Magnetic Quenching of the Positronium Decay

OTTO HALPERN
℅ *American Express Company, Zurich, Switzerland*
(Received January 27, 1954)

The influence of static and alternating magnetic fields on the radiative decay of positronium is investigated theoretically. Equations for the time variation of the probability amplitudes of the *ortho*- and *para*-states are presented and solved for various cases of experimental interest. In particular, formulas are derived which describe the resonance effects occurring in the case of a constant and a perpendicular alternating field.

I. INTRODUCTION

In recent years, experiments by Deutsch[1] and his collaborators have given us a very detailed insight into the structure and decay of positronium. The original experiments have been later on somewhat extended by other authors.[2]

The main achievement of the work of Deutsch was the observation of the theoretically postulated[3] three-quantum decay of the *ortho*-form of positronium, as well as an insight into the modifications of this three-quantum decay which are produced by static or alternating magnetic fields.

It had been shown theoretically[3] that the lifetime of the *ortho*-form of positronium is about 1100 times as large as that of the *para*-form. This is due to the fact that two antiparallel light quanta, the decay product of *para*-positronium, cannot carry a resultant angular momentum of the value $h/2\pi$ with them, which is the original angular momentum of *ortho*-positronium. This leads to a three-quantum decay of *ortho*-positronium with a correspondingly increased life-time as compared with *para*-positronium which decays into two quanta. The observation[1] of the three-quantum decay constituted the first significant experimental advance.

If the positronium is brought into an external magnetic field, the *ortho*- and *para*-states become mixed up. This leads to possible two-quantum decays of *ortho*-states and thereby to a diminution of the observed three-quantum coincidences. Theory and experiment agree in the conclusion that only one of the three *ortho*-states is involved in this change if a homogeneous static magnetic field is applied; only the *ortho*-state with a vanishing component of its angular momentum with respect to the direction of the magnetic field combines with the *para*-state.

A further refinement of the experiment[3] consisted in the use of the by now classical method of applying a small alternating magnetic field perpendicular to the direction of the large constant magnetic field. One can thereby induce transitions between the two *ortho*-states with finite projection of their angular momenta upon the direction of the constant magnetic field and the remaining *ortho*- and *para*-states. Thus an additional quenching of the three-quantum decay can be made observable[3] which is sharply dependent upon the frequency of the alternating magnetic field.

The papers of Deutsch and his collaborators have already given a semiquantitative treatment of these data. Since the basis of this treatment is largely formed by analogies, the meaning of which may not always be unambiguous, it seemed appropriate[4] to present a theoretical discussion of these phenomena which goes back to first principles by studying the time variation of the probability amplitudes of the various states in dependence upon the external magnetic fields. The present paper, the publication of which has been unduly delayed for external reasons, will present such calculations in detail and thereby also cover some limiting cases for which the method of analogies would not seem to be sufficient.

II. THE MATRIX ELEMENTS OF THE PERTURBING MAGNETIC ENERGY

We denote by $\varphi_e(1)$, $\varphi_e(2)$ the eigenfunctions of the electron with its spin parallel and antiparallel to the z axis, with $\varphi_p(1)$, $\varphi_p(2)$ the eigenfunctions of the positron in the corresponding states. If we then neglect very small corrections of relativistic order of magnitude we can write the eigenfunctions of the *para*- and *ortho*-ground states of positronium as follows:

$$\psi_1=\varphi_e(1)\varphi_p(1), \tag{1a}$$

$$\psi_0=[\varphi_e(1)\varphi_p(2)+\varphi_e(2)\varphi_p(1)]/\sqrt{2}, \tag{1b}$$

$$\psi_{-1}=\varphi_e(2)\varphi_p(2), \tag{1c}$$

$$\psi=[\varphi_e(1)\varphi_p(2)-\varphi_e(2)\varphi_p(1)]/\sqrt{2}. \tag{1d}$$

The magnetic perturbation terms of the Hamiltonian in the presence of a constant magnetic field along the z axis and a variable magnetic field along the x axis are given by the expression

$$H_m=\frac{eh}{4\pi mc}[H_z(\sigma_{ez}-\sigma_{pz})+H_x\cos\omega t(\sigma_{ex}-\sigma_{px})]. \tag{2}$$

In (2) the meaning of all symbols is conventional; σ_e and σ_p denote, respectively, the matrices of the spin vector with the eigenvalues ±1.

