by Meek and Saxe that, during the passage of the streamer tip in long sparks, there is a cusp-shaped luminosity that extends out radially at least 10-cm beyond the streamer axis as sensed by their photomultiplier. After this passes, the streamer channel is largely dark until the return stroke illuminates a channel of some 3-mm radius or less. In the case of lighting discharge, the radius of photographed vigorous steps with heavy currents runs out to some 10 meters. In such strokes, the channel at the cloud end of the stroke may remain visible for some little time. In this case the pilot leader invisible to the camera advances 20 to 200 meters with an active conducting channel of $R=10$ cm illuminated by the return stroke from ground some 10 *milliseconds* later, and has as well the expanded channel of positive space charge of radius R' which, with its roughly 5×10^{10} electrons per cm^3, is illuminated only by the step flash from the cloud end within some 10–100 *microseconds*. After ionization and illumination, the step ionization of radius R' decays, and it is only the original channel of $R=10$ cm which has sufficient conductivity owing to the field X_c to carry the return stroke.

It will be noted from Table III that the observed values of what might be R' are less than those computed. This is not surprising, for, as radial expansion continues, the time rate of ionization and accompanying excitation decline, and what is preceived by photomultiplier or photographic plate corresponds to values considerably less than R' depending on the sensitivity of the detector. Amin could observe only at a value of radius greater than 0.05 cm from the streamer axis. That he did not observe luminosity between 0.05 and 0.1 cm is not surprising. That, however, there was a transient luminosity after the intensely luminous peak passed the slit, is shown by the shoulder of luminosity following the tip and of such shape as only to be accounted for by ionization and excitation occurring long-after even an especially broad tip had passed the slit. The resolving power of Meek and Saxe's system was not such as to have revealed the details of the fine structure of the tip luminosity as observed by Amin, while their radial transient tip expansion could well be observed with their heavy currents. From the evidence presented, even though observed luminosity and calculated values R agree only in order of magnitude, it is believed that the nature of the transient tip shape and luminosity is accounted for. The transient space charge expansion described is illustrated schematically in Fig. 2, which is self-explanatory.

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The Scattering of Electromagnetic Waves by Turbulent Atmospheric Fluctuations*

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The statistical theory of turbulence is applied to the problem of density fluctuations in the troposphere and the ionosphere. For suitable wavelengths, for which the so-called similarity region (Kolmogoroff spectrum) of the spectrum of turbulence is relevant, a closed formula can be given for the scattering cross section. It contains as only parameter the turbulent power dissipation S, and its angular dependence is given by $(\sin\frac{1}{2}\theta)^{13/3}$, θ being the scattering angle. The values of S required to explain ionospheric scattering are in excellent agreement with values found from investigations of meteor trails. Tropospheric data cannot be fitted with the assumptions of dry-air turbulence alone. The inference is that humidity fluctuations play an essential part in tropospheric scattering. A preliminary study of these latter fluctuations gives satisfactory results. Further investigations (and experimental data) are needed, however, to work out a quantitative theory.

LIGHT waves are scattered by random fluctuations of the refractive index. In what follows we derive the scattering of elementary waves by random fluctuations which are produced by turbulent perturbations. The general idea underlying this study has been suggested by a number of authors, especially Megaw[1] and Booker.[2] It will be shown that, under certain conditions, the scattering produced by these fluctuations can be expressed in terms of only one parameter, the turbulent energy S dissipated per cm^3 per sec. The conditions of validity of this relation are well fulfilled for the scattering of meter waves in the E layer of the ionosphere; for tropospheric scattering other parameters such as the inhomogeneity of potential temperature and specific humidity play an important part.

A. SIMPLE DERIVATION OF THE SCATTERING FORMULA

1. Scattering Cross Section

In this section we derive the expressions by simple qualitative arguments and leave the exact derivations for Sec. B.

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[1] E. C. S. Megaw, Nature **166**, 1100 (1950), and Proc. Inst. Elec. Engrs. (London) **100**, 7 (1953).

[2] H. Booker and W. E. Gordon, Proc. Inst. Radio Engrs. **38**, 401 (1950).

