilon_{\nu'}-\epsilon_\nu)}(\nu'|\tau_\rho\sigma_m|\nu), \\ (\nu'|A_m{}^{\rho}|\nu) &= -f\frac{\epsilon_{\nu'}-\epsilon_\nu}{\omega_0+(\epsilon_{\nu'}-\epsilon_\nu)}(\nu'|(-1)^m\tau_{-\rho}\sigma_{-m}|\nu), \end{aligned} \tag{27}$$

respectively.

Because of the factor $\epsilon_{\nu'}-\epsilon_\nu$ on the right-hand side of Eq. (27), these operators and consequently V cause only those transitions connecting two eigenstates of H_0 with different energy.[11] Numerical values of these are calculated by using the wave functions obtained in Sec. III.

Equation (22) is transformed to an equation for an amplitude $C(\nu;u,\rho,m)$ of Ψ, where the bound system is in a state ν and an unbound meson is in an u orbit having charge $e\rho$ and projection of angular momentum m:

$$(E-\epsilon_\nu-\omega_u)C(\nu;u,\rho,m) =\sum_{\nu'}\sum_{u',\rho',m'}(\nu;u,\rho,m|U|\nu';u',\rho',m') \times C(\nu';u',\rho',m'), \tag{28}$$

where

$$(\nu;u,\rho,m|U|\nu';u',\rho',m') =\omega_{0\,u}{}^*\omega_{0,\,u'}(\nu;\rho,m|W|\nu';\rho',m'), \tag{29}$$

and

$$(\nu;\rho,m|W|\nu',\rho',m')=\sum_{\nu''}\left\{\frac{(\nu|A_m{}^{\rho}|\nu'')(\nu''|A_{m'}{}^{\rho'*}|\nu')}{E-\epsilon_{\nu''}} +\frac{(\nu|A_{m'}{}^{\rho'*}|\nu'')(\nu''|A_m{}^{\rho}|\nu')}{E-\epsilon_{\nu''}-\omega_u-\omega_{u'}}\right\}. \tag{30}$$

The first term in the bracket is due to a process of absorption of a "u" meson followed by emission of a u meson and the second one due to a process in which emission precedes absorption. Hereafter we shall replace ω_u and $\omega_{u'}$ in the denominator of the second term by some average value which, for simplicity, is taken to be equal to the incident energy p_0 of the pion.[12] The total energy is equal to a sum of p_0 and $\epsilon_{1/2,\,1/2,\,0}$ (or simple ϵ_0).

If we neglect the inelastic part of the effective potential i.e., the part connecting eigenstates ν and ν' of H_0 with different energy, Eq. (28) becomes an equation for the elastic scattering amplitude $C(\nu=0;u,\rho,m)$. We shall transform the representation of the unbound field from the u representation to the ordinary momentum representation[13] and furthermore introduce the constant of motion: the total angular momentum J and total isotopic spin T of the whole system as well as their projection J_z and T_3.

TABLE III. Calculated values of $\epsilon_{T,J,n}$, $\Delta\epsilon_{T,J,n}$ and $C(T,J,n;S,L,N,\Lambda)$ for $T=J=3/2$ and $n=1$.

$C(3/2,3/2,n;S,L,N,\Lambda)$ N, S, L, $\Lambda(\Lambda')$	$f=0.2$ $n=1$	$f=0.4$ $n=1$	$f=0.6$ $n=1$
1 1 1 0	−0.8530	−0.7058	−0.6117
2 1 1—0	0.1198	0.1915	0.2326
2 2 2 0	0.4853	0.5976	0.6099
3 1 1 22	−0.0715	−0.1582	−0.2157
3 1 1 0	−0.0207	−0.0501	−0.0729
3 1 2 0	−0.0436	−0.0773	−0.0867
3 2 1 0	−0.0436	−0.0773	−0.0867
3 2 2 0	−0.1098	−0.2372	−0.3170
4 1 1 26	0.0084	0.0312	0.0533
4 1 1 0	0.0129	0.0420	0.0654
4 1 2 0	0.0059	0.0180	0.0262
4 2 1 0	0.0059	0.0180	0.0262
4 2 2 (0)	0.0106	0.0370	0.0604
4 2 2 (4)	0.0231	0.0774	0.1231
4 2 2 (6)	0.0099	0.0399	0.0717
$\epsilon_{3/2,\,3/2,\,1}/\omega_0$	0.5779	−0.2740	−1.2750
$\Delta\epsilon_{3/2,\,3/2,\,1}/\omega_0$	0.8958	0.7639	0.6768

[10] See Eq. (27) in reference 2.

