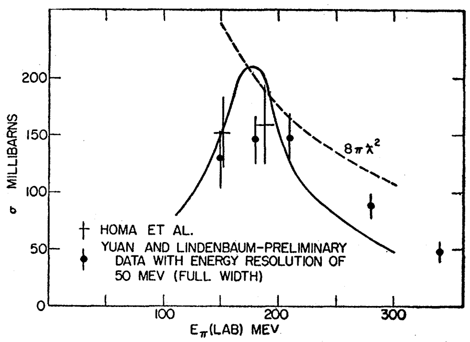

FIG. 8. Total cross section for scattering of positive mesons from protons. Only $p_{\frac{3}{2}}$ and $s_{\frac{1}{2}}$ phase shifts, taken from upper curve Fig. 3 and from Eq. (12), are considered. For experimental work see reference 19.

lower energies, in disagreement with experiment for positive mesons.

There is a rough consistancy between this phenomenological theory and experiment. The limitations of the theory are also quite evident. In the region near 310 Mev, where the assumption of energy independence should introduce only small errors, the agreement between theory and experiment is only within 10 or 20 percent. Of course absolute errors of this order may be present in the data, but this seems an unlikely explanation of some of the present difficulties. Improvement in the theory might be obtained by adjusting the coupling constant (i.e., see Fig. 5). Also the less important angular momentum isotopic spin states could be considered more fully. It is further seen, for example in $d\sigma^+/d\Omega(90°)$ (Fig. 5), how the theory breaks down completely at the high and low ends of the energy region. That the matrix elements should decrease in this region is indicated by examination of the Born approximation term;[3,20] and there is, perhaps, room for extension of the theory by making a detailed examination of the energy dependence of the various terms. Rapid changes are not indicated, however, and it seems clear that unless fairly rapid change of parameters with energy should be predicted, very good agreement with experiment would not be obtained. As our Born approximation term is only calculated in the weak coupling approximation, such difficulty is not surprising.

The author would like to thank Professor H. A. Bethe for his interest in this work.

[20] In the spirit of the present theory, the photoproduction reduces to the weak coupling limit as threshold is approached (see Fig. 4 with $G^2/4\pi=16$). This is in accord with the idea of N. M. Kroll and M. A. Ruderman, Phys. Rev. **93**, 233 (1954). The cross section $d\sigma^+/d\Omega(90°)=6$ at about E_γ(lab) = 175 Mev, recently reported by Bernardini (reference 16), would be fitted in this theory by a coupling constant of about 14.

PHYSICAL REVIEW VOLUME 94, NUMBER 2 APRIL 15, 1954

A Covariant Treatment of Meson-Nucleon Scattering

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(Received January 4, 1954)

A covariant equation for the meson-nucleon system is presented, in which the renormalization of divergent processes is carried out to all orders. A closed expression is given for their contribution to the wave function after renormalization, while the contribution coming from finite processes still involves a series expansion. Exact formulas are derived for the scattering phase shifts.

I. INTRODUCTION

RECENT experiments on pion-nucleon scattering[1] have made apparent the inadequacy of the Born approximation for the calculation of this process and the necessity of a theoretical analysis based on more elaborate methods.

Several attempts have been made[2] to analyze the data by means of the Tamm-Dancoff[3] nonadiabatic method, or an improved form of it.[4] Although this method seems to yield results which are in qualitative agreement with experiment, at least for the p wave, its defects are even more apparent here than in the treatment of nuclear forces.[5] A rapid calculation shows indeed that, even for low-energy scattering, high momenta play a decisive role in intermediate states, and that, consequently, the convergence of the interaction expansion can be expected to be very poor. Moreover, the main contribution to the scattering cross sections

[1] Barnes, Angell, Perry, Miller, Ring, and Nelson, Phys. Rev. **92**, 1327 (1953); Bodansky, Sachs, and Steinberger, Phys. Rev. **93**, 918 (1954); Anderson, Fermi, Martin, and Nagle, Phys. Rev. **91**, 155 (1953).

[2] G. F. Chew, Phys. Rev. **89**, 591 (1953); J. S. Blair and G. F. Chew, Phys. Rev. **90**, 1065 (1953); S. Fubini, Nuovo cimento **10**, 564 (1953); Dyson, Schweber, and Visshér, Phys. Rev. **90**, 372 (1953); Sundaresan, Salpeter, and Ross, Phys. Rev. **90**, 372 (1953); N. Fukuda, Proceedings of the International Conference of Kyoto, September, 1953 (unpublished).

[3] I. Tamm, J. Phys. U.S.S.R. **9**, 449 (1945); S. M. Dancoff, Phys. Rev. **78**, 382 (1950).

[4] F. J. Dyson, Phys. Rev. **91**, 1543 (1953).

[5] M. M. Lévy, Phys. Rev. **88**, 72, 725 (1952); A. Klein, Phys. Rev. **90**, 1101 (1953).

comes from radiative corrections which cannot be handled correctly within the framework of the Tamm-Dancoff method.

These difficulties are partly removed if one uses, instead, a covariant nonadiabatic two-body integral equation, analogous to the one proposed by Salpeter and Bethe,[6] which can easily be extended to the present case. (As in Karplus *et al.*,[7] it will be called, in the following, the M.N. equation.) However, as in the case of quantum electrodynamics, renormalization—that is, the unambiguous elimination of unobservable divergent quantities—has still to be performed on the integral equation, using the methods of Feynman[8] and Dyson.[9]

In the present problem, however, special difficulties appear in the course of the renormalization process, which are essentially due to the fact that π mesons can play simultaneously the dual role of virtual field quanta and of real interacting particles. Mathematically, the difficulties arise through the fact that integral equations which possess perfectly finite kernels do not yield finite solutions, because they involve summation over a series of virtual processes, some of which include radiative effects. Partial solutions to these difficulties have been proposed by Karplus *et al.*[7] and Fubini.[10] In the treatment of these authors, however, renormalization is performed through a special device, the validity of which is limited to the lowest order, and it is not easily seen how the solution can be extended *to all orders*. Moreover, they work in terms of the Feynman two-body kernel, which is really convenient only when total cross sections need to be computed. When a phase-shift analysis of the experimental data is necessary, it is much easier to work in terms of the M.N. "wave function," as will be seen later.

The purpose of the present paper is to provide a framework within which scattering phase shifts and cross sections can be computed to all orders without renormalization difficulties. All the infinite diagrams are separated out, and a *closed* expression is given for their contribution to the wave function after renormalization. The calculation of the contribution coming from finite processes, however, still involves a series expansion, the convergence of which is not discussed here.[11]

The removal of divergences in the M.N. equation is most easily understood if, at first, only the two lowest-order diagrams (which are of the second order in the coupling constant) are included in the interaction kernel. This is done in Sec. II. The solution is presented, however, in such a form that the extension to all orders (Sec. III) is almost immediate. Exact expressions for

[6] E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951); M. Gell-Mann and F. Low, Phys. Rev. **84**, 350 (1951).
[7] Karplus, Kivelson, and Martin, Phys. Rev. **90**, 1072 (1953).
[8] R. P. Feynman, Phys. Rev. **76**, 749, 769 (1949).
[9] F. J. Dyson, Phys. Rev. **75**, 486, 1736 (1949).
[10] S. Fubini, Nuovo cimento **10**, 851 (1953).
[11] See, however, the concluding remarks of Sec. V.

