Conservation Theorems in Modified Electrodynamics*

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A system of interacting particles with variable rest masses is considered, whose motion is governed by a variational principle of general form, in which there are self-action terms instead of the usual specifically inertial terms. For suitable restrictions of the action principle it is shown that, when the particles are at distances large compared to the classical electron radius, the new theory reduces to classical electrodynamics and the variable rest masses become constants of the motion. The conservation laws of energy momentum and of angular momentum are derived from the Lorentz invariance of the general action principle.

1. INTRODUCTION

IN a recent paper[1] one of us proposed an action at a distance theory of interacting particles with variable rest masses. Here we wish to consider theories of this type based on action integrals of the general form,

$$J_*^{**}=\tfrac{1}{2}\sum_{ab}\int_{u_a^*}^{u_a^{**}}\int_{u_b^*}^{u_b^{**}}\Lambda_{ab}du_a du_b, \tag{1}$$

$$\Lambda_{ab}=\Lambda(\xi_{ab}{}^\mu,\dot{x}_a{}^\mu,\dot{x}_b{}^\mu). \tag{2}$$

The notation is as follows: Latin subscripts label the different particles of the system, Greek suffixes label space-time components and are subject to the summation convention, $x_a{}^\mu$ are the space-time coordinates of particle a,

$$\xi_{ab}{}^\mu=x_a{}^\mu-x_b{}^\mu, \tag{3}$$

$$\dot{x}_a{}^\mu=dx_a{}^\mu/du_a, \tag{4}$$

and u_a is a physically significant parameter along the world line of particle a. In terms of this parameter the variable rest mass m_a is defined by[2]

$$m_a{}^2=\dot{x}_a{}^\mu\dot{x}_{a\mu}. \tag{5}$$

In Eq. (1), $u_a{}^*$ refers to a set of points obtained by choosing a point on each of the world lines of the system, and $u_a{}^{**}$ refers to any other such set, the only restriction being that $u_a{}^{**}\geqslant u_a{}^*$.

In Eq. (1), the double summation includes self-action terms for which $a=b$. Here, and throughout, such a term is to be interpreted as

$$\tfrac{1}{2}\int_{u_a^*}^{u_a^{**}}\int_{u_a^*}^{u_a^{**}}\Lambda(\xi_{aa'}{}^\mu,\dot{x}_a{}^\mu,\dot{x}_{a'}{}^\mu)du_a du_a{}', \tag{6}$$

where u_a and $u_a{}'$ are the parameters of two independent points $x_a{}^\mu$ and $x_{a'}{}^\mu$ on the same world line a, and

$$\xi_{aa'}{}^\mu=x_a{}^\mu-x_{a'}{}^\mu,\quad \dot{x}_{a'}{}^\mu=dx_{a'}{}^\mu/du_a{}'. \tag{7}$$

Because of the double summation in Eq. (1) and the double integration in the self-action terms (6), we can assume, without loss of generality, that the interaction function Λ_{ab} is symmetric in the two particles:

$$\Lambda(\xi_{ab}{}^\mu,\dot{x}_a{}^\mu,\dot{x}_b{}^\mu)=\Lambda(-\xi_{ab}{}^\mu,\dot{x}_b{}^\mu,\dot{x}_a{}^\mu),$$
$$\Lambda_{ab}=\Lambda_{ba}. \tag{8}$$

We also assume that Λ is Lorentz invariant, so that the theory will be relativistic.

2. VARIATION AND EQUATIONS OF MOTION

Consider a variation of the world lines of the particles and of their parametrizations:

$$x_a{}^\mu(u_a)\rightarrow x_a{}^\mu(u_a)+\delta x_a{}^\mu(u_a). \tag{9}$$

Equation (5) shows that such a variation implies not only a variation of the classical variables of motion, but also a variation of the rest masses of the particles.