We calculate the scattering which a light beam with the wave vector $\mathbf{k}_0$ suffers in a region of space of volume V, which contains a medium whose dielectric constant ϵ fluctuates: $\epsilon=\epsilon_0+\Delta\epsilon$, where $\Delta\epsilon$ is a function of space. We assume that the time variations are sufficiently slow and do not affect the scattering. Then the scattered electric field amplitude $(\mathbf{E}=\mathbf{E}_s e^{i(\omega t-\mathbf{k}_1\cdot\mathbf{r})})$ at a distance R from the scattering volume $(R\gg V^{\frac{1}{3}})$ is given by

$$E_s=\frac{E_0}{4\pi R\lambda\!\!\!^{-2}}\left|\int_V d\mathbf{r}\Delta\epsilon(\mathbf{r})e^{i\mathbf{K}\cdot\mathbf{r}}\right|\sin\chi. \tag{1}$$

Here E_0 is the electric-field amplitude of the incident wave $(\mathbf{E}_{\text{inc}}=\mathbf{E}_0e^{i(\omega t-\mathbf{k}_0\cdot\mathbf{r})})$, $\lambda\!\!\!^-$ is the wavelength divided by 2π, χ is the angle between the direction of the incident field and the scattering, and

$$K=|\mathbf{K}|=|\mathbf{k}_0-\mathbf{k}_1|=2k\sin\tfrac{1}{2}\theta, \tag{2}$$

where

$$k=|\mathbf{k}_0|=|\mathbf{k}_1|=\lambda\!\!\!^{-\,-1}.$$

We introduce the cross section $\sigma d\Omega$ for the scattering per unit volume of the scattering volume into the solid angle $d\Omega$:

$$\sigma d\Omega=(R^2/V)(E_s^2/E_0^2),$$

and get

$$\sigma d\Omega=|M|^2\sin^2\chi d\Omega/V\lambda\!\!\!^{-4}(4\pi)^2, \tag{3}$$

with

$$M=\int d\mathbf{r}\Delta\epsilon(\mathbf{r})e^{i\mathbf{K}\cdot\mathbf{r}}. \tag{3a}$$

For the ionosphere, we assume that $\Delta\epsilon$ is due to fluctuations in electron concentration, proportional to the density fluctuations $\Delta\rho$ of the carrier medium:

$$\Delta\epsilon=g\Delta\rho/\rho, \tag{4}$$

and

$$g=-\omega_N^2/\omega^2;\quad \omega_N^2=(1/\epsilon_0)(Ne^2/m), \tag{5}$$

where N is the density of free electrons in the medium. In the troposphere, we will have to cope with fluctuations in density, potential temperature and humidity. Because of the large dipole moment of the water molecule,[3] this latter contribution is particularly important. The dielectric constant may be written as

$$\epsilon-1=1.55\times10^{-4}\frac{p}{T}\left(\frac{\text{mb}}{{}^\circ\text{K}}\right)+0.74\frac{p'}{T^2}\left(\frac{\text{mb}}{{}^\circ\text{K}^2}\right). \tag{6}$$

In (6), p is the pressure of air, p' the partial pressure of water vapor. Since $p/T\sim\rho$, we best express $\Delta\epsilon$ in terms of density fluctuations:

$$\Delta\epsilon=0.45\Delta\rho\left(\frac{\text{g}}{\text{cm}^3}\right)+3.4\times10^3\left[\frac{\Delta\rho'}{T}-\frac{\rho'}{T^2}\Delta T\right]\left(\frac{\text{g/cm}^3}{{}^\circ\text{K}}\right). \tag{6a}$$

In (6a), ρ refers to the density of air, ρ' to the density of water vapor, both in g/cm^3. In what follows, we shall develop a description of the density fluctuations $\Delta\rho$ from a statistical dynamics of turbulence. Since the problems involved in the discussion of the humidity fluctuations [second term in (6a)] are of rather different nature we shall discuss them separately, and presently we shall discuss the density fluctuations in dry and isentropic air only.