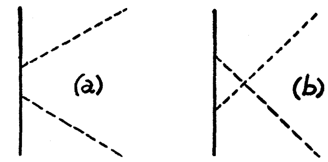

FIG. 1. The two basic diagrams of meson-nucleon scattering to the second order in the coupling constant.

the scattering phase-shifts are given in Sec. IV. In the handling of the M.N. "wave function," a good deal is implicitly used of a covariant theory of scattering, which starts from a relativistic four-dimensional wave equation, and is therefore the logical relativistic extension of the standard Faxén-Holtzmarck treatment[12] of the Schrödinger equation. This theory, which also yields a variational principle for the scattering phase-shifts, will be discussed in a subsequent paper.

The application of the formal results contained in the present paper to the analysis of low- and high-energy pion-nucleon scattering data will be presented later. The connection between meson-nucleon scattering and nuclear forces will also be discussed at that time.

II. COVARIANT EQUATION TO THE LOWEST ORDER IN THE COUPLING CONSTANT

In this section, only physical processes corresponding to an infinite number of iterations of the two basic diagrams of Fig. 1 will be considered. We shall call $K_M(x,y)$ and $K_N(x,y)$, respectively, the meson and nucleon propagation functions between two points x and y in space-time, their Fourier transforms being defined as follows:

$$K_M(x,y)=(2\pi)^{-4}\int K_M(p)\exp[ip(x-y)]d^4p,$$
$$K_N(x,y)=(2\pi)^{-4}\int K_N(p)\exp[ip(x-y)]d^4p. \quad (1)$$

These definitions imply therefore the following connections with the well known S_F and Δ_F functions, as defined by Dyson:[9]

$$K_M(p)=\tfrac{1}{2}\Delta_F(p)=-i(p^2+\mu^2)^{-1},$$
$$K_N(p)=-\tfrac{1}{2}S_F(p)=-i(i\gamma p+M)^{-1},$$

where μ and M are the meson and nucleon masses. The Feynman two-body kernel (see, for example, reference 6) will be written as $K(x,\xi;y,\eta)$.

For the sake of definiteness, we shall assume that we are dealing with a symmetrical mixture of pseudoscalar mesons with pseudoscalar coupling to the nucleons, writing therefore the interaction Hamiltonian as

$$H_{\text{int}}=iG\bar{\psi}\gamma_5\tau_k\psi\varphi_k, \quad (2)$$

where G is the coupling constant, ψ and φ_k the nucleon and meson fields, respectively. The theory can be, how-

[12] See, for example, N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Oxford University Press, London, 1949), second edition, Chap. II.

ever, easily generalized to all "renormalizable" types of interaction.

1. Unrenormalized Equations

The integral equation for the Feynman two-body kernel corresponding to an infinite number of repetitions of Figs. 1(a) and 1(b) can be written as

$$\begin{aligned} K(x,\xi_i;\,y,\eta_j) &= K_N(x,y)K_M(\xi_i,\eta_j) \\ &\quad -iG^2\int K_N(x,\xi')K_M(\xi_i,\xi_k')\gamma_5\tau_k K_N(\xi',\eta') \\ &\qquad \times\gamma_5\tau_l K(\eta',\eta_l';\,y,\eta_j)d^4\xi'd^4\eta' \\ &\quad -iG^2\int K_N(x,\xi')K_M(\xi_i,\eta_k')\gamma_5\tau_l K_N(\xi',\eta') \\ &\qquad \times\gamma_5\tau_k K(\eta',\xi_l';\,y,\eta_j)d^4\xi'd^4\eta', \end{aligned} \tag{3}$$

where the isotopic spin indices have been written as subscripts to the meson variables. Equation (3) contains implicitly divergent higher-order effects such as, for example, those described by the two *reducible*[13] diagrams of Fig. 2. It leads to an unrenormalized equation for the wave function $\psi(x,\xi_i)$ defined as follows:

$$\psi(x,\xi_i)=\lim_{t_y,t_\eta\to-\infty}\int K(x,\xi_i;\,y,\eta_j)\gamma_4\psi_0(y,\eta_j)d\mathbf{y}d\mathbf{n}, \tag{4}$$

where ψ_0 is the free wave function of the system,

$$\psi_0(x,\xi_i)=\lim_{t_y,t_\eta\to-\infty}\int K_N(x,y)K_M(\xi_i,\eta_j)\times\gamma_4\psi_0(y,\eta_j)d\mathbf{y}d\mathbf{n}, \tag{5}$$

and which can be written

$$\begin{aligned} \psi(x,\xi_i) &= \psi_0(x,\xi_i) \\ &\quad -iG^2\int K_N(x,\xi')K_M(\xi_i,\xi_k')\gamma_5\tau_k K_N(\xi',\eta') \\ &\qquad \times\gamma_5\tau_l\psi(\eta',\eta_l')d^4\xi'd^4\eta' \\ &\quad -iG^2\int K_N(x,\xi')K_M(\xi_i,\eta_k')\gamma_5\tau_l K_N(\xi',\eta') \\ &\qquad \times\gamma_5\tau_k\psi(\eta',\xi_l')d^4\xi'd^4\eta'. \end{aligned} \tag{6}$$

2. Separation of the Divergences

We split the two-body kernel into two parts:

$$K=K_a+K_b, \tag{7}$$

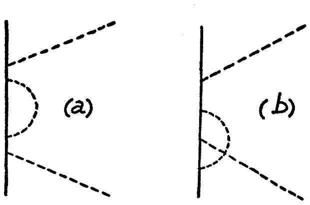

FIG. 2. Two reducible graphs of the meson-nucleon scattering matrix, which result from the combination, in high orders, of diagrams (a) and (b) of Fig. 1 and include divergent effects.

[13] A graph of the meson-nucleon scattering matrix is called *irreducible* if it is not possible to draw a line through it, cutting one nucleon line and one meson line only. Otherwise, it is called *reducible.*

where K_b contains only the *convergent* graphs *which result from the iteration of graph (1,b) alone* (examples are given in Fig. 3). The equation for K_b is simply obtained by suppressing the second term of the right-hand side of (3) and by writing K_b instead of K:

$$\begin{aligned} K_b(x,\xi_i;\,y,\eta_j) &= K_N(x,y)K_M(\xi_i,\eta_j) \\ &\quad -iG^2\int K_N(x,\xi')K_M(\xi_i,\eta_k')\gamma_5\tau_l K_N(\xi',\eta') \\ &\qquad \times\gamma_5\tau_k K_b(\eta',\xi_l';\,y,\eta_j)d^4\xi'd^4\eta'. \end{aligned} \tag{8}$$

The kernel function K_a contains all remaining divergent graphs, which essentially consist of three parts: a self-energy part which is an arbitrary combination of two types of diagrams, the general terms of which are represented in Figs. 4(a) and 4(b); and a vertex part on each side, belonging, respectively, to the general types described in Figs. 4(α) and 4(β). $K_a(x,\xi_i;\,y,\eta_j)$ can therefore be written formally as

$$\begin{aligned} K_a(x,\xi_i;\,y,\eta_j) &= -iG^2\int K_N(x,x')K_M(\xi_i,\xi_k') \\ &\quad \times\Gamma_5^{(\beta)}(\xi_k';\,x',y')\tau_k K_N'(y',x'')\tau_l\Gamma_5^{(\alpha)}(\eta_l';\,x'',y'') \\ &\quad \times K_N(y'',y)K_M(\eta_l',\eta_j)d^4x'd^4x''d^4y'd^4y''d^4\xi'd^4\eta'. \end{aligned} \tag{9}$$

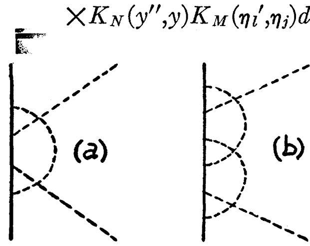

FIG. 3. Two reducible graphs corresponding to convergent processes and resulting from the iteration of diagram (b) of Fig. 1.

