Green's Functions in Quantum Electrodynamics*

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Dyson has shown that the evaluation of the S matrix for quantum electrodynamics can be reduced to the problem of evaluating certain quantities, S'^F, D'^F, and Γ_μ. By making use of a formula relating the T product of an operator with its corresponding N product, integro-differential equations for S'^F and D'^F are obtained. These equations are identical in form with those given by Schwinger for his Green's functions, and hence it is concluded that the two formalisms are equivalent. In addition it is shown that all of the quantities introduced by Schwinger can be expressed in terms of a single quantity, S_vac, the vacuum expectation value of the S matrix. The renormalization problem is not discussed.

I. INTRODUCTION AND REVIEW

IN a recent work,[1] Schwinger has proposed a theory of Green's functions which appears to be applicable to many of the problems of quantum electrodynamics and similar theories. However, since his theory is based on his own formalism of quantized fields,[2] it is not immediately evident which connection exists between his results and those of conventional field theories.[3] It is with this relationship that the present paper will deal. Specifically we will show how the results of Schwinger's theory can be derived from Dyson's theory[3] of the S matrix.[4] It is hoped that such a demonstration will serve two purposes; first, to make Schwinger's results readily available in terms of an already familiar formalism, and second, to afford a set of rules whereby one may calculate directly from S-matrix theory the various Green's functions introduced by Schwinger.

In the work to follow, we shall make use of many of the results of S-matrix theory, and hence, for the sake of convenience, we will conclude this section by enumerating them. In the interaction representation, the state vector $\Psi[\sigma]$ of the system, consisting of electromagnetic and electron-positron fields in interaction, satisfies the equation

$$i\frac{\delta\Psi[\sigma]}{\delta\sigma(x)}=\mathfrak{H}(x)\Psi[\sigma],$$

where $\mathfrak{H}(x)$ is the interaction Hamiltonian density, given by[5]

$$\mathfrak{H}(x)=-A_\mu(x)\{j_\mu(x)+J_\mu(x)\}.$$

Here $j_\mu(x)$ is the field current, which for convenience we write as

$$j_\mu(x)=\tfrac{1}{2}e\gamma_{\alpha\beta}{}^\mu[\bar\psi_\alpha(x)\psi_\beta(x)-\psi_\beta(x)\bar\psi_\alpha(x)],$$

whereas $J_\mu(x)$ is some arbitrarily given external current.[6]

In what follows, we will be mainly interested in that operator S, the "so-called" S matrix, which transforms a state of the system at $t=-\infty$, into the corresponding state at $t=+\infty$. Feynman[7] has shown that S can be written in the form

$$S=T\left[\exp\left\{-i\int_{-\infty}^{+\infty}dt\int d^3x\,\mathfrak{H}(x)\right\}\right]=T[\mathfrak{S}]. \quad (1)$$

The T product appearing in Eq. (1) and first introduced by Wick[8] is defined by

$$T(UV\cdots Z)=\delta_P XY\cdots W, \quad (2)$$

the factors U, V, $\cdots$ being arranged in chronological order in the ordinary product on the right. By chronological order, we mean that if two operators in the T product of Eq. (2) correspond to points separated by either a time-like or a zero interval, then in the ordinary product, that operator operates first which corresponds to the earlier time. The sign factor δ_P is the signature ± 1 of the permutation [between left-hand side and right-hand side of Eq. (2) of the electron-positron operators only].

II. THE ORDERING OPERATORS

Several methods[3,7,8] have been given for actually evaluating the matrix elements of a T product such as appear in Eq. (2). All of these methods essentially involve the transformation of the T product into an N product, where the N product is defined by

$$N(UV\cdots Z)=\delta_P XY\cdots W,$$

the ordinary product on the right containing the same factors U, V, $\cdots$ ordered in such a manner that all creation operators stand to the left of all destruction operators. In this form, the matrix elements of any

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[1] J. Schwinger, Proc. Natl. Acad. Sci. **37**, 452 (1951).

[2] J. Schwinger, Phys. Rev. **82**, 914 (1951).

[3] For an excellent summary and bibliography, see F. J. Dyson, Phys. Rev. **75**, 486 and 1736 (1949).

[4] For an alternate treatment see Utiyama, Sunakawa, and Imamura, Progr. Theoret. Phys. (Japan) **8**, 77 (1952).

[5] Throughout this paper we shall employ Schwinger's notation and use units where $\hbar=1$, $c=1$.

[6] The interaction Hamiltonian employed here differs from that employed by Schwinger in that we dispense with the external spinor fields $\eta(x)$ and $\eta'(x)$. Since these quantities were introduced by Schwinger as a mathematical convenience and are eventually set equal to zero, the results obtained here will be equivalent to those obtained by Schwinger.

[7] R. P. Feynman, Phys. Rev. **84**, 108 (1951).

[8] G. C. Wick, Phys. Rev. **80**, 268 (1950).

T product can be readily calculated. In this section we will derive the fundamental expression (21), giving the relation between an arbitrary T product of electromagnetic and electron-positron field operators and its corresponding N product.

We shall treat first the case of a T product which is a functional only of the electromagnetic field operator $A_\mu(x)$. In order to introduce creation and destruction operators into the theory, we decompose $A_\mu(x)$ into two parts according to[9]

$$A_\mu(x) = A_\mu^+(x) + A_\mu^-(x),$$

where $A_\mu^+(x)$ contains only photon destruction operators, while $A_\mu^-(x)$ contains only photon creation operators. These auxiliary fields satisfy the following commutation relations

$$[A_\mu^+(x), A_\nu(x')] = i\delta_{\mu\nu}[D_A^+(x-x') - D_R^+(x-x')],$$
$$[A_\mu^-(x), A_\nu(x')] = i\delta_{\mu\nu}[D_A^-(x-x') - D_R^-(x-x')], \quad (3)$$
$$[A_\mu^+(x), A_\nu^+(x')] = [A_\mu^-(x), A_\nu^-(x')] = 0,$$

where the only property of the D functions which we shall need is that

$$D_R^\pm(x) = 0, \quad \text{for} \quad x < 0,$$
$$D_A^\pm(x) = 0, \quad \text{for} \quad x > 0, \quad (4)$$

and

$$D^F(x) \equiv -i\{D_A^-(x) + D_R^+(x)\} \quad (5)$$

satisfies

$$\square D^F(x) = i\delta(x). \quad (6)$$

With the help of Eqs. (3–6), we shall now prove the following statement: the N product corresponding to a given T product can be obtained by substituting for every field operator $A_\mu(x)$ the quantity $A_\mu'(x)$ given by[10]

$$A_\mu'(x) = A_\mu(x) + \int d^4x' D_F(x-x') \frac{\delta}{\delta A_\mu(x')} \quad (7)$$

in the T product, considered now as an ordinary c-number functional of the $A_\mu(x)$, and performing the indicated differentiations. The resulting expression is then to be considered as an N product, which, in fact, is equivalent to the original T product. The statement is of course trivially true when the T product to be reordered is just $A_\mu(x)$ itself. To prove it for the more

[9] For a detailed discussion of this method of decomposition, see J. Schwinger, Phys. Rev. **75**, 651 (1949).