2. Density Fluctuations in Homogeneous Turbulence

First we connect the density fluctuations with the velocity fluctuations by means of Bernoulli's law: Local velocity differences Δv will be accompanied by pressure differences Δp, and density differences $\Delta\rho$, according to

$$\Delta\rho/\rho\cong\Delta p/p=(\Delta v)^2/v_M^2, \tag{7}$$

where v_M^2 is the average square of the molecular velocity. We now concentrate our attention on the velocity fluctuations under conditions of homogeneous turbulence.[4] We may picture this situation as follows: there are external causes which constantly produce large eddies of a certain dimension L_0 and velocity v_0. These eddies divide soon into smaller eddies, say, of size $L_1=\alpha L_0$, with $\alpha<1$. The velocity of these smaller eddies embedded in the larger ones is v_1. (Measurement of v_1 is relative to the environment, i.e., relative to the motion of the large eddy.) These eddies divide again into smaller ones $L_2=\alpha^2L_0$, with a velocity v_2 relative to their surrounding. In this dividing process, the energy fed into the largest eddy is transferred to smaller and smaller ones. A constant amount of energy per volume and time is fed into the larger eddies from an outside source. The dividing process ends with that eddy size at which the effect of the molecular viscosity is large enough to dissipate the energy into heat.

We now establish the quantitative relations which exist in this process. The energy per unit volume contained in one eddy is of the order ρv_n^2. The lifetime of such an eddy will be of the order $t_n=L_n/v_n$. Hence, the eddies of size L_n lose energy to smaller ones (size L_{n+1}) with a rate (energy/time volume):

$$S_n\sim\rho v_n^3/L_n. \tag{8}$$

Since we have a stationary process, the loss S_{n-1} of the eddies one size larger to those of size n must be equal to S_n; hence, S_n is a constant independent of n:

$$\rho v_n^3/L_n=S. \tag{8a}$$

S is the energy transmitted through the eddies from the energy source (larger eddies) all the way down to the smaller ones. The order of magnitude of S is given by the transfer from the largest eddies:

$$S\cong\rho v_0^3/L_0. \tag{8b}$$

[3] C. M. Crain, Phys. Rev. **74**, 691 (1948).

[4] The theory of turbulence as used here was first suggested by A. N. Kolmogoroff [Compt. rend. acad. sci. U.R.S.S. **30**, 301 (1941)]. Our treatment is closest to the ideas of G. F. v. Weizsäcker [Z. Physik **124**, 614 (1948)] and W. Heisenberg [Z. Physik **124**, 628 (1948)].

We can use the magnitude of S in order to determine the size of the smallest eddies: the energy per cm^3 and sec dissipated by molecular viscosity is given by $\sim\eta(dv/dx)^2$, where dv/dx is the rate of change of v per unit length. The smallest eddy is the one in which this energy reaches S. Since $(dv/dx)\sim(v_n/L_n)$, we get for the smallest eddy, whose velocity is v_S and whose L is L_S,

$$\eta v_S^2/L_S^2 \cong S.$$

Using the relation (8a): $\rho v_S^3/L_S=S$, and also (8b), we get

$$L_0/L_S=(\rho v_0 L_0/\eta)^{3/4}. \tag{9}$$

In words, the ratio of the sizes of the largest to the smallest eddies is the $\frac{3}{4}$-power of the Reynolds number, associated with the eddy-producing large scale kinematics.

We can identify the velocities v_n with the magnitude Δv used in (6). We consider regions in space of the linear size L, L fulfilling the relation

$$L_S<L<L_0. \tag{10}$$

Let us then assume that $L_n=L$; then $\Delta v=v_n$ is the average deviation from its surrounding of the velocity in that region. We then get from (6) the average density deviation $(\Delta\rho)_L$ from its surrounding in a region L:

$$(\Delta\rho)_L=\rho(v_n/v_M)^2=\rho(v_0/v_M)^2(L/L_0)^{\frac{2}{3}}. \tag{11}$$

Let us illustrate these results by a few examples:

(a) In the *troposphere*, at 10-km height, we have $\rho=3.6\times10^{-4}$ g cm^{-3}, $\eta=2\times10^{-4}$ g cm^{-1} sec^{-1}. We estimate the order of magnitude of size and velocities in turbulent gusts to 1 km and 10 m/sec, respectively. Hence we may put

$$v_0=10^3\text{ cm/sec},\quad L_0=10^5\text{ cm}, \tag{12}$$

and, therefore,

$$R=1.8\times10^8,\quad L_0/L_S=1.55\times10^6,\quad L_S=0.65\text{ mm},$$
$$S=0.36\text{ erg cm}^{-3}\text{ sec}^{-1}.$$

The smallest eddies are a fraction of a cm large.