The variation of the action integral is given by

$$\delta J_*^{**}=\sum_{ab}\left[\int_{u_b^*}^{u_b^{**}}\frac{\partial\Lambda_{ab}}{\partial\dot{x}_a{}^\mu}du_b\delta x_a{}^\mu\right]_{u_a^*}^{u_a^{**}}+\sum_{ab}\int_{u_a^*}^{u_a^{**}}\int_{u_b^*}^{u_b^{**}}\Lambda_{ab\mu}\delta x_a{}^\mu du_a du_b, \tag{10}$$

where

$$\Lambda_{ab\mu}=\frac{\partial\Lambda_{ab}}{\partial\xi_{ab}{}^\mu}-\frac{d}{du_a}\frac{\partial\Lambda_{ab}}{\partial\dot{x}_a{}^\mu}. \tag{11}$$

The equations of motion are determined by the action principle

$$\delta J=0. \tag{12}$$

Here J is the total action, obtained from J_*^{**} by putting all

$$u_a{}^*=-\infty,\quad u_a{}^{**}=+\infty. \tag{13}$$

Equation (12) is to hold for arbitrary variations $\delta x_a{}^\mu$ which vanish identically outside arbitrary but finite intervals $(\bar{u}_a, \bar{\bar{u}}_a)$ on their respective world lines. From Eq. (10) we find that the *equations of motion* of particle a are

$$\sum_b\int_{-\infty}^{\infty}\Lambda_{ab\mu}du_b\equiv\sum_b\int_{-\infty}^{\infty}\left(\frac{\partial\Lambda_{ab}}{\partial\xi_{ab}{}^\mu}-\frac{d}{du_a}\frac{\partial\Lambda_{ab}}{\partial\dot{x}_a{}^\mu}\right)du_b=0. \tag{14}$$

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[1] A. Schild, Phys. Rev. **92**, 1009 (1953).

[2] The metric of space time is $ds^2=\eta_{\mu\nu}dx^\mu dx^\nu=-(dx^1)^2-(dx^2)^2-(dx^3)^2+(dx^4)^2$.

3. CONSERVATION OF LINEAR MOMENTUM AND ENERGY

From the Lorentz invariance of the action integral it follows that

$$\delta J_*^{**}\equiv 0 \tag{15}$$

is satisfied identically for variations δx_a^μ which are induced by infinitesimal Lorentz transformations. This gives rise to ten identities. In this section and the next it will be shown that, by virtue of the equations of motion, these identities can be put into the form of conservation theorems. Because of the difference between the action J_*^{**} in Eq. (15) and the total action J in Eq. (12), this last fact is not obvious *a priori*, as it is in classical mechanics or in a field theory.

To obtain the conservation laws of linear momentum and energy, we consider the variations induced by an infinitesimal space-time translation,

$$\delta x_a^\mu = \epsilon^\mu, \tag{16}$$

where ϵ^μ is a set of four constants. By Eqs. (10) and (15), this gives the identities

$$\sum_{ab}\left[\int_{u_b^*}^{u_b^{**}}\frac{\partial\Lambda_{ab}}{\partial\dot{x}_a^\mu}du_b\right]_{u_a^*}^{u_a^{**}}+\sum_{ab}\int_{u_a^*}^{u_a^{**}}\int_{u_b^*}^{u_b^{**}}\times\Lambda_{ab\mu}du_adu_b\equiv 0. \tag{17}$$

If the equations of motion (14) are used, this becomes

$$\sum_{ab}\left[\int_{u_b^*}^{u_b^{**}}\frac{\partial\Lambda_{ab}}{\partial\dot{x}_a^\mu}du_b\right]_{u_a^*}^{u_a^{**}}-\sum_{ab}\int_{u_a^*}^{u_a^{**}}\left(\int_{-\infty}^{u_b^*}+\int_{u_b^{**}}^{\infty}\right)\times\Lambda_{ab\mu}du_adu_b=0. \tag{18}$$

On substituting from (11) and integrating by parts, this simplifies to

$$\sum_a\left[\sum_b\int_{-\infty}^{\infty}\frac{\partial\Lambda_{ab}}{\partial\dot{x}_a^\mu}du_b\right]_{u_a^*}^{u_a^{**}}-\sum_{ab}\int_{u_a^*}^{u_a^{**}}\left(\int_{-\infty}^{u_b^*}+\int_{u_b^{**}}^{\infty}\right)\frac{\partial\Lambda_{ab}}{\partial\xi_{ab}^\mu}du_adu_b=0. \tag{19}$$