[10] The functional derivatives $\delta/\delta A_\mu(x)$ are defined through the equation
$$\delta\mathfrak{F}(A_\mu(x)) = \mathfrak{F}(A_\mu(x) + \delta A_\mu(x)) - \mathfrak{F}(A_\mu(x))$$
$$\equiv \int \frac{\delta\mathfrak{F}}{\delta A_\mu(x)} \delta A_\mu(x) d^4x.$$

trictly speaking, this definition is meaningless since the quantities appearing therein are operators which do not commute with each other. We give meaning to the equation by assuming that the $\delta A_\mu(x)$ are c numbers. When so defined, the functional derivatives have the property that
$$\left[\frac{\delta}{\delta A_\mu(x)}, \frac{\delta}{\delta A_\nu(x')}\right] = 0.$$

general case, let us proceed by assuming that it is true for some T product, $T[\mathfrak{F}(A_\mu)]$, i.e., that

$$T[\mathfrak{F}(A_\mu)] = N[\mathfrak{F}(A_\mu')]. \quad (8)$$

Then, if we can prove it for $T[A_\lambda(x)\mathfrak{F}(A_\nu)]$, we can conclude, by induction, that the above-mentioned statement is indeed valid. This last assertion follows from the fact that every operator functional can be expressed as a functional power series in the $A_\mu(x)$. To proceed with the proof, let us assume that the value of x_0 lies somewhere between the times to which the operators in $\mathfrak{F}(A)$ correspond. We express this by the equation

$$T[A_\mu(x)\mathfrak{F}(A)] = X(A)A_\mu(x)Y(A),$$

or equivalently, by

$$T[A_\mu(x)\mathfrak{F}(A)] = X\{A_\mu^-(x) + A_\mu^+(x)\}Y(A), \quad (9)$$

where

$$X(A)Y(A) = \mathfrak{F}(A).$$

In order to convert Eq. (9) into an N product we must commute $A_\mu^-(x)$ through X and $A_\mu^+(x)$ through Y. This can be done by making use of the commutation relations (3) and yields

$$T[A_\mu(x)\mathfrak{F}(A)] = A_\mu^-(x)XY$$
$$+ \left\{ -i\int d^4x' D_A^-(x-x') \frac{\delta}{\delta A_\mu^+(x}X \right\}Y + XYA_\mu^+(x)$$
$$+ X\left\{ -i\int d^4x D_R^+(x-x') \frac{\delta}{\delta A_\mu^-(x')}Y \right\}.$$

However, because of the nature of $D_A^-(x)$ and $D_R^+(x)$ and because all times in Y precede x_0 which in turn precedes all times in X, we can rewrite this equation as

$$T[A_\mu(x)F(A)] = A_\mu^-(x)XY + XYA_\mu^+(x)$$
$$+ \left\{ -i\int d^2x' D_A^-(x-x') \frac{\delta}{\delta A_\mu^+(x')} \right.$$
$$\left. -i\int d^4x' D_R^+(x-x') \frac{\delta}{\delta A_\mu^-(x')} \right\}XY. \quad (10)$$

When we substitute from Eq. (8) into Eq. (10) we obtain

$$T[A_\mu(x)\mathfrak{F}(A)] = A_\mu^-(x)N[\mathfrak{F}(A')] + N[\mathfrak{F}(A')]A_\mu^+(x)$$
$$+ \left\{ -i\int d^4x' D_A^-(x-x') \frac{\delta}{\delta A_\mu^+(x'} \right.$$
$$\left. -i\int d^4x' D_R^+(x-x') \frac{\delta}{\delta A_\mu^-(x')} \right\}N[\mathfrak{F}(A')]. \quad (11)$$

Since now the quantities $A_\mu^-(x)$ and $A_\mu^+(x)$ appear in $\mathfrak{F}(A)$ only in the combination $A_\mu^+(x) + A_\mu^-(x)$, we

note that

$$\frac{\mathfrak{d}}{\mathfrak{d}A_\mu{}^-(x)}N[\mathfrak{F}(A')] = N\left[\frac{\mathfrak{d}\mathfrak{F}(A')}{\mathfrak{d}A_\mu(x)}\right],$$

and

$$\frac{\mathfrak{d}}{\mathfrak{d}A_\mu{}^+(x)}N[\mathfrak{F}(A')] = N\left[\frac{\mathfrak{d}\mathfrak{F}(A')}{\mathfrak{d}A_\mu(x)}\right].$$

Therefore, Eq. (11) becomes

$$T[A_\mu(x)\mathfrak{F}(A)] = N[\{A_\mu{}^+(x)+A_\mu{}^-(x)\}\mathfrak{F}(A')]$$

$$-i\int d^4x'[D_R{}^+(x-x')+D_A{}^-(x-x')]N\left[\frac{\mathfrak{d}\mathfrak{F}(A')}{\mathfrak{d}A_\mu(x')}\right].$$

Finally, remembering the definition of $D^F(x)$, we have

$$T[A_\mu(x)\mathfrak{F}(A)]$$

$$= N\left[\left\{A_\mu(x)+\int d^4x' D^F(x-x')\frac{\mathfrak{d}}{\mathfrak{d}A_\mu(x')}\right\}\mathfrak{F}(A')\right]$$

$$= N[A_\mu{}'(x)\mathfrak{F}(A')],$$

which was to be proved.

We can reformulate this result by noting that Eq. (7) can be rewritten in the form

$$A_\mu{}'(x) = e^\Delta A_\mu(x)e^{-\Delta},$$

where

$$\Delta = \frac{1}{2}\int d^4x d^4x' \delta_{\mu\nu}D^F(x-x')\frac{\mathfrak{d}}{\mathfrak{d}A_\mu(x)}\frac{\mathfrak{d}}{\mathfrak{d}A_\nu(x')}, \quad (12)$$

as can readily be verified by direct calculation. Therefore, if

$$T[\mathfrak{F}(A)] = N[\mathfrak{F}(A')],$$

then it is also true that

$$T[\mathfrak{F}(A)] = N[e^\Delta \mathfrak{F}(A)e^{-\Delta}].$$

In particular, if we are interested only in the matrix elements of $T[\mathfrak{F}(A)]$ we can disregard the factor $e^{-\Delta}$, since there is nothing for it to operate on, and write finally

$$\langle T[\mathfrak{F}(A)]\rangle = \langle N[e^\Delta \mathfrak{F}(A)]\rangle. \quad (13)$$

In what follows, we shall refer to Δ as the photon ordering operator.