(b) In the E-layer of the *ionosphere* at 100-km height, we have $\rho=2\times10^{-9}$ g cm^{-3}. The viscosity is independent of pressure, and, therefore, again $\eta=2\times10^{-4}$g cm^{-1} sec^{-1}. From the very scant data about velocities and eddy sizes we infer roughly:

$$v_0=5\times10^3\text{ cm sec}^{-1},\quad L_0=5\times10^5\text{ cm}, \tag{13a}$$

and hence[5]

$$R=2.5\times10^4,\quad L_0/L_S=2\times10^3,\quad L_S=2.5\times10^2\text{ cm},$$
$$S=0.5\times10^{-3}\text{ erg cm}^{-3}\text{ sec}^{-1}. \tag{13b}$$

We now proceed to discuss the *humidity fluctuations*. A rough estimate of the order of magnitudes indicates that the effects of humidity in the troposphere are probably much stronger than the effects of the fluctuations of the density of the dry air. As can be inferred from (7) and (19), the average mean square fluctuation $\langle\Delta\rho^2\rangle_{\text{Av}}$ of the latter density is of order $\rho(v_0/v_M)^4$. With the data as given by (12), and with $v_M\sim4\times10^4$ cm/sec, we get

$$(1/\rho^2)\langle\Delta\rho^2\rangle_{\text{Av}}\sim0.4\times10^{-6}. \tag{14}$$

We infer from (6a) that for $\rho'\sim10^{-2}\rho$, $T\sim300°$, a $\langle\Delta\rho'^2\rangle_{\text{Av}}/\rho'^2$ or $\langle\Delta T^2\rangle_{\text{Av}}/T^2$ of magnitude (14) produces a $\langle\Delta\epsilon^2\rangle_{\text{Av}}$ twice as large as the dry-air density fluctuation of (14). Meteorological evidence indicates, however, that the fluctuations $\langle\Delta\rho'^2\rangle_{\text{Av}}$ and $\langle\Delta T^2\rangle_{\text{Av}}$ may be considerably larger: Measured $\langle\Delta T^2\rangle_{\text{Av}}/T^2$ values are of order of magnitude 10^{-5};[6] unfortunately humidity measurements are scarce, but indicate[7] fluctuations that may be considerably larger. Practically nothing is known about the process of turbulent mixing of inhomogeneously humid air. Thus no information about short-range correlation of humidity in turbulent air is available. With some knowledge of the dynamics of turbulence it should, however, be possible to calculate approximate correlation functions from measured large-scale fluctuations (over dimensions of the order L_0). Thus, for instance, we may assume that, in an atmosphere with a negative gradient of potential temperature, convection produces large "wet" eddies (they need not be actual clouds) embedded in comparatively dryer air. The dissipation of this "surplus humidity" into the surrounding medium is then to follow closely the dissipation of the kinetic energy of the wet eddy by the breaking-up process described above. This leads to the tentative conclusion of a spectrum of humidity fluctuations very similar to the spectrum of the fluctuations of v^2, and this again is equal to the density fluctuation spectrum. Should this indeed be the case—a close study of this point is under way—the knowledge of large-scale humidity variations $(\Delta\rho')_0$ would serve to determine the humidity fluctuations $(\Delta\rho')_L$ in a volume of order L compared to its surrounding.

In fact we would get from (11)

$$(\Delta\rho')_L/(\Delta\rho')_0=(\Delta\rho)_L/(\Delta\rho)_0=v_n^2/v_0^2=(L/L_0)^{\frac{2}{3}}. \tag{14a}$$

Unfortunately very little is known about the large-scale humidity fluctuations $(\Delta\rho')_0$.

3. Calculation of the Scattering Cross Section

We now determine the magnitude M appearing in (1) and defined by (2). With (4)[8] we write

$$M=\frac{g}{\rho_0}\int d\mathbf{r}\Delta\rho(\mathbf{r})e^{i\mathbf{K}\cdot\mathbf{r}}, \tag{14b}$$

[5] This estimate is in good agreement with the values given by C. deJager [Mem. soc. roy. sci., Liège, Series 4, **XXI**, 223 (1952)] and obtained from an analysis of meteor trails.

[6] See Electrical Engineering Research Laboratory, University of Texas, Reports Nos. 47, 1950, and 53, 1953 (unpublished).

[7] Electrical Engineering Research Laboratory, University of Texas, Reports Nos. 54, 1951, and 6-01, 1953 (unpublished).