[11] This is due to our special choice of ψ_0 Eq. (8).

[12] If our nonrelativistic treatment has any sense, the main contribution will come from low-energy u mesons in real or virtual states comparable with the incident energy, and so this substitution is consistent with our treatment.

[13] See reference 2, p. 618.

Then the elastic scattering amplitude $C(T_3,J_z;\,T,J,k)$ for the state with definite value of J, T, J_z, T_3 and having a meson with a momentum k is given by the following equation:

$$(p_0-k_0)C(T_3,J_z;\,T,J,k)=[-1+(k_0-\omega_0)W_{T,J}]\psi_0(k)\times\int_0^\infty k_0'\psi_0^*(k')C(T_3,J_z;\,T,J,k')dk',\quad(31)$$

where $W_{T,J}$[14] is defined as a diagonal element of W [Eq. (30)] for a state having the prescribed values of T, J and $\nu(=\nu'=0)$.

The solution of Eq. (31) under the boundary condition that only the incident incoming wave and the scattered outgoing wave are present in the asymptotic region is

$$C(T_3,J_z;\,T,J,k)=\delta(p-k)-[(k_0-p_0)^{-1}+i\pi\delta(k_0-p_0)]\times\frac{F_{T,J}(k_0)\psi_0(k)\psi_0^*(p)}{F_{T,J}(p_0)\int[(k_0'-p_0)^{-1}+i\pi\delta(k_0'-p_0)]|\psi_0(k')|^2dk'-W_{T,J}},\quad(32)$$

and the phase shift of p-wave scattering for a pure T and J state is given by

$$\tan\delta_{T,J}=-\frac{\pi p_0}{p}\times\frac{F_{T,J}(p_0)|\psi_0(p)|^2}{F_{T,J}(p_0)\int(k_0'-p_0)^{-1}|\psi_0(k')|^2dk'-W_{T,J}},\quad(33)$$

where

$$F_{T,J}(k_0)=1-(k_0-\omega_0)W_{T,J}.\quad(34)$$

If we write explicitly the energy dependence of $W_{T,J}$, we have, from Eq. (30),

$$W_{T,J}=\sum_\nu\frac{\mathfrak{A}(\nu;\,T,J)}{p_0-\Delta\epsilon_\nu}+\sum_\nu\frac{\mathfrak{B}(\nu;\,T,J)}{-p_0-\Delta\epsilon_\nu}.\quad(35)$$

$\mathfrak{A}(\nu;\,T,J)$ is the square of a matrix element of A^* from the ground state of the bound system to the excited state ν, while $\mathfrak{B}(\nu;\,T,J)$ is that of A for the same transition.

In the weak coupling limit $g\to0$ (i.e., $f\to0$), the ground state and the nth excited states of the bound system are given by states in which we find no bound meson and n bound mesons, respectively.[15] Also A^* and A become identical with a^{0*} and a^0, respectively, and can cause only those transitions in which one meson is emitted or absorbed. Therefore, the effect of A^* on the ground state of the bound system is to produce only the first excited state, and when A operates on the ground state the result vanishes. From the above consideration we find, in the limit $g\to0$,

$$\mathfrak{A}(\nu=1;\,T,J)\to1,\quad\mathfrak{A}(\nu\neq1;\,T,J)\to0,\quad\mathfrak{B}(\nu;\,T,J)\to0\quad(g\to0),\quad(36)$$

and so $F_{T,J}$ and $\delta_{T,J}$ vanish. This vanishing phase shift comes from the cancellation of two different scattering amplitudes, one due to the "potential" scattering and the other due to the "resonance" scattering, as it should be.