We can also derive an expression for an electron-positron ordering operator. In order to do so we must decompose the electron-positron operator according to

$$\psi(x) = u(x)+\bar{v}(x),$$

$$\bar{\psi}(x) = \bar{u}(x)+v(x),$$

where $u(\bar{u}=u^\dagger\beta)$ destroys (creates) electrons, and $v(\bar{v}=\beta v^\dagger)$ destroys (creates) positrons.[11] For our purposes, we shall need the following table of anti-

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[11] We represent the positrons by "negative-energy" electron wave functions and not by the charge-conjugate functions.

commutators:

$$\{u_\alpha(z),\bar{u}_\beta(y)\} = -i\{S_{\alpha\beta}{}^{A+}(z-y)-S_{\alpha\beta}{}^{R+}(z-y)\},$$

$$\{\bar{v}_\beta(y),v_\alpha(z)\} = -i\{S_{\alpha\beta}{}^{A-}(z-y)-S_{\alpha\beta}{}^{R-}(z-y)\},$$

with all other possible anticommutators vanishing. The Feynman kernel function $S_{\alpha\beta}{}^F(x)$ is defined as

$$S_{\alpha\beta}{}^F(x) = -i\{S_{\alpha\beta}{}^{R+}(x)+S_{\alpha\beta}{}^{A-}(x)\}, \quad (14)$$

and satisfies

$$L_{\alpha\beta}{}^0(x)S_{\beta\gamma}{}^F(x) = i\delta_{\alpha\gamma}\delta(x), \quad (15)$$

where

$$L_{\alpha\beta}{}^0(x) \equiv (-i\gamma_{\alpha\beta}{}^\mu\partial_\mu+\delta_{\alpha\beta}m). \quad (16)$$

We can, in complete analogy with the method used to arrive at Eq. (7), derive a similar set of expressions for the electron-positron operators. We will not repeat the details of the proof, but simply assert that for any functional, $\mathfrak{H}(\psi,\bar{\psi})$,

$$T[\mathfrak{H}(\psi,\bar{\psi})] = N[\mathfrak{H}(\psi',\bar{\psi}')],$$

where

$$\psi_\alpha{}'(x) = \psi_\alpha(x) - \int d^4x' S_{\alpha\beta}{}^F(x-x')\frac{\mathfrak{d}}{\mathfrak{d}\bar{\psi}_\beta(x')}, \quad (17a)$$

and

$$\bar{\psi}_\beta{}'(x') = \bar{\psi}_\beta(x') + \int d^4x \frac{\mathfrak{d}}{\mathfrak{d}\psi_\alpha(x)}S_{\alpha\beta}{}^F(x-x'). \quad (17b)$$

Now, however, due to the fact that the electron-positron operators anticommute with each other, we must give a slightly different meaning to the functional derivative as employed in Eqs. (17a and b). A moment's consideration shows that the correct definition is given by the following:

$$\left\{\frac{\mathfrak{d}}{\mathfrak{d}\psi_\alpha(x)},\bar{\psi}_\beta(x')\right\} = 0; \quad \left\{\frac{\mathfrak{d}}{\mathfrak{d}\bar{\psi}_\beta(x)},\psi_\alpha(x')\right\} = 0,$$

$$\left\{\frac{\mathfrak{d}}{\mathfrak{d}\psi_\alpha(x)},\psi_\beta(x')\right\} = \delta_{\alpha\beta}\delta(x-x'),$$

$$\left\{\frac{\mathfrak{d}}{\mathfrak{d}\bar{\psi}_\alpha(x)},\bar{\psi}_\beta(x')\right\} = \delta_{\alpha\beta}\delta(x-x'),$$

from which it immediately follows that

$$\left\{\frac{\mathfrak{d}}{\mathfrak{d}\psi_\alpha(x)},\frac{\mathfrak{d}}{\mathfrak{d}\bar{\psi}_\beta(x')}\right\} = 0.$$

As before, we can introduce an ordering operator Σ defined by

$$\Sigma = \int d^4x d^4x' \frac{\mathfrak{d}}{\mathfrak{d}\psi_\alpha(x)}S_{\alpha\beta}{}^F(x-x')\frac{\mathfrak{d}}{\mathfrak{d}\bar{\psi}_\beta(x')}, \quad (18)$$

and write, in place of Eqs. (17a,b),

$$\psi_\alpha{}'(x) = e^\Sigma\psi_\alpha(x)e^{-\Sigma}, \quad (19a)$$

and
$$\bar{\psi}_\beta'(x) = e^\Sigma \bar{\psi}_\beta(x) e^{-\Sigma}. \tag{19b}$$

Hence,
$$T[\mathfrak{H}(\psi,\bar{\psi})] = N[e^\Sigma \mathfrak{H}(\psi,\bar{\psi}) e^{-\Sigma}],$$

and again, if we are interested only in the matrix elements of $\mathfrak{H}(\psi,\bar{\psi})$ we can write

$$\langle T[\mathfrak{H}(\psi,\bar{\psi})] \rangle = \langle N[e^\Sigma \mathfrak{H}(\psi,\bar{\psi})] \rangle. \tag{20}$$

Upon combining the results contained in Eqs. (13), (20), which we can do since the two fields commute, we obtain the equation linking a given T product to its corresponding N product, namely,[12]

$$\langle T[\mathfrak{G}(A,\psi,\bar{\psi})] \rangle = \langle N[e^\Sigma e^\Delta \mathfrak{G}(A,\psi,\bar{\psi})] \rangle. \tag{21}$$

As we shall see later on, this result will greatly simplify our work in treating the various Green's functions to be introduced.