By using Eqs. (4), (7), (8), and (9), the wave function $\psi(x,\xi_i)$ can now be written

$$\psi(x,\xi_i)=\psi_a+\psi_b, \tag{10}$$

where ψ_b is the finite solution of the convergent integral equation

$$\begin{aligned} \psi_b(x,\xi_i) &= \psi_0(x,\xi_i)-iG^2\int K_N(x,\xi')K_M(\xi_i,\eta_k') \\ &\qquad \times\gamma_5\tau_l K_N(\xi',\eta')\gamma_5\tau_k\psi_b(\eta',\xi_l')d^4\xi'd^4\eta', \end{aligned} \tag{11}$$

and ψ_a is defined by the expression

$$\begin{aligned} \psi_a(x,\xi_i) &= -iG^2\int K_N(x,x')K_M(\xi_i,\xi_k') \\ &\quad \times\Gamma_5^{(\beta)}(\xi_k';\,x',y')\tau_k K_N'(y',x'')\tau_l\Gamma_5^{(\alpha)}(\eta_l',;\,x'',y'') \\ &\quad \times\psi_0(y'',\eta_l')d^4x'd^4x''d^4y'd^4y''d^4\xi'd^4\eta'. \end{aligned} \tag{12}$$

All the infinities are now concentrated in the vertex functions $\Gamma_5^{(\alpha)}$ and $\Gamma_5^{(\beta)}$, and in the modified nucleon propagation function K_N'. Our remaining task is to calculate these functions in closed form in terms of

K_b and ψ_b, and to renormalize them by means of the usual methods.

3. Calculation and Renormalization of $\Gamma_5^{(\alpha)}$ and $\Gamma_5^{(\beta)}$

Let us write for $\rho=\alpha,\beta$

$$\Gamma_5^{(\rho)}(\xi_i;x,y)=\gamma_5\delta(x-\xi)\delta(y-\xi)+\Lambda_5^{(\rho)}(\xi_i;x,y). \quad (13)$$

Calculating the successive contributions of the graphs contained in $\Lambda_5^{(\alpha)}$, one can write a power series expansion in G^2:

$$\Lambda_5^{(\alpha)}(\xi_i;x,y)=-iG^2\gamma_5\tau_k\Big\{K_N(x,\xi)K_M(x_k,y_k)$$
$$-iG^2\int K_N(x,\xi')K_M(x_k,\eta_k')\gamma_5\tau_lK_N(\xi',\eta')$$
$$\times\gamma_5\tau_kK_N(\eta',\xi)K_M(\xi_l',y_l)d^4\xi'd^4\eta'$$
$$+\cdots\Big\}\gamma_5\tau_iK_N(\xi,y)\gamma_5\tau_k, \quad (14)$$

and, by comparison with the corresponding expansion of $K_b(x,\xi_i;y,\eta_j)$, one obtains the following relation:

$$\Lambda_5^{(\alpha)}(\xi_i;x,y)=-iG^2\gamma_5\tau_kK_b(x,x_k;\xi,y_l)$$
$$\times\gamma_5\tau_iK_N(\xi,y)\gamma_5\tau_l. \quad (15)$$

Similarly, $\Lambda_5^{(\beta)}$ is expressed in terms of K_b by the equation

$$\Lambda_5^{(\beta)}(\xi_i;x,y)=-iG^2\gamma_5\tau_lK_N(x,\xi)$$
$$\times\gamma_5\tau_iK_b(\xi,x_l;y,y_k)\gamma_5\tau_k. \quad (16)$$

In order to perform the renormalization of these two vertex operators, we introduce first their Fourier transforms, which can be written, for $\rho=\alpha,\beta$, as

$$\Lambda_{5,i}^{(\rho)}(p,q)=\int\Lambda_5^{(\rho)}(\xi_i;x,y)$$
$$\times\exp[-ip(x-\xi)+iq(y-\xi)]d^4xd^4y. \quad (17)$$

The renormalized operators are then obtained by the usual method:

$$\Lambda_{5,i}^{*(\rho)}(p,q)=\Lambda_{5,i}^{(\rho)}(p,q)-\Lambda_{5,i}^{(\rho)}(p_0,p_0), \quad (18)$$

where p_0 is the energy momentum of a free nucleon satisfying the relation $(i\gamma p_0+M)=0$. In configuration space, Eq. (18) can be written as

$$\Lambda_5^{*(\rho)}(\xi_i;x,y)=\Lambda_5^{(\rho)}(\xi_i;x,y)$$
$$-\Lambda_{5,i}^{(\rho)}(p_0,p_0)\delta(x-\xi)\delta(y-\xi), \quad (19)$$

where $\Lambda_{5,i}^{(\rho)}(p_0,p_0)$, for $\rho=\alpha,\beta$, are two divergent constants which have to be calculated. Equation (17) gives

$$\Lambda_{5,i}^{(\rho)}(p_0,p_0)=\int\Lambda_5^{(\rho)}(\xi_i;x,y)$$
$$\times\exp[ip_0(y-x)]d^4xd^4y, \quad (20)$$

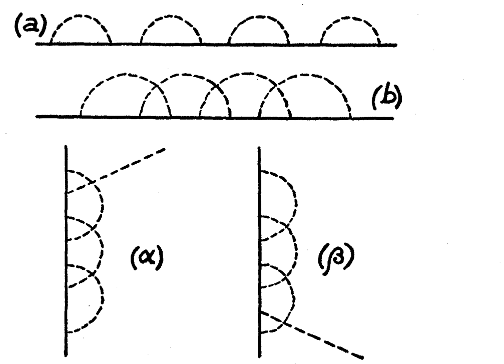

FIG. 4. General form of the divergent self-energy and vertex parts of diagrams corresponding to processes included in the kernel function K_a.

and consequently

$$\Lambda_{5,i}^{(\alpha)}(p_0,p_0)=-iG^2\gamma_5\tau_k\int K_b(x,x_k;\xi,y_l)\gamma_5\tau_iK_N(\xi,y)$$
$$\times\gamma_5\tau_l\exp[ip_0(y-x)]d^4yd^4x,$$
$$\Lambda_{5,i}^{(\beta)}(p_0,p_0)=-iG^2\gamma_5\tau_l\int K_N(x,\xi)\gamma_5\tau_iK_b(\xi,x_l;y,y_k)$$
$$\times\gamma_5\tau_k\exp[ip_0(y-x)]d^4yd^4x. \quad (21)$$

These relations can be simplified by expressing them in energy-momentum space, where K_b is defined by the equation

$$K_b(x,\xi_i;y,\eta_j)=(2\pi)^{-12}\int K_b^{(i,j)}(p,q;p'q')$$
$$\times\delta(p+q-p'-q')\exp[i(px+q\xi-p'y-q'\eta)]$$
$$\times d^4pd^4qd^4p'd^4q'. \quad (22)$$

The results are

$$\Lambda_{5,i}^{(\alpha)}(p_0,p_0)=-iG^2\gamma_5(2\pi)^{-8}\tau_k\int K_b^{(k,l)}$$
$$\times(p,p_0-p;q,p_0-q)\gamma_5\tau_iK_N(q)\gamma_5\tau_kd^4pd^4q,$$
$$\Lambda_{5,i}^{(\beta)}(p_0,p_0)=-iG^2\gamma_5\tau_k(2\pi)^{-8}\int K_N(p)\gamma_5\tau_iK_b^{(k,l)}$$
$$\times(p,p_0-p;q,p_0-q)\gamma_5\tau_ld^4pd^4q. \quad (23)$$