For a low-energy scattering only that part of W due to $\mathfrak{A}(\nu=T,\,J,\,n=1;\,T,J)$ gives the energy-dependent part of W and the other parts are almost independent of the incident energy p_0. So we replace $\Delta\epsilon_\nu$ in the latter parts by some average value $\langle\Delta\epsilon_{T,J}\rangle$,[16] and apply the closure rules to W. The necessary values of $\mathfrak{A}$ and $\mathfrak{B}$ are calculated by using the wave functions obtained in Sec. III and listed in Table IV.

The phase shifts $\delta_{3/2,3/2}$, $\delta_{1/2,3/2}$,[17] and $\delta_{1/2,1/2}$ are calculated for $f=0.4$ and $f=0.6$ by assuming various values of cut-off momenta K_c. ($K_c=3\mu$, 4μ, or 5μ; the corresponding value of ω_0 is 2.45μ, 3.07μ, or 3.71μ, respectively.) For low incident energy of a meson the phase shift for the $T=J=3/2$ state turned out to be positive, indicating the presence of an attractive force; while phase shifts for other states became negative, indicating the presence of a repulsive force. By using the calculated phase shifts, the total cross section of

TABLE IV. Values of $\mathfrak{A}(\nu;\,T,J)$ and $\mathfrak{B}(\nu;\,T,J)$.

f	$T,\ J$	$\mathfrak{A}(\nu=1;\,T,J)$	$\Sigma_\nu\mathfrak{A}(\nu=1;\,T,J)-\mathfrak{A}(\nu=1;\,T,J)$	$\Sigma_\nu\mathfrak{B}(\nu=1;\,T,J)$
0.2	1/2, 1/2	1.1955	0.4858	0.0225
	1/2, 3/2 (3/2, 1/2)	0.9982	0.0034	0.0334
	3/2, 3/2	0.9132	0.0173	0.0407
0.4	1/2, 1/2	1.0339	1.509	0.0900
	1/2, 3/2 (3/2, 1/2)	0.9239	0.0572	0.1481
	3/2, 3/2	0.7634	0.2406	0.2000
0.6	1/2, 1/2	0.6170	2.8021	0.2025
	1/2, 3/2 (3/2, 1/2)	0.4864	0.1958	0.3500
	3/2, 3/2	0.7671	0.8184	0.4952

[14] $W_{T,J}$ does not depend on T_3 and J_z because of the spherically symmetric property of W in both charge and ordinary space. So Eq. (31) has the same form for different values of T_3 and J_z.

[15] Of course, there are degeneracies for larger n, even if we specify the angular momentum, isotopic spin, and their projections.

[16] We assume $\langle\Delta\epsilon_{3/2,3/2}\rangle=2\omega_0$, $\langle\Delta\epsilon_{1/2,1/2}\rangle=3\omega_0$, and $\langle\Delta\epsilon_{1/2,3/2}\rangle=\langle\Delta\epsilon_{3/2,1/2}\rangle=3\omega_0$.

[17] $\delta_{1/2,3/2}=\delta_{3/2,1/2}$ is a direct result from the symmetric property of the interaction term in Eq. (1) for an interchange of the ordinary space with the isotopic space.

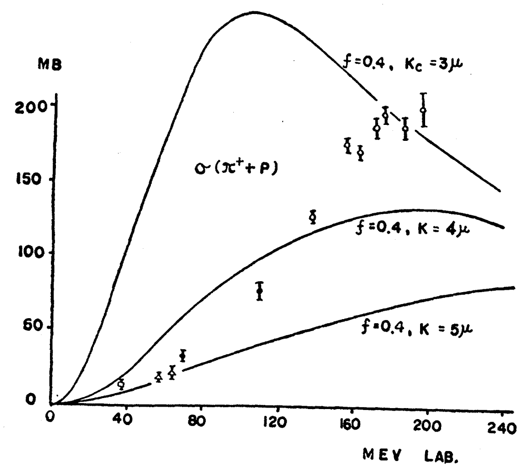

FIG. 2. Theoretical and experimental total cross section *vs* the incident energy of a pion for $\pi^+ + p$ scattering. ○: Ashkin, Blaser, Feiner, Gorman, and Stern, reports at The Fourth Annual Rochester High-Energy Conference in January, 1954 (University of Rochester, Rochester, to be published); ●: Anderson, Fermi, Martin, and Nagle, Phys. Rev. **91**, 155 (1953); Δ: Bodansky, Sachs, and Steinberger, Phys. Rev. **91**, 467 (1953); □: J. P. Perry and C. E. Angell, Phys. Rev. **91**, 1289 (1953).