In passing, it is interesting to note that it is possible to derive from Eq. (21) the more conventional rules for the transformation of a T product into its corresponding N product.[3,8] For the sake of simplicity, we will carry out the demonstration only for the electromagnetic field, and leave the case of the electron-positron field to the interested reader. We shall first state the rules whereby the transformation can be effected. In any T product of the $A_\mu(x)$ we pick out a certain even number of factors, either none or all or any intermediate number, and associate them together in pairs. We replace each pair of factors $A_\mu(x)$, $A_\nu(y)$ by $\delta_{\mu\nu}D^F(x-y)$ and multiply the result by the remaining factors of the T product arranged in a normal order. For instance, in the T product $T[A_\mu(x_1), A_\nu(x_2), A_\lambda(x_3), A_\tau(x_4)]$, the possible results for such factor pairings are given below:

$$N[A_\mu(x_1)A_\nu(x_2)A_\lambda(x_3)A_\tau(x_4)],$$

$$\delta_{\mu\nu}D^F(x_1-x_2)N[A_\lambda(x_3)A_\tau(x_4)],$$
$$\delta_{\mu\lambda}D^F(x_1-x_3)N[A_\nu(x_2)A_\tau(x_4)], \cdots,$$

$$\delta_{\mu\nu}D^F(x_1-x_2)\delta_{\lambda\tau}D^F(x_3-x_4),$$
$$\delta_{\mu\lambda}D^F(x_1-x_3)\delta_{\nu\tau}D^F(x_2-x_4), \cdots.$$

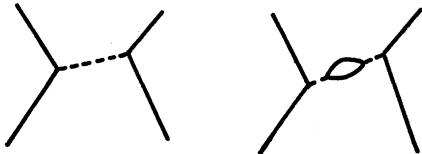Wick has then shown that a given T product is equal to the sum of the results of all such factor pairings, i.e.

FIG. 1. Scattering of electron by electron in second and fourth orders. The photon (dotted) line is a λ line.

[12] S. Hori, Progr. Theoret. Phys. 7, 578 (1952) has obtained a similar formula but only for the special case where $\mathfrak{G}$ is the S matrix itself. We have developed Eq. (21) since we will need it to treat more general forms of $\mathfrak{G}$.

for the example given, the T product is equal to the sum of all the expressions in the above table.

In order to show that Eq. (21) is equivalent to these rules, we need merely to expand e^Δ in a power series. The first term in the expansion is just unity and hence we simply arrange all of the factors $A_\mu(x)$ appearing in $\mathfrak{G}$ in a normal order: for this first term we make no factor pairings. The next term in the expansion is just Δ. Because of the nature of the functional derivatives, the net effect of operating with it on $\mathfrak{G}$ is to pick out, in all possible ways, two factors $A_\mu(x)$ and $A_\nu(y)$ and insert in their place $\delta_{\mu\nu}D^F(x-y)$. Upon rearranging the remaining factors in a normal order, as the equation tells us to do, we obtain the sum of all the results of x single factor pairings mentioned above. In a similar manner we see that the third term in the expansion will give us the sum of the results obtained by making all possible pairings involving four factors $A_\mu(x)$, and so on. The factorials compensate for the fact that in the nth term of the expansion each particular pairing occurs $n!$ times.

III. THE ONE-ELECTRON GREEN'S FUNCTION

In dealing with Eq. (21), it is often convenient to employ the so-called Feynman graphs. The Feynman graph for a particular term in the expansion of the right-hand side of Eq. (21) can be drawn as follows: For every factor $D^F(x_i - x_i')$ a dotted (photon) line is drawn connecting the points x_i and x_i'; for every factor $S_{\alpha\beta}^F(x_j - x_j')$ a directed (electron, positron) line is drawn from x_j to x_j'; for the factors $\bar{\psi}_\alpha(x_k)$, $\psi_\beta(x_k')$ directed lines are drawn leading out from x_k to the edge of the diagram, and in from the edge of the diagram to x_k'; for every factor $A_\mu(x_1)$ a line is drawn connecting the edge of the diagram with x_1. As sometimes happens, graphs corresponding to two different terms in the expansion of Eq. (21) differ from each other by only one or more self-energy parts. In Fig. 1 we have an example of such a situation. Here the single line connecting the points x and x' in one graph is replaced, in the other graph, by a subgraph which is unconnected to the rest of the diagram except by two lines running from it to x and x'. Graphs which contain self-energy portions are termed "reducible" graphs and always correspond to some particular "irreducible" or "primitive" graph. A particular line in a primitive graph, which has as its counterpart in the associated reducible graph, a self-energy subgraph, will be denoted as a λ line. Dyson has shown[3] that the sum of all of the terms in Eq. (21) which correspond to a given primitive graph, plus all of its associated reducible graphs, can be reduced to a single term to be associated with the primitive graph. This term is obtained in the following manner: In the term associated with the primitive graph we replace each factor $S_{\alpha\beta}^F(x-x')$ or $D^F(x-x')$ as the case may be, which corresponds to a λ line, by the new factors $G_{\alpha\beta}(x,x')$, $\mathfrak{K}_{\mu\nu}(x,x')$, respectively. Substitutions

must also be made for external λ lines but for our purposes we need not consider them. These new factors are given by

$$G_{\alpha\beta}(x,x') = \langle T[\bar{\psi}_\beta(x')\psi_\alpha(x)\mathfrak{S}]\rangle_0 S_{\text{vac}}^{-1}, \quad (22)$$

and

$$\mathfrak{K}_{\mu\nu}(x,x') = \langle T[A_\mu(x)A_\nu(x')\mathfrak{S}]\rangle_0 S_{\text{vac}}^{-1}, \quad (23)$$

where the subscript 0 indicates that we are to take the vacuum expectation value of the quantity appearing between the brackets and where $\mathfrak{S}$ is defined through Eq. (1). S_{vac} is just the vacuum expectation value of the S matrix.

In this section we shall derive a closed-form expression for $G_{\alpha\beta}(x,x')$ in terms of S_{vac}, the vacuum expectation value of the S matrix and at the same time derive the differential equation satisfied by G. As we shall see, this equation will be identical with the one given by Schwinger for his one-electron Green's function, and hence we will be able to conclude that Schwinger's G is identical to the one defined in Eq. (22). With this fact in mind, we can also call our G the one-electron Green's function.

In order to investigate the properties of G, let us apply the results of Eq. (21) to Eq. (22). We have then that

$$G_{\alpha\beta}(x,x') = \langle N[e^\Delta e^\Sigma \bar{\psi}_\beta(x')\psi_\alpha(x)\mathfrak{S}]\rangle_0 S_{\text{vac}}^{-1}. \quad (24)$$

In the investigation of such quantities as appear in the right-hand side of Eq. (24) it is convenient to know the commutation properties of Σ with respect to $\psi_\alpha(x)$ and $\bar{\psi}_\beta(x')$. By combining the results of Eqs. (17a,b; 19a,b) we obtain directly that

$$[e^\Sigma,\psi_\alpha(x)] = -\int d^4y \, S_{\alpha\sigma}^F(x-y)\frac{\delta}{\delta\bar{\psi}_\sigma(y)}e^\Sigma, \quad (25a)$$

and

$$[e^\Sigma,\bar{\psi}_\beta(x')] = \int d^4y' \, S_{\rho\beta}^F(y'-x')\frac{\delta}{\delta\psi_\rho(y')}e^\Sigma. \quad (25b)$$