[8] In case of tropospheric scattering, the dry-air term of (6a) gives $g=0.45\rho$(g/cm^3).

where ρ_0 is the average density, and we try to determine this magnitude on the basis of our picture of homogeneous turbulence. These concepts will be applicable only if the length K^{-1} lies between L_S and L_0:

$$L_S < (2k \sin\tfrac{1}{2}\theta)^{-1} < L_0. \tag{15}$$

The integral appearing in (14) is a Fourier coefficient of the density fluctuations. We can expand the density $\rho(\mathbf{r})$ in a Fourier series within the scattering volume V (which we may assume to be a cube):

$$\rho(\mathbf{r}) = \frac{1}{V} \sum_i \rho(\mathbf{k}_i) e^{-i\mathbf{k}_i\cdot\mathbf{r}}, \tag{16}$$

with the inversion

$$\rho(\mathbf{k}_i) = \int_V d\mathbf{r}\rho(\mathbf{r}) e^{i\mathbf{k}_i\cdot\mathbf{r}}. \tag{17}$$

It is then clear that M, as given by (14), is nothing but

$$M = (g/\rho_0)\rho(\mathbf{K}). \tag{18}$$

We shall now have to establish the relation between $\rho(\mathbf{k})$ and the quantity $(\Delta\rho)_L$ introduced in Sec. (2). Now $(\Delta\rho)_L$ is built up by the fluctuations whose spatial period is of order L. Because of the random nature of these fluctuations, different wavelengths will not interfere, and we have

$$(\Delta\rho)_L{}^2 = \frac{1}{V^2} \sum_{\substack{\mathbf{k} \\ k\sim 1/L}} |\rho(\mathbf{k})|^2 \cong \frac{1}{V^2}\left(\frac{V}{2\pi^2}\right) \int_{k'}^{k''} k^2 dk |\rho(\mathbf{k})|^2.$$

The interval $k''-k'$ is of order $1/L$ itself and centered about $K=1/L$. $(V/2\pi^2)k^2dk$ is the number of Fourier components with wavelength between k and $k+dk$. This gives us roughly:

$$(\Delta\rho)_L{}^2 \cong \frac{1}{6\pi^2}\frac{1}{VL^3}|\rho(1/L)|^2. \tag{19}$$

Hence,

$$|M|^2 = 6\pi^2\left(\frac{g}{\rho_0}\right)^2 VL^3(\Delta\rho)_L{}^2\Big|_{L=1/K},$$

and inserting $(\Delta\rho)_L{}^2$ from (11), we get

$$|M|^2 = 6\pi^2 g^2 V\left(\frac{v_0}{v_M}\right)^4 \frac{L^{13/3}}{L_0{}^{4/3}}\Big|_{L=1/K}.$$

Inserting this into (3) gives

$$\sigma d\Omega \cong g^2\left(\frac{v_0}{v_M}\right)^4 \frac{L^{13/3}}{L_0{}^{4/3}}\frac{\sin^2\chi}{\lambda\!\!\!^{-4}}d\Omega, \tag{20}$$

where a factor of order unity (3/8) is omitted. Setting $L=K^{-1}$ and using (2) gives, finally,

$$\sigma = \frac{g^2}{L_0}\left(\frac{v_0}{v_M}\right)^4\left(\frac{\lambda\!\!\!^-}{L_0}\right)^{1/3}(2\sin\tfrac{1}{2}\theta)^{-13/3}\sin^2\chi. \tag{21}$$

The validity of this expression is limited by the condition (10). It is only applicable as long as

$$L_0{}^{-1} < 2k \sin\tfrac{1}{2}\theta < L_S{}^{-1}. \tag{22}$$

We find that the turbulent conditions enter the cross section only in the form $v_0{}^4/L_0{}^{4/3}$; hence, the cross section is proportional to $(S)^{4/3}$ and depends only upon the energy dissipation S.

We note the characteristic angular dependence $(\sin\frac{1}{2}\theta)^{-13/3}$. The dependence upon the wavelength is different in the troposphere and in the ionosphere. In the former, g^2 is independent of λ, and we get $\sigma \sim \lambda^{1/3}$. In the latter, we get from (5b) a wavelength dependence $\sim\lambda^{13/3}$.