[1] M. Deutsch and S. C. Brown, Phys. Rev. 85, 1047 (1952); there are also references to earlier literature.
[2] J. Wheatley and D. Halliday, Phys. Rev. 88, 424 (1952).
[3] A. Ore and J. L. Powell, Phys. Rev. 75, 1696 (1949).
[4] O. Halpern, Phys. Rev. 88, 164 (1952).

One notices by inspection that the magnetic part H_m of the Hamiltonian has no diagonal elements; this shows that the Zeeman effect of positronium is at least quadratic in the magnetic field-strength. The physical reason for this fact is to be found in the opposite signs of the magnetic moments of the electron and the positron in (2).

The z component of the magnetic field has, as was to be expected, only one nonvanishing (nondiagonal) matrix element

$$M_z=\psi_0 H_m\psi=2ehH_z/4\pi mc. \tag{3}$$

The x component has nonvanishing matrix elements only in reference to states with different total angular momentum:

$$M_x=\psi_1 H_m\psi=\psi_{-1}H_m\psi=\sqrt{2}ehH_x\cos\omega t/4\pi mc. \tag{4}$$

We are now prepared to study the effect of the various magnetic fields on the decay of the two forms of positronium. To make things simpler, we first consider only the case of a static magnetic field.

III. QUENCHING IN A CONSTANT MAGNETIC FIELD

We denote by a_1, a_0, a_{-1}, and a the amplitudes of the four different positronium spin states. The subscripts indicate the magnitude of the projection of j on the z axis in the *ortho*-states; a without subscript stands for the amplitude of the *para*-state ($j=0$). The energy difference between the *ortho*-states and the *para*-state in the absence of external perturbations shall be denoted by $h\omega_0/2\pi$. Since, as shown in (3), there exist no matrix elements of the Hamiltonian between states of unequal m, the amplitudes a_1 and a_{-1} decay as they do in the field-free case.

For the amplitudes a_0 and a we now have the following equations:

$$\frac{da_0}{dt}=-\frac{2\pi i}{h}M_z a e^{i\omega_0 t}-\gamma_1 a_1, \tag{5a}$$

$$\frac{da}{dt}=-\frac{2\pi i}{h}M_z a_0 e^{-i\omega_0 t}-\gamma a. \tag{5b}$$

γ_1 and γ denote, respectively, the amplitude decay constants of the *ortho*- and *para*-states. Since the lifetimes of these states are 1.3×10^{-7} sec and 1.2×10^{-10} sec, respectively, we have for γ_1 and γ the values 3.7×10^6 sec^{-1} and 4×10^9 sec^{-1}.

Introducing the expressions,

$$a_0=b_0e^{-\gamma_1 t}, \tag{6a}$$

$$a=be^{-\gamma t}, \tag{6b}$$

one obtains

$$\frac{db_0}{dt}=-\frac{2\pi i}{h}M_z b e^{i\omega_0 t}e^{-(\gamma-\gamma_1)t}, \tag{7a}$$

$$\frac{db}{dt}=-\frac{2\pi i}{h}M_z b_0 e^{-i\omega_0 t}e^{(\gamma-\gamma_1)t}. \tag{7b}$$

Because the exponents in the exponentials of (7a) and (7b) have opposite signs, this system of linear differential equations of the first order with time-dependent coefficient can be transformed into a linear equation of the second order with constant coefficients. One solves (7a) for b, differentiates with respect to the time, and inserts the expression for db/dt thus obtained into (7b). This leads to

$$\ddot{b}_0+\dot{b}_0(i\omega_0+\gamma')+b_0\frac{4\pi^2}{h^2}M_z^2=0, \tag{8}$$

$$\gamma'=\gamma-\gamma_1\gg\gamma_1. \tag{9}$$

Equation (8) gives for b_0 the expression

$$b_0=C_1e^{\alpha_1 t}+C_2e^{\alpha_2 t}, \tag{10}$$

with

$$\alpha_1=-\frac{i\omega_0+\gamma'}{2}+\frac{1}{2}\left((i\omega_0+\gamma')^2-\frac{16\pi^2}{h^2}M_z^2\right)^{\frac{1}{2}}, \tag{11a}$$

$$\alpha_2=-\frac{i\omega_0+\gamma'}{2}-\frac{1}{2}\left((i\omega_0+\gamma')^2-\frac{16\pi^2}{h^2}M_z^2\right)^{\frac{1}{2}}. \tag{11b}$$