Let us now use Eq. (25b) to interchange the positions of e^Σ and $\bar{\psi}_\beta(x')\psi_\alpha(x)$ in Eq. (24). The net effect of this interchange is simply that

$$G_{\alpha\beta}(x-x') = S_{\alpha\beta}^F(x-x')$$

$$- \int d^4y d^4y' \, S_{\alpha\sigma}^F(x-y)S_{\rho\beta}^F(y'-x')$$

$$\times \left\langle N\left[\frac{\delta}{\delta\psi_\rho(y')}\frac{\delta}{\delta\bar{\psi}_\sigma(y)}e^\Sigma e^\Delta\mathfrak{S}\right]\right\rangle_0 S_{\text{vac}}^{-1}. \quad (26)$$

Because of the form of Σ, we also have that

$$\frac{\delta \ln S_{\text{vac}}}{\delta S_{\rho\sigma}^F(y'-y)} = \left\{\frac{\delta}{\delta S_{\rho\sigma}^F(y'-y)}\langle N[e^\Delta e^\Sigma\mathfrak{S}]\rangle_0\right\}S_{\text{vac}}^{-1}$$

$$= \left\langle N\left[\frac{\delta}{\delta\psi_\rho(y')}\frac{\delta}{\delta\bar{\psi}_\sigma(y)}e^\Delta e^\Sigma\mathfrak{S}\right]\right\rangle_0 S_{\text{vac}}^{-1}. \quad (27)$$

Direct substitution in Eq. (27) into Eq. (26) then yields

$$G_{\alpha\beta}(x,x') = S_{\alpha\beta}^F(x-x') + \int\int d^4y d^4y' \, S_{\alpha\sigma}^F(x-y)$$

$$\times \frac{\delta \ln S_{\text{vac}}}{\delta S_{\sigma\rho}^F(y-y')}S_{\rho\beta}^F(y'-x'). \quad (28)$$

Thus we see that the one-electron Green's function can be expressed entirely in terms of the single quantity S_{vac}.

At this point a word should be said concerning S_{vac}. In Eq. (27) we have treated S_{vac} as though it were a functional of $S^F(x)$. Actually, since S^F is a specified function of its arguments, S_{vac} is just a number. Strictly speaking then, the functional derivative of S_{vac} with respect to $S^F(x)$ has no meaning. However, we can give it meaning if we consider $S^F(x)$, and also $D^F(x)$, to be arbitrary functions of their arguments. It is only after we carry out the prescribed operations on S_{vac} that they assume their actual functional dependence.

Although we have obtained an expression for the one-electron Green's function, in terms of S_{vac}, it is sometimes convenient to know the differential equation satisfied by $G_{\alpha\beta}$. We can derive this equation by again making use of Eq. (24). It will prove convenient however, to interchange the order of $\psi_\alpha(x)$ and $\bar{\psi}_\beta(x')$ and write

$$G_{\alpha\beta}(x,x') = -\langle N[e^\Delta e^\Sigma\psi_\alpha(x)\bar{\psi}_\beta(x')\mathfrak{S}]\rangle_0 S_{\text{vac}}^{-1}.$$

Let us now commute e^Σ through $\psi_\alpha(x)$. If we refer to Eq. (25a), we see that the result is

$$G_{\alpha\beta}(x,x') = \left\langle N\left[\int d^4y \, S_{\alpha\rho}^F(x-y)\right.\right.$$

$$\times \left.\left.\frac{\delta}{\delta\bar{\psi}_\rho(y)}e^\Delta e^\Sigma\bar{\psi}_\beta(x')\mathfrak{S}\right]\right\rangle_0 S_{\text{vac}}^{-1}. \quad (29)$$

(The term which contains the factor $\psi_\alpha(x)$ standing to the left of e^Σ vanishes since we are taking the vacuum expectation value of the entire expression.) Now, since the functional derivative $\delta/\delta\bar{\psi}_\rho$ (y) commutes with e^Σ, we can perform the indicated differentiation in Eq. (29) and so obtain

$$G_{\alpha\beta}(x,x') = S_{\alpha\beta}^F(x-x') - \int d^4y \, S_{\alpha\rho}^F(x-y)\langle N\{iee^\Delta e^\Sigma$$

$$\times \gamma_{\rho\sigma}{}^\mu\bar{\psi}_\beta(x')\psi_\sigma(y)A_\mu(y)\mathfrak{S}\}\rangle_0 S_{\text{vac}}^{-1}. \quad (30)$$

Let us now make use of the fact that

$$\delta\mathfrak{S}/\delta J_\mu(y) = iA_\mu(y)\mathfrak{S}$$

to rewrite Eq. (30) in the form

$$G_{\alpha\beta}(x,x') = S_{\alpha\beta}^F(x-x') + e\int d^4y \, S_{\alpha\rho}^F(x-y)\frac{\delta}{\delta J_\mu(y)}$$

$$\times \langle N\{e^\Delta e^\Sigma\bar{\psi}_\beta(x')\psi_\sigma(y)\mathfrak{S}\}\rangle_0\gamma_{\rho\sigma}{}^\mu S_{\text{vac}}^{-1}. \quad (31)$$

Now substitute the result of Eq. (28) into the bracketed part of Eq. (31), and obtain

$$G_{\alpha\beta}(x,x') = S_{\alpha\beta}{}^F(x-x') + e \int d^4y S_{\alpha\rho}{}^F(x-y)\gamma_{\rho\sigma}{}^\mu$$

$$\times \frac{\delta}{\delta J_\mu(y)}\left\{ S_{\beta\sigma}{}^F(x'-y)S_{\text{vac}} + \int\int d^4z d^4z' S_{\beta\lambda}{}^F(x-z) \right.$$

$$\left. \times \frac{\delta S_{\text{vac}}}{\delta S_{\lambda\tau}{}^F(z-z')}S_{\tau\sigma}{}^F(z'-y) \right\} S_{\text{vac}}^{-1},$$

which becomes, after a slight rearrangement of terms,

$$G_{\alpha\beta}(x,x') = S_{\alpha\beta}{}^F(x,x') - e\int d^4y S_{\alpha\rho}{}^F(x-y)$$

$$\times \gamma_{\rho\sigma}{}^\mu S_{\sigma\beta}{}^F(y-x')\frac{\delta \ln S_{\text{vac}}}{\delta J_\mu(y)} - e\int d^4y S_{\alpha\rho}{}^F(x-y)$$

$$\times \gamma_{\rho\sigma}{}^\mu \int\int d^4z d^4z' S_{\sigma\tau}{}^F(y-z)\frac{\delta \ln S_{\text{vac}}}{\delta S_{\tau\lambda}{}^F(z-z')}$$

$$\times S_{\lambda\beta}{}^F(z'-x)\frac{\delta \ln S_{\text{vac}}}{\delta J_\mu(y)} - e\int d^4y S_{\alpha\rho}{}^F(x-y)\gamma_{\rho\sigma}{}^\mu$$

