Effect of Atomic Electron Screening on the Shape of Forbidden Beta Spectra

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In Fermi's theory of beta decay the influence of the atomic electrons on the shape of the spectra is often disregarded since the influence is small except at low energies. However, there is some interest in knowing exactly how small it is. This paper gives an approximate calculation of the screening effect, both for allowed and for forbidden spectra. The Thomas-Fermi model for the atomic electrons and WKB approximations for the wave functions of the emitted particles are used. The results for the allowed spectra agree with those of Rose and of Longmire and Brown. The calculations for the forbidden spectra lead to a simple rule for finding the screening effect in any specific case.

I. INTRODUCTION

IN Fermi's original paper on beta decay[1] he neglected the effects of the atomic electrons on the process. This has also been done in many of the subsequent developments of the theory; it is ordinarily assumed that the beta particle is created into the Coulomb field of the daughter nucleus. The justification for this simplifying assumption is that the atomic electrons have only a small influence on the electric field at the nucleus, where the beta particle is created. One expects the atomic electrons to be of importance only when the energy of the emitted beta particle is so small that the wavelength is comparable to the size of the atom. The atomic electrons make the effective electric field smaller than the field of the bare nucleus so that, when they are taken into account, a decreased electron emission and an increased positron emission are predicted by the theory.

The effect of the atomic electron screening was first calculated by Rose[2] and later by Longmire and Brown[3] using methods based on the WKB approximation. Most recently Reitz[4] has published tables of the screening correction, found by solving the equations numerically. All these authors have discussed the allowed beta spectra only. However there is a need for similar information about the forbidden spectra also;[5] the purpose of this paper is to discuss the screening effect with emphasis on the forbidden spectra.

The calculations here are similar to those of Rose and Longmire and Brown,[2,3] and for the allowed spectra the results found here agree essentially with theirs. WKB approximations to the wave functions of the emitted beta-particle are used, and the potential due to the atomic electrons is found from the Thomas-Fermi statistical model.[6,7] A simple procedure is given for finding the effect of the screening on any specific forbidden spectrum. It is found that ordinarily the effect for a forbidden spectrum is of the same order of magnitude as for the allowed spectrum.

II. DERIVATION OF THE SCREENING EFFECT

In the following discussion the notation of Konopinski and Uhlenbeck[8] and of Smith,[9] is used. The Konopinski-Uhlenbeck paper gives formulas for the first- and second-forbidden beta spectra for each of five possible interaction types. Smith's letter gives the cross terms which arise when a linear combination of the five interaction terms is used, assuming no Fierz[10] interference (cross terms have also been calculated by Pursey[11] and by Al-Ghita[12]). Both Konopinski-Uhlenbeck and Smith expressed their results in terms of quantities L, M, N, L^-, M^-, N^- which depend quadratically on wave functions of the electron evaluated at the nuclear radius, and then they substituted the wave functions for an electron in the Coulomb field of the daughter nucleus. The screening will be taken into account here by using the Coulomb field modified by the atomic electron cloud. A consequence of treating the screening this way is that different values of L, M, N, L^-, M^-, N^- are obtained but the formulas expressing the spectrum shapes in terms of these quantities are unchanged. Serber and Snyder[13] have shown that this approach to the problem takes proper account of the binding energies of the atomic electrons.

From this point of view, the beta particle is emitted into the potential field of the neutral parent atom plus the Coulomb field of the single extra charge that it leaves on the nucleus:

$$V_{sc}(Z) = \pm V_{TF}(Z \mp 1) - \alpha r^{-1}. \tag{1}$$

Here the upper signs apply for electron emission, the lower for positron emission, Z is the atomic number of the daughter nucleus, V_{sc} is the screened potential energy of the emitted particle, and $V_{TF}(Z\mp 1)$ is the

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[2] M. E. Rose, Phys. Rev. **49**, 727 (1936).
[3] C. Longmire and H. Brown, Phys. Rev. **75**, 264, 1102 (1949).
[4] J. R. Reitz, Phys. Rev. **77**, 10 (1950).
[5] See, for example, Freedman, Wagner, and Engelkemeir, Phys. Rev. **88**, 1155 (1952).
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[7] E. Fermi, Z. Physik **48**, 73 (1928).
[8] E. J. Konopinski and G. E. Uhlenbeck, Phys. Rev. **60**, 308 (1941).
[9] A. M. Smith, Phys. Rev. **82**, 955 (1951).
[10] M. Fierz, Z. Physik **104**, 553 (1937).
[11] D. L. Pursey, Phil. Mag. **42**, 1193 (1951).
[12] M. K. Al-Ghita, thesis, University of Michigan (Publication No. 3709, University Microfilms, Ann Arbor, Michigan, 1952).
[13] R. Serber and H. S. Snyder, Phys. Rev. **87**, 152 (1952).