By (8) and (3), the integrand $\partial\Lambda_{ab}/\partial\xi_{ab}^\mu$ is skew-symmetric in a and b. It follows that the integral operator in the second term of Eq. (19) can be skew-symmetrized, since there is a double summation over a and b. By doing this and using the operator identity

$$\int_{u_a^*}^{u_a^{**}}\left(\int_{-\infty}^{u_b^*}+\int_{u_b^{**}}^{\infty}\right)-\left(\int_{-\infty}^{u_a^*}+\int_{u_a^{**}}^{\infty}\right)\int_{u_b^*}^{u_b^{**}}\equiv\left(\int_{u_a^*}^{\infty}\int_{-\infty}^{u_b^*}-\int_{-\infty}^{u_a^*}\int_{u_b^*}^{\infty}\right)-\left(\int_{u_a^{**}}^{\infty}\int_{-\infty}^{u_b^{**}}-\int_{-\infty}^{u_a^{**}}\int_{u_b^{**}}^{\infty}\right), \tag{20}$$

Eq. (19) becomes

$$\left[\sum_{ab}\int_{-\infty}^{\infty}\frac{\partial\Lambda_{ab}}{\partial\dot{x}_a^\mu}du_b+\tfrac{1}{2}\sum_{ab}\left(\int_{u_a}^{\infty}\int_{-\infty}^{u_b}-\int_{-\infty}^{u_a}\int_{u_b}^{\infty}\right)\times\frac{\partial\Lambda_{ab}}{\partial\xi_{ab}^\mu}du_adu_b\right]_*^{**}=0. \tag{21}$$

Since the u_a^{**} and the u_a^* are two independent sets of points, we obtain the laws of *conservation of linear momentum and energy:*

$$\sum_{ab}\int_{-\infty}^{\infty}\frac{\partial\Lambda_{ab}}{\partial\dot{x}_a^\mu}du_b+\tfrac{1}{2}\sum_{ab}\left(\int_{u_a}^{\infty}\int_{-\infty}^{u_b}-\int_{-\infty}^{u_a}\int_{u_b}^{\infty}\right)\times\frac{\partial\Lambda_{ab}}{\partial\xi_{ab}^\mu}du_adu_b=\text{constant}, \tag{22}$$

i.e., the left side is independent of the choice of the points u_1, u_2, u_3, $\cdots$, on each of the world lines of the system.

4. CONSERVATION OF ANGULAR MOMENTUM

We consider the variations δx_a^μ induced by an infinitesimal space-time rotation

$$\delta x_a^\mu=\epsilon^\mu{}_\nu x_a^\nu, \tag{23}$$

where $\epsilon_{\mu\nu}=\eta_{\mu\rho}\epsilon^\rho{}_\nu$ is a set of six skew-symmetric constants:

$$\epsilon_{\mu\nu}=-\epsilon_{\nu\mu}. \tag{24}$$

First we establish a simple identity which will be required later. From the Lorentz invariance of the interaction function Λ we obtain

$$\Lambda(\xi_{ab}^\mu+\epsilon^\mu{}_\nu\xi_{ab}^\nu,\ \dot{x}_a^\mu+\epsilon^\mu{}_\nu\dot{x}_a^\nu,\ \dot{x}_b^\mu+\epsilon^\mu{}_\nu\dot{x}_b^\nu)\equiv\Lambda(\xi_{ab}^\mu,\ \dot{x}_a^\mu,\ \dot{x}_b^\mu),$$

correct to the first order in $\epsilon^\mu{}_\nu$. Expanding in powers of $\epsilon_{\mu\nu}$ and retaining first powers only, we find

$$\epsilon_{\mu\nu}\left(\frac{\partial\Lambda_{ab}}{\partial\xi_{ab\mu}}\xi_{ab}^\nu+\frac{\partial\Lambda_{ab}}{\partial\dot{x}_{a\mu}}\dot{x}_a^\nu+\frac{\partial\Lambda_{ab}}{\partial\dot{x}_{b\mu}}\dot{x}_b^\nu\right)\equiv 0.$$