$\pi^+ - p$ scattering and $\pi^- - p$ scattering (including the charge exchange scattering) are calculated, and the results are shown in Figs. 2 and 3 for $f=0.4$. The agreement with the experimental results is rather poor although most of the qualitative feature of the cross sections are obtained. The results for $f=0.6$ give worse agreement than those for $f=0.4$. As we neglected the S-wave scattering, our results cannot be applied to the scattering near zero energy.

The excited energies $\Delta\epsilon_{T,J,n}$ are proportional to ω_0 for a given value of the coupling constant f, and so, for a larger cut-off momentum K_c, $\Delta\epsilon_{T,J,n}$ become larger. The shift of the maximum in the phase shifts to higher energy with increasing cut-off momentum is explained by this increase of $\Delta\epsilon_{T,J,n}$. A real resonance is obtained when the denominator of the right-hand side of Eq. (33) becomes zero for a certain incident energy p_0, which is different from $\Delta\epsilon_{T,J,n}$. For example we find the scattering resonance for $T=J=3/2$ at a little higher energy p_0 than $\Delta\epsilon_{3/2,3/2,n}$.

For small incident energies ($\lesssim 200$ Mev) the predominant phase-shift is $\delta_{3/2,3/2,n}$. A linearly increasing character of $\delta_{1/2,1/2}$, $\delta_{1/2,3/2}$, and $\delta_{3/2,1/2}$ with increasing incident energy between 100 and 200 Mev was found, which is due to a factor $\psi_0^2(p)$ in Eq. (74); hence it is a characteristic of the bound wave function chosen here. For a more slowly increasing function $\psi_0^2(p)$, these phase shifts can remain smaller than calculated for the energy interval involved here; also a steeper decrease of the total scattering cross section from the maximum predicted by experiment will be obtained. The broader maximum found for the scattering cross section compared to the experimental results indicates that the attractive force is not strong enough. This may be due partly to the fact that in our approximation the attractive force is not fully taken into account, because the actual wave function of the bound system contains mesons in unbound orbits and the excited energy will be smaller than our calculated values.

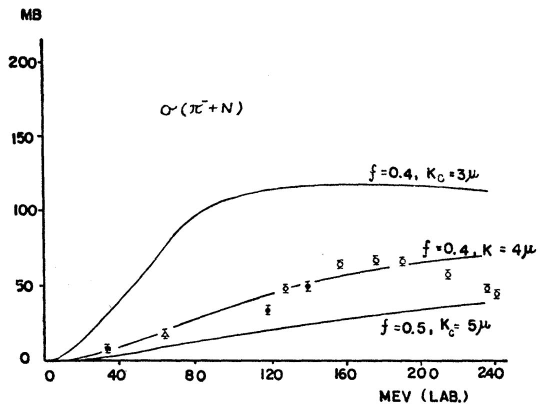

FIG. 3. Theoretical and experimental total cross section *vs* the incident energy of a pion for $\pi^- + p$ scattering including charge exchange scattering. ○: Ashkin, Blaser, Feiner, Gorman, and Stern, reports at the Fourth Annual Rochester High-Energy Conference in January, 1954 (University of Rochester, Rochester, to be published); ●: Anderson, Fermi, Martin, and Nagle, Phys. Rev. **91**, 155 (1953); Δ: Bodansky, Sachs, and Steinberger, Phys. Rev. **91**, 467 (1953); ■: J. Ring and D. N. Nelson, Phys. Rev. **91**, 1289 (1953).

We wish to express our very sincere appreciation to Dr. J. L. Gammel for his excellent computational work on Eq. (19), and Dr. Y. Miyamoto, Dr. D. Ito, Dr. B. A. Jacobsohn, and Dr. T. D. Lee for private discussions about the content of Sec. III. We are also indebted to Professor S. Tomonaga for informing us of his paper before publication and to Professor R. G. Sachs and Professor K. M. Watson for their kind interest and valuable discussions.