Thomas-Fermi potential energy for an electron in the parent atom of atomic number $(Z\mp1)$. It is assumed that the beta particle has escaped from the atom before any rearrangements in the atomic electrons take place. The connection between this screened potential and the Coulomb field of the daughter nucleus is seen by expanding the Thomas-Fermi potential about the nucleus. Using Fermi's formulation of the statistical potential,[7] Baker's expansion about the nucleus,[14] and converting to relativistic units, one finds easily that

$$V_{TF}(Z\mp1)=-\frac{\alpha(Z\mp1)}{r}\left\{1-1.589\frac{r}{\mu}+O\left[\left(\frac{r}{\mu}\right)^{\frac{3}{2}}\right]\right\}, \quad (2)$$

where

$$\mu=0.885\alpha^{-1}(Z\mp1)^{-\frac{1}{3}} \quad (3)$$

is a characteristic length for the parent atom electron cloud. The screened potential in the neighborhood of the nucleus is then

$$V_{sc}(Z)=\mp\frac{\alpha Z}{r}\pm1.589\frac{\alpha(Z\mp1)}{\mu}\left\{1+O\left[\left(\frac{r}{\mu}\right)^{\frac{1}{2}}\right]\right\}. \quad (4)$$

The first term is the usual Coulomb potential. In what follows, a first approximation to the screening effect will be obtained by taking into account the second term, disregarding the terms of order $(r/\mu)^{\frac{1}{2}}$ compared to 1.

It would be difficult to make exact calculations of the electron wave functions in the potential of Eq. (1); the WKB method developed by Uhlenbeck and Bessey[15] for the Dirac radial wave equation will be used to approximate the functions. This approximation has been reviewed in an earlier paper.[16] The use of this WKB method in this problem requires justification because the values of the wave functions at the nuclear radius are required and this radius is in the neighborhood of the pole of the potential function at the origin. However, it has been shown that for the Coulomb potential $V(r)=\mp\alpha Z/r$ the WKB approximation reproduces the exact results within a few percent both for the function $F(Z,W)$ which gives the shape of allowed spectra and for a representative forbidden spectrum correction factor.[16,17] This indicates that the method will give good results in the problem at hand, where the potential also has a simple pole at the origin and then goes monotonically to zero. For use in the formulas of Konopinski-Uhlenbeck and Smith, radial wave functions f_l, g_l satisfying the Dirac radial wave equations, giving an integrable probability density at the origin, and normalized to one particle in a sphere of unit radius, are required. Referring to Eqs. (12) and (13) of reference 16 and putting $u=rf$, $v=rg$, one finds that the WKB approximations to these functions are, when $r<r_1$,

$$f_l=-\frac{1}{2r}\frac{|l+1|}{l+1}\left(\frac{p\psi(r)}{WR(r)}\right)^{\frac{1}{2}} \times\exp\int_r^{r_1}\left[-\left(1+\frac{\xi\psi'(\xi)}{\psi(\xi)}\right)\frac{l+1}{2\xi^2R(\xi)}-R(\xi)\right]d\xi, \quad (5)$$

$$g_l=\frac{1}{2r}\left(\frac{p\phi(r)}{WR(r)}\right)^{\frac{1}{2}} \times\exp\int_r^{r_1}\left[\left(1+\frac{\xi\phi'(\xi)}{\phi(\xi)}\right)\frac{l+1}{2\xi^2R(\xi)}-R(\xi)\right]d\xi, \quad (6)$$

and are, when $r>r_1$,

$$f_l=-\frac{1}{r}\frac{|l+1|}{l+1}\left(\frac{p\psi(r)}{WQ(r)}\right)^{\frac{1}{2}} \times\cos\left\{\int_{r_1}^{r}\left[Q(\xi)-\left(1+\frac{\xi\psi'(\xi)}{\psi(\xi)}\right)\frac{l+1}{2\xi^2Q(\xi)}\right] \times d\xi-\frac{\pi}{4}\right\}, \quad (7)$$

$$g_l=\frac{1}{r}\left(\frac{p\phi(r)}{WQ(r)}\right)^{\frac{1}{2}} \times\cos\left\{\int_{r_1}^{r}\left[Q(\xi)+\left(1+\frac{\xi\phi'(\xi)}{\phi(\xi)}\right)\frac{l+1}{2\xi^2Q(\xi)}\right] \times d\xi-\frac{\pi}{4}\right\}, \quad (8)$$

where

$$\phi(r)=W-V(r)+1, \quad (9)$$

$$\psi(r)=W-V(r)-1, \quad (10)$$

$$Q(r)=[\phi\psi-(l+1)^2r^{-2}]^{\frac{1}{2}}, \quad (11)$$

$$R(r)=[(l+1)^2r^{-2}-\phi\psi]^{\frac{1}{2}}, \quad (12)$$

$V(r)$ is an arbitrary potential, and r_1 is the positive root of $[r^2\phi\psi-(l+1)^2]$. Zwann's method[18] was used to connect the approximations across the turning point r_1.

The next step is to evaluate the approximate wave functions, Eqs. (5) to (8), at the nuclear radius ρ and with the potential of Eq. (1). Since ρ is always less than r_1, formulas (5) and (6) apply with r replaced by ρ. The result then depends only on the value of the potential in the region $r<r_1$. It is easily seen that $r_1<\mu$, so one may use the expansion of the potential for small r/μ [Eq. (4)]; as a further approximation the higher-order terms

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[15] R. J. Bessey, thesis, University of Michigan, 1942 (unpublished).