This must hold for arbitrary skew symmetric $\epsilon_{\mu\nu}$ and therefore the bracketed expression must be symmetric in μ and ν. Writing this down and rearranging terms, we find that

$$\frac{\partial\Lambda_{ab}}{\partial\xi_{ab\mu}}x_a^\nu-\frac{\partial\Lambda_{ab}}{\partial\xi_{ab\nu}}x_a^\mu+\frac{\partial\Lambda_{ab}}{\partial\dot{x}_{a\mu}}\dot{x}_a^\nu-\frac{\partial\Lambda_{ab}}{\partial\dot{x}_{a\nu}}\dot{x}_a^\mu\equiv-\left(\frac{\partial\Lambda_{ab}}{\partial\xi_{ba\mu}}x_b^\nu-\frac{\partial\Lambda_{ab}}{\partial\xi_{ba\nu}}x_b^\mu+\frac{\partial\Lambda_{ab}}{\partial\dot{x}_{b\mu}}\dot{x}_b^\nu-\frac{\partial\Lambda_{ab}}{\partial\dot{x}_{b\nu}}\dot{x}_b^\mu\right), \tag{25}$$

i.e., the left side is skew symmetric in the particle variables a and b.

Substituting (23) in Eqs. (10) and (15), we obtain

$$\sum_{ab}\left[\int_{u_b*}^{u_b**}\left(\frac{\partial\Lambda_{ab}}{\partial\dot{x}_{a\mu}}x_a^\nu-\frac{\partial\Lambda_{ab}}{\partial\dot{x}_{a\nu}}x_a^\mu\right)du_b\right]_{u_a*}^{u_a**}$$
$$+\sum_{ab}\int_{u_a*}^{u_a**}\int_{u_b*}^{u_b**}(\Lambda_{ab}{}^\mu x_a^\nu-\Lambda_{ab}{}^\nu x_a^\mu)du_adu_b\equiv 0. \quad (26)$$

If the equations of motion (14) and integration by parts are used, this becomes

$$\sum_a\left[\sum_b\int_{-\infty}^{\infty}\left(\frac{\partial\Lambda_{ab}}{\partial\dot{x}_{a\mu}}x_a^\nu-\frac{\partial\Lambda_{ab}}{\partial\dot{x}_{a\nu}}x_a^\mu\right)du_b\right]_{u_a*}^{u_a**}$$
$$-\sum_{ab}\int_{u_a*}^{u_a**}\left(\int_{-\infty}^{u_b*}+\int_{u_b**}^{\infty}\right)\left(\frac{\partial\Lambda_{ab}}{\partial\xi_{ab\mu}}x_a^\nu-\frac{\partial\Lambda_{ab}}{\partial\xi_{ab\nu}}x_a^\mu\right.$$
$$\left.+\frac{\partial\Lambda_{ab}}{\partial\dot{x}_{a\mu}}\dot{x}_a^\nu-\frac{\partial\Lambda_{ab}}{\partial\dot{x}_{a\nu}}\dot{x}_a^\mu\right)du_adu_b=0. \quad (27)$$

Since, by Eq. (25), the last integrand is skew symmetric in a and b, we can now proceed exactly as we did from Eq. (19) to Eq. (22). We obtain the law of *conservation of angular momentum:*