For $\gamma'=0$, (11) reduces to

$$ih\alpha_{1,2}=\frac{h\omega_0}{2}\pm\left(\left(\frac{h\omega_0}{2}\right)^2+M_z^2\right)^{\frac{1}{2}}. \tag{12}$$

The original amplitudes a_0 and a are then given by expressions of the type:

$$a_0=C_1e^{(\alpha_1-\gamma_1)t}+C_2e^{(\alpha_2-\gamma_1)t}, \tag{13a}$$

$$a=D_1e^{(\alpha_1+i\omega_0-\gamma_1)t}+D_2e^{(\alpha_2+i\omega_0-\gamma_1)t}, \tag{13b}$$

$$C_1\sim D_1,\quad C_2\sim D_2. \tag{14}$$

Since $\omega_0\gg\gamma'$, α_1 and α_2 are to a first approximation given by

$$\alpha_1=i\frac{4\pi^2M_z^2}{h^2\omega_0}-\gamma'\frac{4\pi^2M_z^2}{h^2\omega_0^2}, \tag{15a}$$

$$\alpha_2=-(i\omega_0+\gamma')-\alpha_1. \tag{15b}$$

It follows from (13a) and (13b) that one can form linear combinations of the wave functions (1b) and (1d) which decay according to a simple exponential law; if we put for $t=0$,

$$C_2=0,\quad \psi_0=\frac{C_1}{(C_1^2+D^2)^{\frac{1}{2}}},\quad \psi=\frac{D_1}{(C_1^2+D_1^2)^{\frac{1}{2}}} \tag{16a}$$

or

$$C_1=0,\quad \psi_0=\frac{C_2}{(C_2^2+D_2^2)^{\frac{1}{2}}},\quad \psi=\frac{D_2}{(C_2^2+D_2^2)^{\frac{1}{2}}}, \tag{16b}$$

then these aggregates have decay constants $\gamma_1+(\gamma'4\pi^2M_z^2/h^2\omega_0^2)$ and $\gamma-(\gamma'4\pi^2M_z^2/h^2\omega_0^2)$, respectively.

The physical interpretation of this result is clear. In the absence of decay, $\gamma_1=\gamma=0$, the expressions (11), (12) and (16) are the well-known wave functions of a magnetic system which has vanishing diagonal and nonvanishing off-diagonal matrix elements of the magnetic Hamiltonian. The energy-splitting given by (12) is the familiar generalization of the Landé formula.

The discussion for nonvanishing γ_1 and γ can best be carried out remembering the order of magnitude of the various quantities involved. We have $\omega_0\sim 10^{12}$ sec^{-1}; γ_1 and γ (or γ') are about $-3.7\ .10^6$ and 4×10^9 sec^{-1} respectively, and $\gamma' 4\pi^2 M_z^2/h^2\omega_0^2$ becomes for $H_z\sim 9000$ gauss about $\gamma' 10^{-2}$. One thus sees easily that the imaginary parts of (11) (the energy splitting) are very little affected by the spontaneous decay. The real part of $\alpha_2-\gamma_1$, i.e., essentially the decay constant of the *para*-state, is hardly changed by the presence of a constant magnetic field. The decay of a_0, on the other hand, is strongly influenced by a sizable magnetic field. One sees readily that for values of H_z larger than, say, 2000 gauss the term $\gamma' 4\pi^2 M_z^2/h^2\omega_0^2$ predominates. This means that the three-quantum decay is progressively replaced by the two-quantum decay of the *para*-state. For $H_z\sim 9000$ gauss only a few percent of the three-quantum decays are left; this means experimentally that only $\frac{2}{3}$ of all three-quantum processes originating from the states with the amplitudes a_1 and a_{-1} remain.

It should perhaps be re-emphasized that it is not just the small fraction $4\pi^2 M_z^2/h^2\omega_0^2$ of *para*-state present in a_0 that decays rapidly. It is, as shown by (13a) and (13b), the whole linear combination (16a) which decays at the rate $\gamma_1+(\gamma 4\pi^2 M_z^2/h^2\omega_0^2)$.