4. Calculation and Renormalization of $K_N'(x,y)$

The modified propagation function introduced in subsection (II,2) is, according to Dyson,[9] the solution of an integral equation of the form

$$K_N'(x,y)=K_N(x,y)+\int K_N(x,z)F(z,z')$$
$$\times K_N'(z',y)dzdz', \quad (24)$$

where $F(z,z')$ contains only the irreducible self-energy graphs, namely those which cannot be split into two parts by cutting only one nucleon line and no meson line. The series of self-energy processes, the general term of which is represented in Fig. 4(a), corresponds therefore to only one (irreducible) term in F, namely,

$$F_1(z,z')=-iG^2\gamma_5\tau_k K_N(z,z')\gamma_5\tau_k K_M(z_k,z_k'). \quad (25)$$

The processes which are represented by diagrams like Fig. (4b) are, on the contrary, all irreducible. Their contribution to F is consequently represented by a power series expansion in G^2:

$$\begin{aligned}F_2(z,z')=&(-iG^2)^2\gamma_5\tau_k\int K_N(z,\xi)K_M(z_k,\eta_k)\\&\times\gamma_5\tau_l K_N(\xi,\eta)\gamma_5\tau_k K_N(\eta,z')K_M(\xi_l,z_l')\gamma_5\tau_l d^4\xi d^4\eta\\&+(-iG^2)^3\gamma_5\tau_k\int K_N(z,\xi)K_M(z_k,\eta_k)\gamma_5\tau_m K_N(\xi,\eta)\\&\times\gamma_5\tau_k K_N(\eta,\xi')\gamma_5\tau_l K_M(\xi_m,\eta_m')\gamma_5\tau_m K_N(\eta',z')\\&\times K_M(\xi_l',z_l')\gamma_5\tau_l d^4\xi d^4\eta d^4\xi' d^4\eta'+\cdots. \quad (26)\end{aligned}$$

A rapid inspection of this expansion shows that it is directly related to $K_b(z,z_k;z',z_l')$, through the equation

$$\begin{aligned}F_2(z,z')=-iG^2\gamma_5\tau_k[&K_b(z,z_k;z',z_l')\\&-K_N(z,z')K_M(z_k,z_l')]\gamma_5\tau_l. \quad (27)\end{aligned}$$

Combining (25) and (27) yields the relation

$$F(z,z')=F_1+F_2=-iG^2\gamma_5\tau_k K_b(z,z_k;z',z_l')\gamma_5\tau_l. \quad (28)$$

Expressing Eq. (24) in momentum space and using (22) and (28), we are led to a purely algebraic expression for $K_N'(p)$:

$$K_N'(p)=K_N(p)/[1-\Sigma(p)K_N(p)], \quad (29)$$

where we have introduced the notation

$$\begin{aligned}\Sigma(p)=-iG^2\gamma_5\tau_k\int K_b^{(k,l)}(q,p-q;q',p-q')\\\times\gamma_5\tau_l d^4q d^4q'. \quad (30)\end{aligned}$$

The only divergent quantity is evidently $\Sigma(p)$ which can be renormalized by the usual method (see reference 9):

$$\begin{aligned}\Sigma^*(p)=&\Sigma(p)-\Sigma(p_0)\\&+\tfrac{1}{4}(i\gamma_\mu p_\mu+M)\left[i\gamma_\nu\frac{\partial\Sigma(p)}{\partial p_\nu}\right]_{p=p_0}, \quad (31)\end{aligned}$$

where p_0 is again the energy-momentum of a free nucleon satisfying the relation: $i\gamma p_0+M=0$. The renormalized modified propagation function is then obtained by putting $\Sigma^*(p)$ instead of $\Sigma(p)$ in Eq. (29).

5. Final Expression of $\psi_a(x,\xi_i)$

Having obtained finite expressions for $\Gamma_{5,i}^{(\alpha)}$, $\Gamma_{5,i}^{(\beta)}$ and K_N', we are now in a position to calculate K_a and ψ_a in a closed finite form in terms of K_b and ψ_b:

$$\begin{aligned}K_a(x,\xi_i;y,\eta_j)=&-iG^2\int K_N(x,\xi')K_M(\xi_i,\eta_k')\\&\times\{\gamma_5\tau_k[1-\Lambda_{0k}^{(\beta)}]\delta(\xi'-\eta')\delta(\xi''-\eta')\\&-iG^2\gamma_5\tau_m K_N(\xi',\eta')\gamma_5\tau_k K_b(\eta',\xi_m';\xi'',\xi''_h)\gamma_5\tau_h\}\\&\times K_N'(\xi'',\eta'')\{\gamma_5\tau_l[1-\Lambda_{0l}^{(\alpha)}]\delta(\eta''-\xi''')\delta(\xi'''-\eta''')\\&-iG^2\gamma_5\tau_r K_b(\eta'',\eta_r'';\xi''',\eta_s''')\gamma_5\tau_l\\&\times K_N(\xi''',\eta''')\gamma_5\tau_s\}K_N(\eta''',y)K_M(\xi''',\eta_j)\\&d^4\xi'\cdots d^4\xi'''d^4\eta'\cdots d^4\eta''', \quad (32)\end{aligned}$$

where we have put $\Lambda_{5i}^{(\rho)}(p_0,p_0)=\gamma_5\Lambda_{0i}^{(\rho)}$, for $\rho=\alpha,\beta$. Equation (32) can be simplified by integrating over the δ functions and by making use of the integral equation (8) for $K_b(x,\xi_i;y,\eta_j)$, as well as another one which is completely equivalent to (8), namely,

$$\begin{aligned}K_b(x,\xi_i;y,\eta_j)=&K_N(x,y)K_M(\xi_i,\eta_j)\\&-iG^2\int K_b(x,\xi_i;\xi',\eta_k')\gamma_5\tau_l K_N(\xi',\eta')\\&\times\gamma_5\tau_k K_N(\xi',y)K_M(\eta_l',\eta_j)d\xi'd\eta'. \quad (33)\end{aligned}$$

The resulting expression for K_a is given by

$$\begin{aligned}K_a(x,\xi_i;y,\eta_j)=&-iG^2\int[K_b(x,\xi_i;\xi',\xi_k')\\&-\Lambda_{0k}^{(\beta)}K_N(x,\xi')K_M(\xi_i,\xi_k')]\gamma_5\tau_k\\&\times K_N'(\xi',\eta')\gamma_5\tau_l[K_b(\eta',\eta_l';y,\eta_j)\\&-\Lambda_{0l}^{(\alpha)}K_N(\eta',y)K_M(\eta_l',\eta_j)]d\xi'd\eta'. \quad (34)\end{aligned}$$

Making use of Eqs. (4) and (5), one obtains next an equation for $\psi_a(x,\xi_i)$:

$$\begin{aligned}\psi_a(x,\xi_i)=&-iG^2\int[K_b(x,\xi_i;\xi',\xi_k')\\&-\Lambda_{0k}^{(\beta)}K_N(x,\xi')K_M(\xi_i,\xi_k')]\gamma_5\tau_k K_N'(\xi',\eta')\\&\times\gamma_5\tau_l[\psi_b(\eta',\eta_l')-\Lambda_{0l}^{(\alpha)}\psi_0(\eta',\eta_l')]d\xi'd\eta'. \quad (35)\end{aligned}$$

This expression, which is a kind of inhomogeneous term to be added to ψ_b in order to get the exact wave function ψ, is actually a "contact" term since, in the center-of-mass system, it only involves the value of ψ_b at the origin. This leads to a particularly simple expression for the contribution of ψ_a to the S-matrix elements, as will be seen in Sec. IV.