$$\sum_{ab}\int_{-\infty}^{\infty}\left(\frac{\partial\Lambda_{ab}}{\partial\dot{x}_{a\mu}}x_a^\nu-\frac{\partial\Lambda_{ab}}{\partial\dot{x}_{a\nu}}x_a^\mu\right)du_b$$
$$+\tfrac{1}{2}\sum_{ab}\left(\int_{u_a}^{\infty}\int_{-\infty}^{u_b}-\int_{-\infty}^{u_a}\int_{u_b}^{\infty}\right)\left(\frac{\partial\Lambda_{ab}}{\partial\xi_{ab\mu}}x_a^\nu-\frac{\partial\Lambda_{ab}}{\partial\xi_{ab\nu}}x_a^\mu\right.$$
$$\left.+\frac{\partial\Lambda_{ab}}{\partial\dot{x}_{a\mu}}\dot{x}_a^\nu-\frac{\partial\Lambda_{ab}}{\partial\dot{x}_{a\nu}}\dot{x}_a^\mu\right)du_adu_b=\text{constant}, \quad (28)$$

i.e., the left side is independent of the choice of the points u_1, u_2, u_3, $\cdots$, on each of the world lines of the system.

5. CLASSICAL ELECTROMAGNETIC THEORY

Under suitable conditions on the interaction function Λ, and for charged particles with moderate accelerations which interact at distances large compared to 10^{-13} cm, our theory reduces to classical electrodynamics and, in particular, the rest mass of each particle is a constant of the motion. We shall discuss this for the restricted class of interaction functions of the form

$$\Lambda_{ab}=e_ae_b\dot{x}_a^\mu\dot{x}_{b\mu}f(\xi_{ab}^2,\dot{x}_a^\mu,\dot{x}_b^\mu), \quad (29)$$

$$\xi_{ab}^2=\xi_{ab}^\mu\xi_{ab\mu},\quad |\xi_{ab}|=|(\xi_{ab}^\mu\xi_{ab\mu})^{\frac{1}{2}}|. \quad (30)$$

Here e_a is the charge of a fundamental particle a. The units are chosen so that $e_a=\pm1$ for all charged particles, and the velocity of light $c=1$. All dimensions are expressible in terms of cm, and the rest mass of an electron is

$$m_0=3.6\times10^{12}\text{ cm}^{-1}. \quad (31)$$

We shall use the empirical fact that there exists no charged particle in nature with a rest mass smaller than that of the electron, so that

$$1/\dot{x}_a=1/m_a\leqslant 1/m_0=2.8\times10^{-13}\text{ cm}, \quad (32)$$

$$\dot{x}_a=|(\dot{x}_a^\mu\dot{x}_{a\mu})^{\frac{1}{2}}|. \quad (33)$$

We require that the structure function f have the following properties:

1. It is Lorentz-invariant and symmetric in the two particles [see Eq. (8)]:

$$f(\xi_{ab}^2,\dot{x}_a^\mu,\dot{x}_b^\mu)=f(\xi_{ab}^2,\dot{x}_b^\mu,\dot{x}_a^\mu). \quad (34)$$

2. With respect to its first variable ξ^2, f must approximate a δ function. It must be normalized such that

$$\int_{-\infty}^{\infty}f(\xi^2,\dot{x}_a^\mu,\dot{x}_b^\mu)d\xi^2=1 \quad (35)$$

for all $\dot{x}_a^\mu$, $\dot{x}_b^\mu$. We assume that f (and its integral with respect to ξ^2) is negligibly small outside an interval of order $1/\dot{x}_a$ or $1/\dot{x}_b$ about $\xi=0$:

$$f(\xi^2,\dot{x}_a^\mu,\dot{x}_b^\mu)\sim 0\quad\text{for}\quad|\xi|>k\max\left(\frac{1}{\dot{x}_a},\frac{1}{\dot{x}_b}\right), \quad (36)$$

where k is of the order of magnitude 1. From the normalization condition (35) it follows that f must be large somewhere in the range $|\xi|\leqslant k\max(1/\dot{x}_a,1/\dot{x}_b)$. By Eq. (32), this is an interval of the order of magnitude of 10^{-13} cm or less.

3. When $a=b$, f must satisfy the condition

$$\int_{-\infty}^{\infty}f(\xi^2,\dot{x}_a^\mu,\dot{x}_a^\mu)d\xi=\dot{x}_a, \quad (37)$$

for all $\dot{x}_a^\mu$.