If the conditions (22) are no longer fulfilled, we expect deviations from (21). The case where $L=(2k\sin\frac{1}{2}\theta)^{-1}$ is comparable to or smaller than L_S is of special interest. In this case, the eddies of that size are attenuated by molecular dissipation, and $(\Delta\rho)_L$ is then smaller than (11). In particular, it will fall off stronger than $L^{2/3}$ with decreasing L. Let us put instead of (11), for example,

$$(\Delta\rho)_L = \rho(v_0/v_M)^2(L/L_0)^{2/3}F(L/L_S), \tag{23}$$

where $F(x)$ is a function which is equal to unity for $x\gg 1$ and which falls off rapidly for $x<1$. We then get a factor $|F(\lambda\!\!\!^-/2L_S\sin\frac{1}{2}\theta)|^2$ multiplied into the cross section (21). Hence, the scattering falls off more strongly with decreasing λ or increasing θ if $(2k\sin\frac{1}{2}\theta)^{-1} < L_S$.

B. QUANTITATIVE THEORY

1. Cross Section

With Eqs. (1) and (2), we can write the cross section (1) as

$$\sigma d\Omega = \frac{g^2}{(4\pi)^2}\frac{1}{\lambda\!\!\!^{-4}\rho^2 V}\left|\int_V d\mathbf{r}\Delta\rho(\mathbf{r})e^{i\mathbf{K}\cdot\mathbf{r}}\right|^2 \sin^2\chi d\Omega.$$

Now

$$\frac{1}{\rho^2}\left|\int_V d\mathbf{r}\Delta\rho(\mathbf{r})e^{i\mathbf{K}\cdot\mathbf{r}}\right|^2 = \int d\mathbf{r}e^{i\mathbf{K}\cdot\mathbf{r}}\frac{1}{\rho^2}\int d\mathbf{R}\Delta\rho(\mathbf{R})\Delta\rho(\mathbf{R}+\mathbf{r}) = \int d\mathbf{r}e^{i\mathbf{K}\cdot\mathbf{r}}V\frac{\langle\Delta\rho^2\rangle_{\text{Av}}}{\rho^2}C(\mathbf{r}). \tag{24}$$

Defining the Fourier-transformed $C(\mathbf{k})$ of the correlation function $C(\mathbf{r})$ by

$$C(\mathbf{r}) = \left(\frac{1}{2\pi}\right)^3\int d\mathbf{k}e^{-i\mathbf{k}\cdot\mathbf{r}}C(\mathbf{k}), \tag{25}$$

we get

$$\sigma d\Omega = \frac{g^2}{(4\pi)^2}\frac{1}{\lambda\!\!\!^{-4}}\left(\frac{\langle\Delta\rho^2\rangle_{\text{Av}}}{\rho^2}\right)C(\mathbf{K})\sin^2\chi d\Omega, \tag{26}$$

with $K=2k\sin\frac{1}{2}\theta$.

2. Velocity and Density Spectrum in the Statistical Theory of Turbulence[9]

We shall use a Fourier series expansion of the velocity field as given in Eqs. (16) and (17);[10] this enables us to write the fundamental Navier-Stokes equation in the form

$$\dot{\mathbf{v}}_{\mathbf{k}}+i\sum_{\mathbf{k}'}(\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}'})\mathbf{v}_{\mathbf{k}-\mathbf{k}'}+\frac{\eta}{\rho}k^2\mathbf{v}_{\mathbf{k}}+i\frac{\mathbf{k}}{\rho}p_{\mathbf{k}}=0. \tag{27}$$

We shall assume an incompressible medium

$$(\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}})=0. \tag{27a}$$

This may sound paradoxical in view of our attempt to calculate the density fluctuation $\Delta\rho$. Actually our assumption merely implies that the divergence free part of the velocity field is dynamically much more important than the irrotational components. It also amounts to assuming that the potential energy stored in density variations is much smaller than the kinetic energy of the vortices. With this in mind, we shall now simply calculate the pressure fluctuations Δp, and then calculate $\Delta\rho$ with the aid of

$$p(\mathbf{r})=p_0[\rho(\mathbf{r})/\rho_0]^\gamma, \tag{28}$$

ρ_0, p_0 being average values of density and pressure, and $\gamma=C_p/C_v=1.4$ (for diatomic molecules). Multiplying (27) with $\mathbf{k}$ and realizing that for $k\neq 0$

$$p_{\mathbf{k}}=\Delta p_{\mathbf{k}},$$

we have

$$\Delta p_{\mathbf{k}}=-(\rho_0/k^2)\textstyle\sum_{\mathbf{k}'}(\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}'})(\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}-\mathbf{k}'}),$$

and hence

$$\Delta\rho_k=-(\rho_0/\gamma p_0)^2k^{-2}\textstyle\sum_{\mathbf{k}'}(\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}'})(\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}-\mathbf{k}'}). \tag{29}$$