$$\times \int\int d^4z d^4z' S_{\sigma\tau}{}^F(y-z)\frac{\delta^2 \ln S_{\text{vac}}}{\delta S_{\tau\lambda}{}^F(z-z')\delta J_\mu(y)}$$

$$\times S_{\lambda\beta}{}^F(z'-x).$$

This result can then be expressed in terms of $G_{\alpha\beta}$ as

$$G_{\alpha\beta}(x,x') = S_{\alpha\beta}{}^F(x-x') - e\int d^4y S_{\alpha\rho}{}^F(x-y)$$

$$\times G_{\sigma\beta}(y,x')\gamma_{\rho\sigma}{}^\mu\frac{\delta \ln S_{\text{vac}}}{\delta J_\mu(y)} - e\int d^4y S_{\alpha\rho}{}^F(x-y)$$

$$\times \gamma_{\rho\sigma}{}^\mu\frac{\delta G_{\sigma\beta}(y,x')}{\delta J_\mu(y)}. \quad (32)$$

In order to remain as close to Schwinger's notation as possible, we shall introduce at this point a quantity $\langle A_\mu(x)\rangle$ defined by

$$\langle A_\mu(x)\rangle = -i\frac{\delta \ln S_{\text{vac}}}{\delta J_\mu(x)}.$$

In terms of $\langle A_\mu\rangle$, Eq. (32) then becomes

$$G_{\alpha\beta}(x,x') = S_{\alpha\beta}{}^F(x-x') - ie\int d^4y S_{\alpha\rho}{}^F(x-y)G_{\sigma\beta}(y,x')$$

$$\times \gamma_{\rho\sigma}{}^\mu\langle A_\mu(y)\rangle - e\int d^4y S_{\alpha\rho}{}^F(x-y)\gamma_{\rho\sigma}{}^\mu\frac{\delta G_{\sigma\beta}(y,x')}{\delta J_\mu(y)}. \quad (34)$$

Finally, by applying the differential operator $L_{\alpha\beta}{}^0$ to both sides of Eq. (34) we obtain, as the differential equation satisfied by $G_{\alpha\beta}$,

$$\{L_{\alpha\beta}{}^0(x) - e\langle A_\mu(x)\rangle\gamma_{\alpha\sigma}{}^\mu\}G_{\sigma\beta}(x,x')$$

$$+ ie\gamma_{\alpha\sigma}{}^\mu\frac{\delta G_{\sigma\beta}(x,x')}{\delta J_\mu(x)} = i\delta_{\alpha\beta}\delta(x-x'). \quad (35)$$

Upon comparing this equation for $G_{\alpha\beta}$ with the one given by Schwinger, we see that they are identical and hence we are justified in equating our Green's function with that introduced by Schwinger.

We shall conclude this section with a brief discussion of the so-called "mass" operator introduced by Schwinger. Schwinger has assumed that the functional derivative appearing in Eq. (35) can be represented by an integral operator, i.e., that

$$ie\gamma_{\alpha\sigma}{}^\mu\frac{\delta G_{\sigma\beta}(x,x')}{\delta J_\mu(x)} = \int d^4y \Sigma_{\alpha\sigma}(x,y)G_{\sigma\beta}(y,x'). \quad (36)$$

The mass operator $M_{\alpha\beta}(x,y)$ is then defined by the equation

$$M_{\alpha\beta}(x,y) = m\delta_{\alpha\beta}\delta(x-y) + \Sigma_{\alpha\beta}(x,y).$$

As was mentioned above, Eq. (36) is an assumption. It can be justified, however, and we shall do so by deriving an expression for $\Sigma_{\alpha\beta}(x,y)$ in terms of S_{vac}. Let us begin by rewriting Eq. (28) in Schwinger's matrix notation. We have that

$$G = S^F + S^F K S^F, \quad (37)$$

where K is an abbreviation for the functional derivative of $\ln S_{\text{vac}}$ with respect to S^F. Equation (37) can be solved for S^F, giving

$$S^F = (1 + S^F K)^{-1}G.$$

Substitution of this result back into the second term in the right-hand side of Eq. (37) then yields

$$G = S^F + S^F K(1 + S^F K)^{-1}G.$$

Now let us compare this equation with Eq. (34), written in matrix form and in terms of Σ, viz.,

$$G = S^F - ieS^F\gamma^\mu G\langle A_\mu\rangle + iS^F\Sigma G.$$

We see immediately that Σ is equivalent to

$$\Sigma = e\gamma^\mu\langle A_\mu\rangle + K(1 + S^F K)^{-1}. \quad (39)$$

From this we can conclude that a Σ does exist and indeed can be expressed in terms of S_{vac}. Actually, the derivation leading to Eq. (39) rests upon the assumption that $(1 + S^F K)^{-1}$ exists and is nonsingular. However, even if these conditions are not fulfilled we can still use Eq. (39), at least formally, to express Σ as a series expansion in S^F and K.

IV. PROPERTIES OF $\langle A_\mu(x) \rangle$

In Schwinger's original work on Green's functions, the idea was put forth that one might eliminate some of the difficulties inherent in the S-matrix formalism by treating Eq. (35) as a basic equation for $G_{\alpha\beta}$. In other words, one need not consider its antecedents, but attempt to solve it directly for $G_{\alpha\beta}$. In order to carry out this program consistently, one then needs an equation to determine $\langle A_\mu(x) \rangle$. In this section we shall derive such an equation, and, although we do not adhere to Schwinger's program in this paper, we shall find it useful for other purposes. From Eq. (33) we see that

$$\langle A_\mu(x) \rangle = \langle N[e^\Delta e^\Sigma A_\mu(x) \mathfrak{S}] \rangle_0 S_{\text{vac}}{}^{-1}.$$

Upon commuting e^Δ through $A_\mu(x)$ we obtain

$$\langle A_\mu(x) \rangle$$
$$= \left\langle N \left[e^\Delta e^\Sigma \int d^4 y D^F(x-y) \frac{\mathfrak{d}}{\mathfrak{d} A_\mu(y)} \mathfrak{S} \right] \right\rangle_0 S_{\text{vac}}{}^{-1}. \quad (40)$$

By carrying out the indicated functional differentiation, Eq. (40) runs to

$$\langle A_\mu(x) \rangle = i \int d^4 y D^F(x-y) \{ J_\mu(y) + \langle j_\mu(y) \rangle \}, \quad (41)$$

where

$$\langle j_\mu(y) \rangle = \langle T[j_\mu(y) \mathfrak{S}] \rangle_0 S_{\text{vac}}{}^{-1}. \quad (42)$$