IV. THE EFFECT OF AN ADDITIONAL ALTERNATING MAGNETIC FIELD

For the treatment of this problem it is advantageous to change the basic wave functions. It should be noted that the linear combinations (16a) can be made mutually orthogonal and are also orthogonal to the states with the amplitudes a_1 and a_{-1}. If we introduce these expressions in place of (1b) and (1d), we have thereby included the effect of H_z, which as mentioned before does not have matrix elements between states with m different from zero. The equations of variation of constants therefore contain M_x only. We continue to use the notation for the amplitudes of the states $a_1\cdots a$ which will not lead to any misunderstanding.

Keeping in mind the results of Sec. II, one thus arrives at the following equations:

$$\frac{da_1}{dt}=-\frac{2\pi i}{h}\{M_x a_0 e^{-i\omega' t}\beta+M_x a e^{+i\omega_0 t}\}-\gamma_1 a_1, \tag{17a}$$

$$\frac{da_0}{dt}=-\frac{2\pi i}{h}\{M_x\beta e^{i\omega' t}(a_1+a_{-1})\}-\gamma'' a_0, \tag{17b}$$

$$\frac{da_{-1}}{dt}=-\frac{2\pi i}{h}\{M_x\beta a_0 e^{-i\omega' t}+M_x a e^{i\omega_0 t}\}-\gamma_1 a_{-1}, \tag{17c}$$

$$\frac{da}{dt}=-\frac{2\pi i}{h}\{M_x e^{-i\omega_0 t}(a_1+a_{-1})\}-\gamma a, \tag{17d}$$

$$\beta=2\pi M_z/h\omega_0, \tag{18a}$$

$$\gamma''=\gamma_1+\gamma'\beta^2, \tag{18b}$$

$$\omega'=\text{imaginary part of } \alpha_1. \tag{18c}$$

A study of this system of equations shows that no simple reduction to a linear equation of higher order with constant coefficients is possible. This has its origin in the factor $\cos\omega t$ in M_x which cannot be eliminated as was done with the exponentials in III. But closer attention to the physical side of the problem allows us to introduce simplifications which lead in the end to another linear equation of the second order with constant coefficients.

One has to keep in mind that we are looking for resonance effects which will occur when $\omega'=\omega$, or in other words when h times the frequency of the alternating field equals the energy difference between the *ortho*-states $a_{\pm 1}$ and a_0 created by the constant magnetic field H_z. This energy difference is given by the imaginary part $(h/2\pi)\alpha_1$. General dispersion theory leads to the experimentally confirmed view that the influence of the alternating field will be very small off resonance. This means first of all that all terms containing a in (17) can be dropped; in fact the whole equation (17d) may be neglected. The frequency of the alternating field will always be very small compared with ω_0.

Furthermore, if we write

$$\cos\omega t=\tfrac{1}{2}(e^{i\omega t}+e^{-i\omega t}),$$

than we see that in (17a) and (17c) only the term $e^{i\omega t}$, while in (17b) only the terms $e^{-i\omega t}$, can give rise to a resonance phenomenon. The second term produces only dispersion effects which average out during the time of observation. One thus obtains the following three equations:

$$\frac{da_1}{dt}=-\frac{2\pi i}{2h}M_x'\beta a_0 e^{-i(\omega'-\omega)t}-\gamma_1 a_1, \tag{19a}$$

$$\frac{da_0}{dt}=-\frac{2\pi i}{2h}M_x'\beta(a_1+a_{-1})e^{i(\omega'-\omega)t}-\gamma'' a_0, \tag{19b}$$

$$\frac{da_{-1}}{dt}=-\frac{2\pi i}{2h}M_x'\beta a_0' e^{-i(\omega'-\omega)t}-\gamma_1 a_{-1}, \tag{19c}$$

$$M_x=M_x'\cos\omega t. \tag{20}$$

We introduce again in analogy with III the quantities

$$a_1=b_1 e^{-\gamma_1 t}, \tag{21a}$$

$$a_0=b_0 e^{-\gamma'' t}, \tag{21b}$$

$$a_{-1}=b_{-1}e^{-\gamma_1 t}, \tag{21c}$$

and obtain for the quantities b the equations

$$\frac{db_1}{dt}=-\frac{\pi i}{h}M_x'\beta b_0 e^{-i(\omega'-\omega)t}e^{-(\gamma''-\gamma_1)t}, \tag{22a}$$

$$\frac{db_0}{dt}=-\frac{\pi i}{h}M_x'\beta(b_1+b_{-1})e^{i(\omega'-\omega)t}e^{(\gamma''-\gamma_1)t}, \tag{22b}$$

$$\frac{db_{-1}}{dt}=-\frac{\pi i}{h}M_x'\beta b_0 e^{-i(\omega'-\omega)t}e^{-(\gamma''-\gamma_1)t}. \tag{22c}$$