III. RENORMALIZATION TO ALL ORDERS

The treatment given in Sec. II can now be extended to the irreducible processes corresponding to all orders in G^2 by very small modifications of the formalism.

Equations (7) and (10), through which the separation of divergences is performed, are still valid, and so are Eqs. (34) and (35) which express the renormalized functions K_a and ψ_a in terms of K_b and ψ_b. The latter quantities, however, obey now more general integral equations of the form

$$K_b(x,\xi_i;\, y,\eta_j)=K_N(x,y)K_M(\xi_i,\eta_j)+\int K_N(x,z)K_M(\xi_i,\xi_k)I_b(z,\zeta_k;\, z',\zeta_l')\times K_b(z',\zeta_l';\, y,\eta_j)dzdz'd\zeta d\zeta', \quad (36)$$

and

$$\psi_b(x,\xi_i)=\psi_0(x,\xi_i)+\int K_N(x,z)K_M(\xi_i,\zeta_k)\times I_b(z,\zeta_k;\, z',\zeta_l')\psi_b(z',\zeta_l')dzdz'd\zeta d\zeta', \quad (37)$$

where $I_b(z,\zeta_k;\, z',\zeta_l')$ is an interaction function which can be described by means of a series of graphs having the following properties:

(a) they are all irreducible (in the sense defined in reference 13);

(b) they contain no vertex or self-energy parts;[14]

(c) they are all finite and lead individually to finite expressions for K_b and ψ_b (we do not discuss here the convergence of I_b).

I_b can be expanded in a power series in G^2:

$$I_b=I_b^{(1)}+I_b^{(2)}+\cdots+I_b^{(n)}+\cdots, \quad (38)$$

where $I_b^{(n)}$ is proportional to G^{2n}. $I_b^{(1)}$ corresponds to diagram 1(b) and has therefore the expression

$$I_b^{(1)}=-iG^2\gamma_5\tau_l K_N(z,\zeta)\gamma_5\tau_k\delta(z-z')\delta(\zeta-\zeta'). \quad (39)$$

$I_b^{(2)}$ corresponds to only one fourth-order diagram, represented in Fig. 5(a), and is given by the relation

$$I_b^{(2)}=(-iG^2)^2\gamma_5\tau_m K_N(z,\zeta')\gamma_5\tau_l K_N(\zeta',\zeta)\times\gamma_5\tau_k K_N(\zeta,z')\gamma_5\tau_m K_M(z'_m,z_m). \quad (40)$$

$I_b^{(3)}$ corresponds to seven sixth-order diagrams, which are drawn in Fig. ($5b,c,d,\alpha,\beta,\gamma,\delta$), etc.

The above description makes it clear how the interaction function I_b has to be computed. The main point is that going to higher orders does not introduce any new divergence. All divergent processes have been, once and for all, separated out in K_a and ψ_a, and their renormalized contributions expressed in terms of the finite functions K_b and ψ_b.

It is perhaps worth mentioning that we have been concerned, so far, only in the removal of the special types of divergences introduced by the combination, in higher orders, of graph 1(a) with the other graphs of the meson-nucleon scattering matrix. There remain, of course, all the "normal" types of divergences which

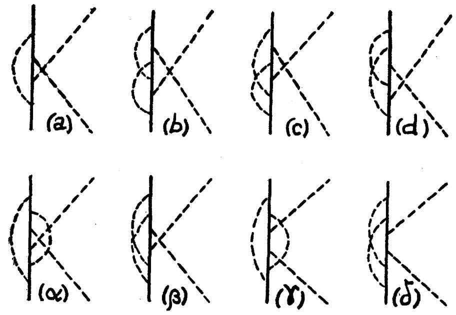

FIG. 5. Irreducible fourth- and sixth-order diagrams included in the interaction function I_b.

can be handled by standard methods: namely, in the final expressions for K_a, K_b, ψ_a, and ψ_b, all the K_N and K_M functions should be replaced by the exact modified propagation functions[15] K_N' and K_M', and γ_5 by the vertex operator Γ_5. Finally, the contributions of processes involving meson-meson scattering should be renormalized by means of an additional nonlinear $\lambda\varphi^{*2}\varphi^2$ term in the interaction Hamiltonian[16] and added to the function I_b introduced in this section.

IV. CALCULATION OF SCATTERING PHASE-SHIFTS

The elements of the S matrix between two free states $\psi_0(x,\xi_i)$ and $\chi_0(x,\xi_j)$ are related to the Feynman two-body kernel by the relation

$$(\mathbf{k}_1',\mathbf{k}_2',j|S-1|\mathbf{k}_1,\mathbf{k}_2,i)=\lim_{\substack{t_x,t_\xi\to+\infty\\ t_y,t_\eta\to-\infty}}\int\bar{\chi}_0(x,\xi_j)\gamma_4\times\{K(x,\xi_j;\, y,\eta_i)-K_N(x,y)K_M(\xi_j,\eta_i)\}\times\gamma_4\psi_0(y,\eta_i)d\mathbf{x}d\mathbf{y}d\xi d\mathbf{n}, \quad (41)$$

where the initial and final states are defined by the equations

$$\psi_0(x,\xi_i)=\psi_0^{(i)}(\mathbf{k}_1,\mathbf{k}_2)\exp[ik_1x+ik_2\xi],$$
$$\chi_0(x,\xi_j)=\chi_0^{(i)}(\mathbf{k}_1',\mathbf{k}_2')\exp[ik_1'x+ik_2'\xi]. \quad (42)$$

Using Eqs. (4) and (5), putting $S-1=R=R_a+R_b$, we can write (41) as follows:

$$(\mathbf{k}_1',\mathbf{k}_2',j|R_a|\mathbf{k}_1,\mathbf{k}_2,i)=\lim_{t_x,t_\xi\to+\infty}\int\bar{\chi}_0(x,\xi_j)\times\gamma_4\psi_a(x,\xi_j)d\mathbf{x}d\xi, \quad (43)$$

$$(\mathbf{k}_1',\mathbf{k}_2',j|R_b|\mathbf{k}_1,\mathbf{k}_2,i)=\lim_{t_x,t_\xi\to+\infty}\int\bar{\chi}_0(x,\xi_j)\gamma_4\times\{\psi_b(x,\xi_j)-\psi_0(x,\xi_j)\}d\mathbf{x}d\xi. \quad (44)$$

[14] This means that they are also *irreducible* in the sense defined by Dyson (reference 9), whose definition is different from the one given in reference 13.

[15] This modified propagation function K_N' (as well as the vertex operator Γ_5) should not be confused with the functions K_N', $\Gamma_5^{(\alpha)}$ and $\Gamma_5^{(\beta)}$ which we have introduced in Sec. II. These functions have been expressed, once and for all, in terms of the unmodified functions K_N, K_M and γ_5 through Eqs. (13), (15), (16), (29), (30), and (36).

[16] A. Salam, Phys. Rev. **86**, 731 (1952).