Equation (41) can be put into the form of a differential equation by operating on it with $-i\square$ to give

$$\square \langle A_\mu(x) \rangle = -J_\mu(x) - \langle j_\mu(x) \rangle. \quad (43)$$

Furthermore, because of the definition of $G_{\alpha\beta}$,

$$\langle j_\mu(x) \rangle = e\gamma_{\alpha\rho}{}^\mu G_{\rho\alpha}(x,x), \quad (44)$$

Eq. (43) can be put in the form

$$\square \langle A_\mu(x) \rangle = -J_\mu(x) - e\gamma_{\alpha\rho}{}^\mu G_{\rho\alpha}(x,x).$$

Hence we can use Eq. (45) together with Eq. (35) to determine both $\langle A_\mu(x) \rangle$ and $G_{\alpha\beta}(x,x')$.[13]

Before concluding this section, it will be of some interest to consider the particular case where $J_\mu(x) = 0$. Since originally $J_\mu(x)$ was introduced into the theory as a mathematical artifice to enable us to derive Eq. (35), we see that this is the case with which we shall have to deal in all final applications of the theory. In cases where differentiation with respect to $J_\mu(x)$ is indicated we must perform the differentiation before we approach

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[13] For a discussion of boundary conditions to be used in connection with solving these equations, see reference 1.

the limit of $J_\mu(x)$ going to zero. Consider now what happens to $G(x,x')$ in the limit. In general $G(x,x')$ is not an invariant function of its arguments under a translation of the coordinate axis since it depends upon $J_\mu(x)$. However, in the limit $G(x,x')$ must be an invariant function of its arguments and hence becomes a function G^0 of the coordinate difference $x - x'$. We can therefore conclude from Eq. (44) that

$$\langle j_\mu(x) \rangle_{J=0} = e\gamma^\mu G^0(0) \equiv 0,$$

since each component of $\langle j_\mu(x) \rangle$ is now a number quite independent of the coordinate system. The only vector having these properties which is invariant under Lorentz transformations is the null vector.

Upon referring to Eq. (41) we see then that

$$\langle A_\mu(x) \rangle_{J=0} = 0.$$

In this case then, Eq. (38) becomes simply

$$G^0 = S^F + iS^F \Sigma^0 G^0,$$

which is just the equation given by Dyson for his Green's function S'^F [Eq. (63), Phys. Rev. **75**, 1736 (1949)].

V. THE ONE-PHOTON GREEN'S FUNCTION

In this section we shall treat some of the properties of the one-photon Green's function as given by Eq. (23). Actually, we shall follow Schwinger and define the one-photon Green's function as

$$\mathfrak{G}_{\mu\nu}(x,x') = \langle T[A_\mu(x) A_\nu(x') \mathfrak{S}] \rangle_0 S_{\text{vac}}{}^{-1} - \langle A_\mu(x) \rangle \langle A_\nu(x') \rangle. \quad (46)$$

From the results of the preceding section it is evident that in the limit $J_\mu \to 0$, $\mathfrak{G}_{\mu\nu}(x,x')$ is equivalent to $\mathfrak{K}_{\mu\nu}(x,x')$. If we keep in mind now the form of $\mathfrak{G}$, we see that Eq. (46) is equivalent to

$$\mathfrak{G}_{\mu\nu}(x,x') = -\frac{\mathfrak{d}^2 S_{\text{vac}}}{\mathfrak{d} J_\mu(x) \mathfrak{d} J_\nu(x')} S_{\text{vac}}{}^{-1} + \frac{\mathfrak{d} \ln S_{\text{vac}}}{\mathfrak{d} J_\mu(x)} \frac{\mathfrak{d} \ln S_{\text{vac}}}{\mathfrak{d} J_\nu(x')},$$

or

$$\mathfrak{G}_{\mu\nu}(x,x') = \frac{\mathfrak{d}^2 \ln S_{\text{vac}}}{\mathfrak{d} J_\mu(x) \mathfrak{d} J_\nu(x')} = \mathfrak{G}_{\nu\mu}(x',x). \quad (47)$$

If we make use of Eq. (33), we can then put Eq. (47) into the form

$$\mathfrak{G}_{\mu\nu}(x,x') = -i \frac{\mathfrak{d} \langle A_\mu(x) \rangle}{\mathfrak{d} J_\nu(x')}. \quad (48)$$

Let us now take the derivative of both sides of Eq. (41). By making use of the identity (48) we so obtain

$$\mathfrak{G}_{\mu\nu}(x,x') = \delta_{\mu\nu} D^F(x-x') + \int d^4 y D^F(x-y) \frac{\mathfrak{d} \langle j_\mu(y) \rangle}{\mathfrak{d} J_\nu(x')}. \quad (49)$$

Finally, by operating on Eq. (49) with $\Box$, we obtain the fundamental equation satisfied by $\mathfrak{G}_{\mu\nu}$:

$$\Box \mathfrak{G}_{\mu\nu}(x,x') = i\delta_{\mu\nu}\delta(x-x') + i\frac{\mathfrak{d}\langle j_\mu(x)\rangle}{\mathfrak{d}J_\nu(x')}. \quad (50)$$

Let us examine the last term in the right-hand side of Eq. (50). From Eq. (44) we have that

$$i\frac{\mathfrak{d}\langle j_\mu(x)\rangle}{\mathfrak{d}J_\nu(x')} = ie\gamma_{\alpha\rho}{}^\mu\frac{\mathfrak{d}G_{\rho\alpha}(x,x')}{\mathfrak{d}J_\nu(x')}. \quad (51)$$

Now Eq. (43) tells us that $J_\mu(x)$ is equal to a term linear in $\langle A_\mu(x)\rangle$ plus an additional term. Hence, by the chain rule of differentiation, we can rewrite Eq. (51) as

$$i\frac{\mathfrak{d}\langle j_\mu(x)\rangle}{\mathfrak{d}J_\nu(x')} = ie\gamma_{\alpha\rho}{}^\mu\int d^4y\frac{\mathfrak{d}G_{\rho\alpha}(x,x')}{\mathfrak{d}\langle A_\sigma(y)\rangle}\frac{\mathfrak{d}\langle A_\sigma(y)\rangle}{\mathfrak{d}J_\nu(x')}.$$

If then we define a new quantity $P_{\mu\nu}(x,x')$ by the equation

$$P_{\mu\nu}(x,x') = ie\gamma_{\alpha\rho}{}^\mu\frac{\mathfrak{d}G_{\rho\alpha}(x,x')}{\mathfrak{d}\langle A_\nu(x')\rangle},$$

we can put our fundamental Eq. (50) into a more symmetrical form, namely,

$$\Box \mathfrak{G}_{\mu\nu}(x,x') = i\delta_{\mu\nu}\delta(x-x')$$
$$+ i\int d^4y P_{\mu\sigma}(x,y)\mathfrak{G}_{\sigma\nu}(y,x'). \quad (53)$$

In the particular case where $J_\mu = 0$, Eq. (49) becomes, in matrix notation:

$$\mathfrak{G}_{\mu\nu}{}^0 = \delta_{\mu\nu}D^F - iD^F P_{\mu\sigma}{}^0\mathfrak{G}_{\sigma\nu}{}^0.$$

From this expression, we see that $\mathfrak{G}_{\mu\nu}{}^0$ corresponds to Dyson's D'^F, and iP^0 corresponds to the proper photon-self energy Π^*.