So we now turn our attention to the velocity field $\mathbf{v}(\mathbf{k})$. Equation (27) gives, upon multiplication with $\mathbf{v}(-\mathbf{k})$,

$$-\frac{\rho}{2}\frac{d}{dt}(\mathbf{v}_{\mathbf{k}}\cdot\mathbf{v}_{-\mathbf{k}})=i\rho\textstyle\sum_{k'}(\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}'})(\mathbf{v}_{-\mathbf{k}}\cdot\mathbf{v}_{\mathbf{k}-\mathbf{k}'})+\eta k^2(\mathbf{v}_{\mathbf{k}}\cdot\mathbf{v}_{-\mathbf{k}}). \tag{30}$$

We recognize $\frac{1}{2}\rho\mathbf{v}(\mathbf{k})\cdot\mathbf{v}(-\mathbf{k})$ as the mean kinetic energy density carried by the wave number $\mathbf{k}$, since

$$\tfrac{1}{2}\rho\langle\mathbf{v}^2(\mathbf{x})\rangle_{\text{Av}}=\tfrac{1}{2}\rho\textstyle\sum_k(\mathbf{v}_k\cdot\mathbf{v}_{-\mathbf{k}}).$$

Summing over all k's in (30), we get the total *power balance*:

$$S=-\frac{d}{dt}\left(\frac{\rho}{2}\textstyle\sum_k(\mathbf{v}_{\mathbf{k}}\cdot\mathbf{v}_{-\mathbf{k}})\right)$$
$$=i\rho\sum_{\mathbf{k},\mathbf{k}'}(\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}'})(\mathbf{v}_{-\mathbf{k}}\cdot\mathbf{v}_{\mathbf{k}-\mathbf{k}'})+\eta\textstyle\sum_{\mathbf{k}}k^2(\mathbf{v}_{\mathbf{k}}\cdot\mathbf{v}_{-\mathbf{k}}). \tag{30a}$$

The first term on the right-hand side describes the *transfer* of kinetic energy to different wave numbers $\mathbf{k}'$, the second term the *viscous dissipation* of energy. We shall now define a stationary situation within a region (L_0^3) completely enclosed in the domain over which (27) holds: this region then contains a largest eddy of linear dimension L_0 and characteristic velocity v_0. This eddy is coupled to the *outside* of that region by virtue of the "transfer" term in (30a). We assume that the effect of this coupling is to maintain the average velocity v_0 of this eddy by means of a power supply S_0. Provided S_0 is constant in time, an equilibrium will be reached in which the average $\mathbf{v}(\mathbf{k})$ is also independent of time.

From (30a) we see that S_0 must be of the form

$$S_0=\alpha\rho(v_0^3/L_0),$$

with α an absolute constant (independent of the values of v_0 and L_0), depending only on the large scale geometry of the power-supplying region outside L_0^3. Subsequently, we shall put $\alpha=1$ and *define* S_0 by

$$S_0=\rho(v_0^3/L_0). \tag{31}$$

If the Reynolds number $R_0=\rho v_0L_0/\eta$ is sufficiently large, the dissipative term will enter into play only for wave numbers $k\gg 1/L_0$. It will affect eddies of size L_S and velocity v_S, for which the "local" Reynolds number has dropped to 1:

$$\rho L_Sv_S/\eta=1. \tag{32}$$

Once wave numbers of order $1/L_S$ are reached, the frictional power dissipation replaces the power transfer to higher wave numbers. So we get another equation characterizing v_S and L_S:

$$S_0=\eta(v_S/L_S)^2. \tag{32a}$$

We shall use (32) and (32a) to *define* the two quantities v_S and L_S:

$$v_S=(S_0\eta/\rho^2)^{1/4};\quad L_S=(\eta^3/S_0\rho^2)^{1/4}. \tag{33}$$

We see that the three "external" parameters S_0, η, ρ *set an absolute scale of length and velocities* for the problem of energy transfer and dissipation in turbulent flow.