1. Expression of the R-Matrix Elements

The matrix elements of R_a defined by (43) can be expressed in terms of K_b and ψ_b by means of Eq. (35):

$$(\mathbf{k}_1',\mathbf{k}_2',j|R_a|\mathbf{k}_1,\mathbf{k}_2,i)=\lim_{t_x,t_\xi\to+\infty}-iG^2\int\bar{\chi}_0(x,\xi_j)\gamma_4 \times[K_b(x,\xi_j;\xi',\xi_k')-\Lambda_{0k}^{(\beta)}K_N(x,\xi')K_M(\xi_i,\xi_k')] \times\gamma_5\tau_k K_N'(\xi',\eta')\gamma_5\tau_l[\psi_b(\eta',\eta_l')-\Lambda_{0l}^{(\alpha)}\psi_0(\eta',\eta_l')] \times d\mathbf{x}d\xi d\xi'd\eta'. \quad (45)$$

This expression can be simplified further if one introduces the function $\bar{\chi}_b(y,\eta_i)$ defined by

$$\bar{\chi}_b(y,\eta_i)=\lim_{t_x,t_\xi\to+\infty}\int\bar{\chi}_0(x,\xi_j)\gamma_4K_b(x,\xi_j;y,\eta_i)d\mathbf{x}d\xi, \quad (46)$$

which is adjoint to that particular "wave function" of the system which, after an infinite time, transforms into the final free state $\chi_0(y,\eta_i)$. There exists obviously the relation

$$\bar{\chi}_0(y,\eta_i)=\lim_{t_x,t_\xi\to+\infty}\int\bar{\chi}_0(x,\xi_j)\gamma_4K_N(x,y) \times K_M(\xi_j,\eta_i)d\mathbf{x}d\xi, \quad (47)$$

and $\bar{\chi}_b(y,\eta_i)$ satisfies the integral equation

$$\bar{\chi}_b(y,\eta_i)=\bar{\chi}_0(y,\eta_i)+\int\bar{\chi}_b(z,\zeta_k)I_b(z,\zeta_k;z',\zeta_l') \times K_N(z',y)K_M(\zeta_l',\eta_i)dzdz'd\zeta d\zeta', \quad (48)$$

which can be obtained from Eq. (36) by multiplying on the right by $\bar{\chi}_0(x,\xi_j)$, integrating over the space components of x and ξ and letting the corresponding times go to infinity. Equation (45) becomes, then,

$$(\mathbf{k}_1',\mathbf{k}_2',j|R_a|\mathbf{k}_1,\mathbf{k}_2,i)=-iG^2\int[\bar{\chi}_b(\xi',\xi_k') -\Lambda_{0k}^{(\beta)}\bar{\chi}_0(\xi',\xi_k')]\gamma_5\tau_kK_N'(\xi',\eta')\gamma_5\tau_l \times[\psi_b(\eta',\eta_l')-\Lambda_{0l}^{(\alpha)}\psi_0(\eta',\eta_l')]d\xi'd\eta'. \quad (49)$$

It is now convenient to separate out the motion of the center of mass, by means of the transformation:

$$X=\alpha x+(1-\alpha)\xi,\quad z=x-\xi, \quad (50)$$

where the parameter α expresses the well-known ambiguity in defining relativistically the position of the center of mass. We can write[17]

$$\psi_b(x,\xi_i)=\varphi_b^{(i)}(z)\exp(iP_\mu X_\mu), \quad (51)$$

where

$P_\mu=(P_0,\mathbf{P})$, $\mathbf{P}=\mathbf{k}_1+\mathbf{k}_2$, $P_0=E_{k1}+\omega_{k2}$, $E_{k1}=(k_1^2+M^2)^{\frac{1}{2}}$, and

$$\omega_{k2}=(k_2^2+\mu^2)^{\frac{1}{2}}.$$

Similarly, we have

$$\bar{\chi}_b(x,\xi_i)=\bar{\rho}_b^{(i)}(z)\exp(-iP_\mu'X_\mu), \quad (52)$$

with $P_\mu'=(P_0',\mathbf{P}')$, $\mathbf{P}'=\mathbf{k}_1'+\mathbf{k}_2'$ and $P_0'=E_{k1'}+\omega_{k2'}$. The resulting expression of the R_a matrix elements is therefore

$$(\mathbf{k}_1',\mathbf{k}_2',j|R_a|\mathbf{k}_1,\mathbf{k}_2,i) =-iG^2(2\pi)^4[\bar{\rho}_b^{(k)}(0)-\Lambda_{0k}^{(\beta)}\bar{\rho}_0^{(k)}(0)]\gamma_5\tau_kK_N'(P)\gamma_5\tau_l \times[\varphi_b^{(l)}(0)-\Lambda_{0l}^{(\alpha)}\varphi_0^{(l)}(0)]\delta(\mathbf{P}-\mathbf{P}')\delta(P_0-P_0'). \quad (53)$$

In the same way, the matrix elements of R_b can be simplified to some extent by transforming them to the coordinate system (50). This gives:

$$(\mathbf{k}_1',\mathbf{k}_2',j|R_b|\mathbf{k}_1,\mathbf{k}_2,i) =(2\pi)^3\delta(\mathbf{P}-\mathbf{P}')\lim_{T\to+\infty}\{\exp[i(P_0-P_0')T]\} \times\int\bar{\rho}_0^{(j)}(\mathbf{z},0)[\varphi_b^{(j)}(\mathbf{z},0)-\varphi_0^{(j)}(\mathbf{z},0)]d\mathbf{z}, \quad (54)$$

where $\rho_0^{(j)}(\mathbf{z},0)\equiv\chi_0(\mathbf{k}_1',\mathbf{k}_2')\exp[i(\mathbf{k}_1'-\alpha\mathbf{P}')\cdot\mathbf{z}]$. In the derivation of Eq. (54), the adiabatic decoupling of the meson and nucleon fields when time goes to infinity has been implicitly assumed, since the limiting process t_x, $t_\xi\to+\infty$ has been performed on the assumption that $t_x-t_\xi\to0$. Using the same procedure as Kita,[17] we introduce the definition

$$(\mathbf{k}_1',\mathbf{k}_2',j|R_b|\mathbf{k}_1,\mathbf{k}_2,i) =[\![\mathbf{k}_1',\mathbf{k}_2',j|R_b|\mathbf{k}_1,\mathbf{k}_2,i]\!]\delta(\mathbf{P}-\mathbf{P}')\delta(P_0-P_0'), \quad (55)$$

and obtain the relation

$$\delta_+(P_0-P_0')[\![\mathbf{k}_1',\mathbf{k}_2',j|R_b|\mathbf{k}_1,\mathbf{k}_2,i]\!]=(2\pi)^3\chi_0^{*(j)}(\mathbf{k}_1',\mathbf{k}_2') \times\int[\varphi_b^{(j)}(\mathbf{z},0)-\varphi_0^{(j)}(\mathbf{z},0)] \times\exp[-i(\mathbf{k}_1'-\alpha\mathbf{P}')\mathbf{z}]d\mathbf{z}, \quad (56)$$

where the identity

$$\lim_{T\to+\infty}\exp(iTx)\delta_+(x)=\delta(x)$$

has been used.

In the system where the center of mass is at rest ($\mathbf{k}_1=-\mathbf{k}_2=\mathbf{k}$, $\mathbf{k}_1'=-\mathbf{k}_2'=\mathbf{k}'$), this gives simply

$$\delta_+(P_0-P_0')[\![\mathbf{k}',-\mathbf{k}',j|R_b|\mathbf{k},-|,i]\!] =(2\pi)^3\chi_0^{*(j)}(\mathbf{k}',-\mathbf{k}') \times\{\varphi_b^{(j)}(\mathbf{k}')-\varphi_0^{(j)}(\mathbf{k})(2\pi)^3\delta(\mathbf{k}-\mathbf{k}')\}, \quad (57)$$

where $\varphi_b^{(j)}(\mathbf{p})$ is the Fourier transform of $\varphi_b^{(j)}(\mathbf{z},0)$ and $\varphi_0^{(j)}(\mathbf{k})\equiv\psi_0^{(j)}(\mathbf{k},-\mathbf{k})$.