Some examples of structure functions which satisfy these three conditions are:

$$f_{ab}=\delta(\xi_{ab}^2-\lambda_{ab}^2), \quad (38)$$

$$f_{ab}=\frac{1}{2\lambda_{ab}^2}\exp(-|\xi_{ab}|/|\lambda_{ab}|)\quad\text{for}\quad\xi_{ab}^2\geqslant0$$
$$=0\qquad\text{for}\quad\xi_{ab}^2<0, \quad (39)$$

where, in each case, λ_{ab} can be any one of the following expressions:

$$\lambda_{ab}^2=1/(\dot{x}_a^\mu\dot{x}_{b\mu}), \quad (40)$$

$$\lambda_{ab}^2=e_ae_b/(\dot{x}_a^\mu\dot{x}_{b\mu}), \quad (41)$$

$$\lambda_{ab}^2=2/(\dot{x}_a^2+\dot{x}_b^2), \quad (42)$$

$$\lambda_{ab}^2=2e_ae_b/(\dot{x}_a^2+\dot{x}_b^2), \quad (43)$$

$$\lambda_{ab}^2=4/[(\dot{x}_a^\mu+\dot{x}_b^\mu)(\dot{x}_{a\mu}+\dot{x}_{b\mu})], \quad (44)$$

etc. Any two time-like vectors in Minkowski space satisfy the reversed Schwarz's inequality,

$$(A_\mu B^\mu)^2 \geqslant A_\mu A^\mu B_\nu B^\nu. \quad (45)$$

From this and Eq. (32) it follows that any of the λ_{ab} given above are small,

$$|\lambda_{ab}| \leqslant 2.8\times 10^{-13} \text{ cm}, \quad (46)$$

so that Eq. (36) is satisfied. The remaining conditions on the structure functions can be verified directly.

We now examine a system of charged fundamental particles which move under the following restrictions:

a. Relative to some suitable inertial frame, the distances between all pairs of distinct particles are at all times large compared to 10^{-13} cm.

b. The velocities of the particles may be relativistic, but they are not too close to the velocity of light.

c. For each particle a, the change in the momentum $\dot{x}_a{}^\mu$ during any proper time interval of magnitude 10^{-13} cm or 10^{-23} sec is negligible. We call this the condition of "moderate accelerations," but it includes a condition of moderate rates of change of the rest masses $m_a = \dot{x}_a$.

A typical interaction term in J^{**}_{*} for two distinct particles a and b is, by Eqs. (29), (30),

$$\tfrac{1}{2}e_a e_b \int_{u_a*}^{u_a**}\int_{u_b*}^{u_b**} f_{ab}\dot{x}_a{}^\mu \dot{x}_{b\mu} du_a du_b. \quad (47)$$

The restrictions a, b, c, and condition2 on the structure function, insure that this interaction term can be approximated by

$$\tfrac{1}{2}e_a e_b \int_{u_a*}^{u_a**}\int_{u_b*}^{u_b**} \delta(\xi_{ab}{}^2)\dot{x}_a{}^\mu \dot{x}_{b\mu} du_a du_b. \quad (48)$$

A typical self-action term in J^{**}_{*} is

$$\tfrac{1}{2}\int_{u_a*}^{u_a**}\int_{u_a*}^{u_a**} f(\xi_{aa'}{}^2, \dot{x}_a{}^\mu, \dot{x}_{a'}{}^\mu)\dot{x}_a{}^\mu \dot{x}_{a'\mu} du_a du_a'. \quad (49)$$

Condition 2, Eq. (36), and the restriction c of moderate accelerations imply that we have approximately

$$\dot{x}_{a'}{}^\mu = \dot{x}_a{}^\mu, \quad du_a' = d\xi_{aa'}/\dot{x}_a, \quad (50)$$

in the small interval about $u_a' = u_a$, outside of which $f_{aa'}$ is negligibly small. If we introduce these approximations in (49), condition 3 enables us to perform the integration with respect to u_a', and the self-action term reduces to

$$\tfrac{1}{2}\int_{u_a*}^{u_a**} \dot{x}_a{}^\mu \dot{x}_{a\mu} du_a. \quad (51)$$