[16] R. H. Good, Jr., Phys. Rev. **90**, 131 (1953).

[17] R. H. Good, Jr., thesis, University of Michigan (Publication No. 2597, University Microfilms, Ann Arbor, Michigan, 1951).

[18] See, for example, Edwin C. Kemble, *The Fundamental Principles of Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1937), first edition, p. 95.

will be disregarded so that

$$V_{sc}(Z)=\mp\alpha Zr^{-1}+\Delta, \quad (13)$$

where

$$\Delta=\pm 1.589\alpha\mu^{-1}(Z\mp 1)$$
$$=\pm 1.795\alpha^2(Z\mp 1)^{4/3}. \quad (14)$$

A property of Eqs. (5) and (6) is that W and $V(r)$ always occur in the combination $W-V(r)$ in the integrations. This permits the screened-potential wave functions f_l^{sc}, g_l^{sc} to be expressed in terms of the well-known Coulomb wave functions f_l^{Coul}, g_l^{Coul}. Introducing the notation

$$\bar{K}(Z,W)=K(Z,W-\Delta) \quad (15)$$

for any function $K(Z,W)$, then one sees that

$$f_l^{sc}=(p\bar{W}/\bar{p}W)^{\frac{1}{2}}\bar{f}_l^{\mathrm{Coul}}, \quad (16)$$

$$g_l^{sc}=(p\bar{W}/\bar{p}W)^{\frac{1}{2}}\bar{g}_l^{\mathrm{Coul}}. \quad (17)$$

It is assumed that $W>\Delta+1$. The formula for the allowed spectrum, regardless of the potential used, is [reference 8, Eqs. (1) and (22)]

$$P(W)dW=(G^2/2\pi^3)C_0F(Z,W)pW(W_0-W)^2dW, \quad (18)$$

where

$$F(Z,W)=(g_0^2+f_{-2}^2)p^{-2}(1+s)^{-1}. \quad (19)$$

The result for the screened potential is then easily found to be

$$F^{sc}=(\bar{p}\bar{W}/pW)\bar{F}^{\mathrm{Coul}}. \quad (20)$$

This result is substantially in agreement with the results of Rose[2] and of Longmire and Brown,[3] although they choose a different value for Δ (Reitz's paper[4] shows the connection between their results). The numerical evaluation of this function has been discussed by Fano.[19]

What happens in the forbidden spectra is easily shown by a specific example. For the vector first-forbidden transition the spectrum is [reference 8, Eq. (24)]

$$P(W)dW=dW\frac{G^2}{\pi^2}\frac{q^2W}{p}\left[\left|\int\mathbf{r}\right|^2\left\{\frac{q^2}{3}\frac{g_0^2+f_{-2}^2}{4\pi}+\frac{f_0^2+g_{-2}^2}{4\pi\rho^2}\right\}+\text{other terms}\right]. \quad (21)$$

This is usually written like Eq. (18) except with a correction factor:

$$P(W)dW=(G^2/2\pi^3)F(Z,W)pW(W_0-W)^2dW$$
$$\left[\left|\int\mathbf{r}\right|^2\{\tfrac{1}{3}K^2L_0+M_0\}+\text{other terms}\right], \quad (22)$$

where

$$L_0=(g_0^2+f_{-2}^2)(2p^2F)^{-1}, \quad (23)$$

$$M_0=(f_0^2+g_{-2}^2)(2p^2F\rho^2)^{-1}. \quad (24)$$

To find the effect of the screening, one substitutes Eqs. (16) and (17) into Eq. (21) and puts the results in the form of Eq. (22):

$$P(W)dW=(G^2/2\pi^3)F^{sc}(Z,W)pW(W_0-W)^2dW$$
$$\times\left[\left|\int\mathbf{r}\right|^2\{\tfrac{1}{3}K^2L_0^{sc}+M_0^{sc}\}+\text{other terms}\right], \quad (25)$$

where

$$L_0^{sc}=\bar{L}_0^{\mathrm{Coul}}, \quad (26)$$

$$M_0^{sc}=\bar{M}_0^{\mathrm{Coul}}. \quad (27)$$

This argument can be applied uniformly to all the terms in the forbidden spectra. One may take the screening into account by using the F^{sc} of Eq. (20) in the basic spectrum formula and by replacing W by $W-\Delta$ in the results of Konopinski-Uhlenbeck and Smith for the quantities L, M, N, L^-, M^-, N^- which arise in the correction factors. Ordinarily these quantities do not depend drastically on the energy, so the screening effect for forbidden spectra is of the same order of magnitude as it is for the allowed spectrum. A very simple interpretation of these results has been given by Huster.[20]

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[19] U. Fano, *Tables for the Analysis of Beta Spectra*, Natl. Bur. Standards U. S. Appl. Math. Ser. 13 (U. S. Government Printing Office, Washington, 1952).

[20] E. Huster, Z. Physik **136**, 303 (1953).