Scattering phase-shifts can then easily be obtained by means of a partial wave analysis of Eqs. (53) and (57).

2. Introduction of Three-Dimensional Wave Functions and Connection with the Tamm-Dancoff Method

The calculation of the R_a-matrix elements by means of Eq. (53) implies the knowledge of both functions $\bar{\chi}_b(y,\eta_i)$ and $\psi_b(x,\xi_i)$. It is perhaps more convenient to

[17] H. Kita, Progr. Theoret. Phys. Japan 7, 217 (1952).

obtain the corresponding phase-shifts through a direct analysis of the asymptotic behavior of the function $\psi_a(x,\xi_i)$ for equal times of the interacting particles. The connection with calculations based on the Tamm-Dancoff method will, at the same time, become more apparent.

Let us write, in the system of coordinates (50),

$$\psi_a(x,\xi_i)=\varphi_a^{(i)}(z)\exp(iP_\mu X_\mu), \tag{58}$$

and eliminate the X dependence by means of Eqs. (22) and (51). The resulting expression of $\varphi_a^{(i)}(z)$ is

$$\begin{aligned}\varphi_a^{(i)}(z)=-iG^2(2\pi)^{-4}\exp(-i\alpha Pz)\int K_N(p)K_M^{(i)}\\ \times(P-p)[R_b^{(i,k)}(p;P)-\Lambda_0^{(\beta)}(i,k)]\gamma_5\tau_kK_N'(P)\gamma_5\tau_l\\ \times[\varphi_b^{(l)}(0)-\Lambda_{0l}^{(\alpha)}\varphi_0^{(l)}(0)]\exp(ipz)d^4p,\end{aligned} \tag{59}$$

where the following definition has been introduced:

$$\begin{aligned}(2\pi)^{-4}\int K_b^{(i,j)}(p,P-p;q,P-q)d^4q\\ =K_N(p)K_M^{(i)}(P-p)R_b^{(i,j)}(p;P),\end{aligned} \tag{60}$$

$R_b^{(i,j)}(p;P)$ being a logarithmically divergent function which obeys the following integral equation:

$$\begin{aligned}R_b^{(i,j)}(p;P)=\delta_{ij}-iG^2(2\pi)^{-4}\gamma_5\tau_k\int K_N(u+p-P)\\ \times\gamma_5\tau_iK_N(u)K_M^{(k)}(P-u)R_b^{(k,j)}(u;P)du.\end{aligned} \tag{61}$$

Since, according to Eqs. (23) and (60), $\Lambda_0^{(\beta)}(i,j)$ can be written in the form

$$\begin{aligned}\Lambda_0^{(\beta)}(i,j)=-iG^2(2\pi)^{-4}\gamma_5\tau_k\int K_N(u)\gamma_5\tau_iK_N(u)\\ \times K_M^{(k)}(P-u)R_b^{(k,j)}(u;p_0)du,\end{aligned} \tag{62}$$

it is easily seen that

$$R_b^{*(i,j)}=R_b^{(i,j)}-\Lambda_0^{(\beta)}(i,j), \tag{63}$$

which appears on the right-hand side of (59), is a finite quantity. We now introduce the Fourier transform of $\varphi_a^{(i)}(z)$ through the equation

$$\begin{aligned}\varphi_a^{(i)}(z)=\exp[i(1-\alpha)Pz](2\pi)^{-4}\int\Phi_a^{(i)}(p)\\ \times\exp(ipz)dp,\end{aligned} \tag{64}$$

and consequently,

$$\begin{aligned}\Phi_a^{(i)}(p)=-iG^2K_N(p+P)K_M^{(i)}(p)R_b^{*(i,k)}(p+P;P)\\ \times\gamma_5\tau_kK_N'(P)\gamma_5\tau_l[\varphi_b^{(l)}(0)-\Lambda_{0l}^{(\alpha)}\varphi_0^{(l)}(0)].\end{aligned} \tag{65}$$

Let us now split $\Phi_a^{(i)}(p)$ and $\Phi_b^{(i)}(p)$ (which is related to $\varphi_b^{(i)}(z)$ through an equation analogous to (64)), into their positive and negative energy components by means of the relations

$$\begin{aligned}\Phi_a^{(i)}(p)=\sum_\rho u_\rho(\mathbf{p})(\omega_p)^{-\frac{1}{2}}A_\rho^{(i)}(p),\\ \Phi_b^{(i)}(p)=\sum_\rho u_\rho(\mathbf{p})(\omega_p)^{-\frac{1}{2}}B_\rho^{(i)}(p),\end{aligned} \tag{66}$$

where the Dirac-spinor $u_\rho(\mathbf{p})$, in which the index ρ takes the values 1 and 2, is defined by

$$[i\gamma\mathbf{p}-\gamma_4E_\rho(\mathbf{p})+M]u_\rho(\mathbf{p})=0, \tag{67}$$

with $E_1(\mathbf{p})=-E_2(\mathbf{p})=E_p$. Let us similarly write

$$\varphi_0^{(l)}(0)=\sum_\rho u_\rho(\mathbf{k})(\omega_k)^{-\frac{1}{2}}C_\rho^{(l)}. \tag{68}$$

Equation (65) can then be expressed in the form

$$A_\rho^{(i)}(p,p_0)=-\frac{G^2F_\rho^{(i)}(\mathbf{p},p_0;P_0)}{(\omega_p^2-p_0^2)[E_\rho(\mathbf{p})-p_0-P_0]}, \tag{69}$$

where the function $F_\rho^{(i)}$ is defined by

$$\begin{aligned}F_\rho^{(i)}(\mathbf{p},p_0;P_0)=\omega_p^{\frac{1}{2}}\omega_k^{-\frac{1}{2}}\sum_{\rho'}\bar{u}_\rho(\mathbf{p})R_b^{*(i,k)}\\ \times(\mathbf{p},p_0+P_0;0,P_0)\gamma_5\tau_kK_N'(0,P_0)\\ \times\gamma_5\tau_lu_{\rho'}(\mathbf{k})B_\rho^{*(l)},\end{aligned} \tag{70}$$

with $p_0=-ip_4$, the finite constant $B_\rho^{*(i)}$ being expressed as

$$\begin{aligned}B_\rho^{*(i)}=\bigg\{i(2\pi)^{-4}\sum_{\rho'}\int u_\rho^*(\mathbf{k})u_{\rho'}(\mathbf{p})\\ \times\omega_k^{\frac{1}{2}}\omega_p^{-\frac{1}{2}}B_{\rho'}^{(i)}(\mathbf{p},p_0)d\mathbf{p}dp_0\bigg\}-\Lambda_{0i}^{(\alpha)}C_\rho^{(i)}.\end{aligned} \tag{71}$$

In order to obtain the three-dimensional Fourier-component of the wave function corresponding to equal times of the particles, we must[5] calculate the quantity:

$$a_\rho^{(i)}(p)=-\frac{1}{2\pi i}\int_{-\infty}^{+\infty}A_\rho^{(i)}(\mathbf{p},p_0)dp_0.$$