Thus, for structure functions with the properties 1, 2, 3, and for systems satisfying the restrictions a, b, c, the action principle (12) simplifies to

$$\delta\left[\tfrac{1}{2}\sum_a \int_{-\infty}^{\infty} \dot{x}_a{}^\mu \dot{x}_{a\mu} du_a + \tfrac{1}{2}\sum_{\substack{ab\\a\neq b}} e_a e_b \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(\xi_{ab}{}^2)\dot{x}_a{}^\mu \dot{x}_{b\mu} du_a du_b\right] = 0. \quad (52)$$

The *equations of motion* of particle a are easily seen to be

$$\ddot{x}_{a\mu} = e_a \dot{x}_a{}^\nu F_{a\nu\mu}, \quad (53)$$

where

$$F_{a\nu\mu} = \partial A_{a\nu}/\partial x_a{}^\mu - \partial A_{a\mu}/\partial x_a{}^\nu, \quad (54)$$

$$A_{a\mu} = \sum_{\substack{b\\b\neq a}} e_b \int_{-\infty}^{\infty} \delta(\xi_{ab}{}^2)\dot{x}_{b\mu} du_b. \quad (55)$$

Then, if Eq. (53) is multiplied by $\dot{x}_a{}^\mu$, it follows from the skew symmetry of $F_{a\nu\mu}$ that

$$\frac{d}{du_a}(\dot{x}_a{}^\mu \dot{x}_{a\mu}) = \frac{d}{du_a}(m_a{}^2) = 0. \quad (56)$$

Thus the rest mass m_a of each particle is a constant of the motion. The $A_{a\mu}$ are half-retarded plus half-advanced Maxwellian electromagnetic potentials; Eq. (53) is easily seen to be equivalent to the Lorentz equations of motion for a point charge.

Thus, for a wide range of phenomena, and for a large class of structure functions, our theory reduces to classical electrodynamics.[3]

Under the same assumptions, the law (22) of *conservation of linear momentum and energy* reduces to

$$\sum_a (\dot{x}_a{}^\mu(u_a) + e_a A_a{}^\mu(u_a)) + \sum_{\substack{ab\\a\neq b}} e_a e_b \left(\int_{u_a}^{\infty}\int_{-\infty}^{u_b} - \int_{-\infty}^{u_a}\int_{u_b}^{\infty}\right)\delta'(\xi_{ab}{}^2) \times \xi_{ab}{}^\mu \dot{x}_a{}^\nu \dot{x}_{b\nu} du_a du_b = \text{constant}. \quad (57)$$

This conservation law has been obtained by Fokker[4] and by Wheeler and Feynman,[3] who also showed that it is equivalent to the usual field-theoretic formulation. The law (28) of *conservation of angular momentum* reduces to

$$\sum_a \{x_a{}^\mu(\dot{x}_a{}^\nu + e_a A_a{}^\nu) - x_a{}^\nu(\dot{x}_a{}^\mu + e_a A_a{}^\mu)\}_{u_a} + \sum_{\substack{ab\\a\neq b}} e_a e_b \left(\int_{u_a}^{\infty}\int_{-\infty}^{u_b} - \int_{-\infty}^{u_a}\int_{u_b}^{\infty}\right) \times\{\delta'(\xi_{ab}{}^2)(x_a{}^\nu x_b{}^\mu - x_a{}^\mu x_b{}^\nu)\dot{x}_a{}^\rho \dot{x}_{b\rho} - \tfrac{1}{2}\delta(\xi_{ab}{}^2)(\dot{x}_a{}^\nu \dot{x}_b{}^\mu - \dot{x}_a{}^\mu \dot{x}_b{}^\nu)\} du_a du_b = \text{constant}. \quad (58)$$

[3] See J. A. Wheeler and R. P. Feynman, Revs. Modern Phys. **17**, 157 (1945); **21**, 425 (1949).
[4] A. D. Fokker, Z. Physik **58**, 386 (1929).