Solving (22b) for b_1+b_{-1}, and differentiating with respect to the time, we obtain again from (22a) and (22c) the desired equation

$$\ddot{b}_0+\dot{b}_0[i(\omega-\omega')+\gamma_1-\gamma'']+\frac{2\pi^2}{h^2}\beta^2 M_x'^2 b_0=0. \tag{23}$$

This gives for b_0 the expression

$$b_0\sim e^{-\delta t}, \tag{24}$$

with

$$\delta^2+\delta[i(\omega-\omega')+\gamma_1-\gamma'']+\frac{2\pi^2}{h^2}\beta^2 M_x'^2=0. \tag{25}$$

One finally obtains for a_0 a time dependence of the form

$$a_0=Ae^{(\delta_1-\gamma'')t}+Be^{(\delta_2-\gamma'')t}, \tag{26}$$

$$\delta_1=-\frac{i(\omega-\omega')+\gamma_1-\gamma''}{2}+\frac{1}{2}\left([i(\omega-\omega')+\gamma_1-\gamma'']^2-\frac{8\pi^2}{h^2}\beta^2 M_x'^2\right)^{\frac{1}{2}}, \tag{27a}$$

$$\delta_2=-\frac{i(\omega-\omega')+\gamma_1-\gamma''}{2}-\frac{1}{2}\left([i(\omega-\omega')+\gamma_1-\gamma'']^2-\frac{8\pi^2}{h^2}\beta^2 M_x'^2\right)^{\frac{1}{2}}. \tag{27b}$$

The successive use of (19), (21), and (26) together with the values of δ as given by (27), permits us to determine the decay times of the various states in the presence of both magnetic fields; the parametric dependence on ω gives us the behavior at and near resonance. Two limiting cases are of particular interest and lead to simple results especially at resonance.

(1) H_x small, or, more quantitatively,

$$\gamma''-\gamma_1\ll(3\pi/h)\beta M_x'. \tag{28}$$

We can then expand the square root in (27) and obtain at resonance the following values for δ_1 and δ_2:

$$\delta_1=+\frac{2\pi^2\beta^2 M_x'^2}{h^2(\gamma''-\gamma_1)}, \tag{29a}$$

$$\delta_2=\gamma''-\gamma_1-\frac{2\pi^2\beta^2 M_x'^2}{h^2(\gamma''-\gamma_1)}. \tag{29b}$$

$\delta_1-\gamma''$ obviously gives essentially the decay time of the state with vanishing m, which is only slightly affected by the alternating field. This can be understood physically by observing that the rate of decay is large compared with the rate of transition to one of the other *ortho*-states.

$\delta_2-\gamma''$, on the other hand, leads by resubstitution easily to that quantity which now represents essentially the decay rate of the two *ortho*-states. We find for it:

$$\delta_2-\gamma''=-\gamma_1-\frac{2\pi^2\beta^2 M_x'^2}{h^2(\gamma''-\gamma_1)}.$$

The percentage of the increase of the decay rate is given by

$$\frac{2\pi^2\beta^2 M_x'^2}{h^2\gamma_1(\gamma''-\gamma_1)}\sim\frac{e^2H_x^2}{4m^2c^2\gamma\gamma_1}, \tag{30}$$

This value is well observable for moderate H_x since the decay rate of the pure *ortho*-state is in that case no longer overwhelmingly large compared with the rate of transition to the composite state with $m=0$ from which two-quantum decay processes are now possible.

(2) H_x large, or, more quantitatively,

$$\gamma''-\gamma_1\leq\frac{8\pi^2}{h^2}\beta^2 M_x'^2. \tag{31}$$

Then for the case of resonance the real parts of δ_1 and δ_2 become equal. Substitution into the relevant equations of the text shows that then *all* three states decay with a decay constant $\frac{1}{2}(\gamma_1+\gamma'')$. This means that the states with $m=\pm1$ decay mostly through two-quantum processes; the state $m=0$ is now partly "dequenched" and decays with a larger percentage of three-quantum processes than in the absence of the alternating field. It should be observed that this effect is independent of the magnitude of H_x as long as the relation (31) holds true.