While doing the integration over p_0, however, we can take advantage of the fact that we only need the *asymptotic* form of the Fourier-transform $f_\rho^{(i)}(\mathbf{r})$ of $a_\rho^{(i)}(\mathbf{p})$, so that only the poles exhibited on the right-hand side of (69)—and not those eventually existing in $F_\rho^{(i)}$—will contribute to the integral. The result is

$$a_\rho^{(i)}(\mathbf{p})\sim\frac{G^2\epsilon_\rho F_\rho^{(i)}(\mathbf{p},-\epsilon_\rho\omega_p;P_0)}{\omega_p+\epsilon_\rho[E_\rho(\mathbf{p})-P_0]}, \tag{73}$$

with $\epsilon_1=+1$, $\epsilon_2=-1$, $E_\rho(\mathbf{p})=\epsilon_\rho E_p$. Only the positive energy component ($\rho=1$) of $a_\rho^{(i)}(\mathbf{p})$ does not vanish at infinity. Introducing polar coordinates (p,θ,φ) and (r,Θ,Φ) in momentum and configuration spaces, respectively, we can analyze $a_1^{(i)}(\mathbf{p})$ and $f_1^{(i)}(\mathbf{r})$ into partial waves:

$$\begin{aligned}a_1^{(i)}(p)=\sum a_{1,lm}^{(i)}(p)Y_{l,m}(\theta,\varphi),\\ f_1^{(i)}(r)=\sum f_{1,lm}^{(i)}(r)Y_{l,m}(\Theta,\Phi).\end{aligned} \tag{74}$$

where the sums on the right-hand sides involve also a summation over spin-eigenfunctions. We then get for the asymptotic form of $f_{1,lm}^{(i)}(r)$

$$f_{1,lm}^{(i)}(r)\sim\frac{i^l}{2\pi^2}G^2\int_0^\infty \frac{j_l(pr)F_{1,lm}^{(i)}(p,-\omega_p;P_0)}{\omega_p+E_p-P_0}\times p^2dp, \quad (75)$$

where $j_l(x)=(\pi/2x)^{-\frac{1}{2}}J_{l+\frac{1}{2}}(x)$. Since the main contribution to the asymptotic form comes from the pole $p=k$ (defined by $P_0=\omega_k+E_k$), we can write

$$f_{1,lm}^{(i)}(r)\sim\frac{i^l}{\pi^2}G^2\frac{E_k\omega_k}{P_0}F_{1,lm}^{(i)}(k,-\omega_k;P_0)\int_0^\infty\frac{j_\rho(pr)p^2dp}{p^2-k^2}$$
$$\sim\frac{i^l}{2\pi}G^2\frac{kE_k\omega_k}{P_0}F_{1,lm}^{(i)}(k,-\omega_k;P_0)g_l(kr), \quad (76)$$

with $g_l(x)=(\pi/2x)^{-\frac{1}{2}}J_{-(l+\frac{1}{2})}(x)$. Because of the definition of $\varphi_0^{(l)}(0)$ in Eq. (68), Eq. (76) leads to the following expression for the contribution of ψ_a to the phase shift:

$$\tan\delta_{lm}^{(i)}(a)=\frac{(-1)^lG^2k}{4\pi^{\frac{3}{2}}}\frac{E_k\omega_kF_{1,lm}^{(i)}(k,-\omega_k;P_0)}{P_0(2l+1)^{\frac{1}{2}}C_1^{(i)}}. \quad (77)$$

The same method can be used to calculate the contribution of ψ_b to the scattering phase shifts. In this case, however, the reduction of integral equation (37) to a three-dimensional form involves some manipulations, which can be based, for example, on the iteration method which has been worked out for the nuclear two-body problem[5] and extended to pion-nucleon scattering by Deser and Martin.[18]

A final remark should be made about the factor $\gamma_5\tau_kK_N'(0,P_0)\gamma_5\tau_l$ on the right-hand side of Eq. (70). If the $K_b^{(k,l)}$ function which appears on the right-hand side of Eq. (30) were replaced by its zero-order approximation $K_N(q)K_M(p-q)\delta_{kl}$, this factor would just be the damping coefficient calculated by Brueckner, Gell-Mann, and Goldberger.[19] However, since the exact kernel does probably not have the same dependence on momenta and isotopic spin, the resulting effect might well be entirely different. The results of the present section provide, in fact, a method to ascertain to what extent the damping effects predicted by Wentzel[20] are actually present in pion-nucleon scattering.

V. CONCLUDING REMARKS

It might appear, at first sight, rather surprising that meson-nucleon scattering can be formulated covariantly in a much more compact form than other two-body problems. The main reason is probably that the meson-nucleon system is not properly a two-body system, but should be more appropriately described as a "radiative one-body system." It is this reason which makes the writer feel that it might be possible to formulate the problem in an even more compact way than has been done in the present paper. There might exist, for example, an integral equation for the finite interaction function I_b introduced in Sec. III, which would enable one to solve the problem in a completely closed form. So far, however, only integral equations which take into account parts of the diagrams contained in I_b have been obtained.

Another problem is, of course, to determine to what extent the predictions of the theory agree with experiment. This problem is partly related to the preceding one, since the convergence of the series of diagrams described by I_b for any physically acceptable value of G^2 might well be doubtful. The agreement with experiment also depends on the compared magnitude of the "damping" or "resonance" effects (depending on their sign) in the contributions of ψ_a and ψ_b to the scattering phase shifts. It is likely that the influence of renormalization, which leads to a completely different treatment of these two parts of the wave function, will be felt there very strongly.

The author is indebted to Professor R. E. Marshak for many stimulating discussions.

Additional note: After the completion of this work, we have received, through a private communication, a summary of the results obtained by D. Ito and H. Tanaka, which are similar to those of S. Fubini (reference 10).

Note added in proof.—In order to take properly into account the difficulties of overlapping divergences, the renormalization prescriptions (18) and (31) have to be replaced, respectively, by the following:

$$\Gamma_{5,i}^{*(\rho)}(p,q)=\frac{\Gamma_{5,i}^{(\rho)}(p,q)}{1+\Lambda_{0i}^{(\rho)}}, \quad (a)$$

$$\Sigma^*(p)=\Sigma_1(p)-\Sigma_1(p_0)-(p-p_0)\left[\frac{\partial}{\partial p}\Sigma_1(p)\right]_{p=p_0}, \quad (b)$$

where $\sum_1(p)$ is defined by the equation

$$\Sigma_1(p)=\frac{\Sigma(p)}{1+\Lambda_{0k}^{(\alpha)}+\Lambda_{0l}^{(\beta)}}. \quad (c)$$

These prescriptions preserve the main conclusion of the present paper, namely that the vertex operators $\Gamma_5(\rho)$ and the modified propagation function $K_N'(p)$ can be renormalized in closed form and expressed in terms of the finite functions ψ_b and K_b. Some of the equations of Sec. IV have to be modified accordingly. For example, Eq. (53) becomes

$$(\mathbf{k}_1',\mathbf{k}_2',j|R_a|\mathbf{k}_1,\mathbf{k}_2,j)=-iG^2(2\pi)^4\frac{\bar{\rho}_b^{(k)}(0)}{1+\Lambda_{0k}^{(\beta)}}$$
$$\times\gamma_5\tau_kK_N'(P)\gamma_5\tau_l\frac{\phi_b^{(l)}(0)}{1+\Lambda_{0l}^{(\alpha)}}\delta(\mathbf{P}-\mathbf{P}')\delta(P_0-P_0').$$

Proofs of prescriptions (a), (b), and (c) will be given in a subsequent note.

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