VI. DISCUSSION

In this paper we have demonstrated the equivalence between the Green's functions and associated structures such as the mass operator introduced by Schwinger and the modified propagation functions occuring in the S-matrix theory. In the course of the demonstration a curious result appeared, namely, that all of the various quantities introduced by Schwinger could be expressed in terms of a single quantity S_{vac}. This fact suggests a possible alternate approach to investigating the properties of the Green's functions. Instead of examining each one separately, as is now done, one might begin by examining the properties of S_{vac}, considered as a functional of S^F, D^F, and J_μ. Such a study would entail, in the first place, a knowledge of how S_{vac} depends upon its arguments. By making use of the results of this paper it is possible to derive certain closed-form expressions for S_{vac}. Unfortunately, these expres-

sions all involve the Green's functions themselves, and hence are of little help at present in analyzing S_{vac}. Even if we did know the exact functional dependence of S_{vac} we would still be very far from our goal. We would need in addition a mathematical formalism somewhat akin to analysis as applied to ordinary functions since, as we have seen, the Green's functions all involve functional derivatives of S_{vac} with respect to its arguments. In this respect we can but hope that the mathematicians will become interested in the problem and develop for us a theory of functional analysis.

There also exists a second possibility which appears to be somewhat more manageable. As we have seen, the Green's functions attempt to take into account, in one fell swoop as it were, all of the virtual processes associated with a given real process. The one-electron Green's function, for example, takes into account all of the virtual processes which accompany a free electron in its flight through space. One can take the position, as Schwinger has done, that since these virtual processes are in principle not observable they ought not to appear in the theory. In other words, one should work solely with the Green's functions which are given in terms of some integro-differential equation, and not worry about their antecedents. The chief drawback to such a scheme is that some of the quantities appearing in the integro-differential equations for the Green's functions, such as the mass operator, are themselves extremely complicated objects which up to now have been expressible only as power series in the various propagation functions. An approach of this type has been attempted by Edwards with some degree of success.[14]

There does exist a third approach, however, lying intermediate between the two approaches outlined above, which we would like to outline briefly. Consider for a moment the situation in which an electron is scattered by some external field. Given its initial energy and momentum, we wish to know what will be its final energy and momentum. In attempting to answer this problem from a theoretical standpoint, one of the first questions which arises is how to describe the initial and final states of the electron. In the present form of the theory, the electron is described initially and finally by eigenstates of the free-field Hamiltonian, H_0. However, even before it interacts with the external field, the electron is in interaction with its own virtual photon field. Hence one should, in describing the initial and final states of the electron, include the effects of these virtual photons. As we have seen, our Green's functions include in their description of the electron just these virtual fields. It appears reasonable, therefore, to employ as eigenstates describing the electron, eigenstates of the operator appearing in the equation for the one-electron Green's function. Once one has done this, there exists the possibility that the matrix

[14] S. F. Edwards, Phys. Rev. **90**, 284 (1953).

elements of the S matrix could then be expressed in some simple manner in terms of these eigenstates. The main problem facing the theory would then be shifted to that of calculating these eigenstates, where one would have a much better chance of separating out the infinities which arise in the present theory.

Low-Energy Limits and Renormalization in Meson Theory

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A technique is developed for deriving rigorous expressions for zero-energy mesonic processes. Making use of the ambiguity of mesonic charge renormalization, the coupling constant is defined by zero-energy pion-nucleon scattering. The threshold photomeson production amplitude is also calculated. The experimental value of the coupling constant defined by scattering is at least an order of magnitude less than that of the coupling constant deduced from the photoproduction data. It is shown that pair suppression effects are absent in both cases. The possibility that the results obtained imply disagreement of pseudoscalar theory with experiment is discussed.

I. INTRODUCTION

IT is almost universally accepted at present that π mesons are pseudoscalar, and that to a very good approximation isotopic spin is a good quantum number in mesonic-nucleon processes. Unfortunately, it is not at all clear to what extent the conventional pseudo-scalar symmetrical meson theory correctly describes such processes. Because of the difficulty of carrying out accurate calculations, it has been hard to distinguish between the predictions of the theory and the effects of the particular approximation schemes hitherto employed. In order to avoid extraneous difficulties, it seems advisable to concentrate on understanding low-energy phenomena, where both nucleons and mesons are nonrelativistic. Two such processes, which will be discussed in the present paper, are threshold photo-meson production and meson-nucleon scattering at zero energy.

By the use of special techniques it has been possible to evaluate the matrix elements for these two processes rigorously in the limit of vanishing meson mass. These are, of course, purely formal results; however, as is discussed at length below, one of them, say the meson scattering, may be used to assign a precise numerical value to the coupling constant and the other then in principle serves as a check on the agreement between theory and experiment. Unfortunately we are not able

to evaluate the photo-meson matrix element explicitly, and thus unambiguously settle this crucial question; nor is the present experimental data sufficiently accurate to give a reliable number for the coupling constant. We can, however, say the following: If in the computation of meson-nucleon scattering one may without serious error set the mass of the real mesons equal to zero in the portion of the scattering amplitude which is independent of isotopic spin, one deduces from the existing data the value $g^2/4\pi \sim \frac{1}{3}$. If one may assume that with such a small value of the coupling constant perturbation theory is valid (at least in an asymptotic sense), then one may conclude that the present theory does not correctly describe the behavior of π mesons. Of course, should the value of $g^2/4\pi$ turn out to be much larger or should the neglect of the meson mass mentioned above prove unwarranted, we can make no definite statement.

Before the theory can be compared with experiment, it must, of course, be renormalized. Pseudoscalar theory (with pseudoscalar coupling) is a "renormalizable" theory in the conventional sense of the term; however, in contrast to quantum electrodynamics, the renormalization program may here be carried out in a variety of nonequivalent ways. This fact is of great importance in any attempts to ascertain the physical content of mesonic calculations. Charge renormalization is uniquely defined in electrodynamics;[1] this is closely related to the existence of a charge conservation law and finds its

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[1] G. Källén (to be published).