To carry this idea through in a more quantitative way, let us define a *spectral intensity* distribution $F(k)$ for the velocities:

$$\tfrac{1}{2}\textstyle\sum_{\mathbf{k}}\mathbf{v}_{\mathbf{k}}\cdot\mathbf{v}_{-\mathbf{k}}=\displaystyle\int_0^\infty dkF(k). \tag{34}$$

(In this definition, isotropy of the velocity spectrum has been assumed.) The dissipative term in (30a) is then

$$2\eta\int_0^\infty dkk^2F(k). \tag{35}$$

[9] Much of the content of this section is to be found in G. K. Batchelor's recent publication on the theory of homogeneous turbulence [*The Theory of Homogeneous Turbulence* (Cambridge University Press, London, 1953)] as well as Heisenberg's original paper (reference 4).

[10] We also use $v_k=v(k)$.

Since (35) represents the total power absorbed, we have, with $y=kL_S$:

$$S_0=(2\eta/L_S^3)\int_0^\infty dyy^2F(L_S^{-1}y).$$

Using (32a), we see that $F(L_S^{-1}y)=v_S^2L_SE(y)$ defines a *universal* function $E(y)$ normalized to 1:

$$1=\int_0^\infty dy2y^2E(y). \tag{36}$$

Determination of $E(y)$ or $F(k)$

(a) *Inertial range*

Provided $L_S \ll L_0$, there is a region $k \ll 1/L_S$, in which the shape of the velocity spectrum $F(k)$ is determined by the transfer mechanism and not affected by the viscous dissipation. In this range $F(k)$ should be independent of η, or $v_S^2L_SE(kL_S)$ independent of η. Clearly this can only be satisfied with a power law for E, since L_S contains η. Putting

$$E(kL_S)=\text{const}\times(kL_S)^n,$$

we get, with (33):

$$k^nv_S^2L_S^{n+1}\propto k^n\eta^{\frac{1}{2}+\frac{3}{4}(n+1)},$$

and independence of η gives $n=-5/3$.

To normalize F approximately, we can tentatively assume that this spectrum holds for all k values between $1/L_0$ and $1/L_S$ and that beyond both these limits the spectrum is cut off. On the high wave number side, the viscosity is certainly to act as a cutoff, whereas on the low end the power source produces naturally eddies of a certain maximum size L_0. With these assumptions the normalization condition (36) gives us:

$$\text{const}\times(2\eta/S_0)\int_{1/L_0}^{1/L_S} dkk^{\frac{1}{3}}=1.$$

(We see here that the exact shape of the spectrum at the lower end is immaterial as far as the normalization goes.) Hence,

$$\text{const}\cong\tfrac{2}{3}(S_0/\eta)L_S^{4/3}=\tfrac{2}{3}(S_0/\rho)^{2/3},$$

and

$$F(k)=\tfrac{2}{3}(S_0/\rho)^{\frac{2}{3}}k^{-5/3}. \tag{37}$$

(b) *Tail of Velocity Spectrum*

For this region no really good solution of the problem has been given. The simplest way to deal with the problem is to consider the transfer of kinetic energy from small to large wave numbers as a damping effect on the motion of the large eddies. Thus the concept of an *eddy viscosity* $\eta'(k)$ is introduced to describe the drain of energy on an eddy of size k^{-1} by eddies of size $<k^{-1}$. If we assume that $\eta'(k)$ can at all be expressed in terms of $F(k')$ $(k'<k)$, a unique expression results from dimensional arguments:

$$\eta'(k)=\text{const}\times\rho\int_k^\infty dk'[F(k')/k'^3]^{\frac{1}{2}}. \tag{38}$$

The value of the constant can be estimated from the argument that $\eta'(k)$ should be large compared with η for $k<1/L_S$ and becoming equal η at $k=1/L_S$. Hence,

$$\eta=\text{const}\times\rho\int_{1/L_S}^\infty dk'[F(k')/k'^3]^{\frac{1}{2}};$$

with (37), this gives:

$$\text{const}=(4/3)\sqrt{\tfrac{3}{2}}\cong 1.$$

We can now picture our stationary power transfer as a dissipation process whereby the energy S_k dissipated in eddies $k'\leq k$ *or* transferred to the velocity components $\mathbf{v}_{k'}$, with $k'>k$ is given by

$$S_k=2[\eta'(k)+\eta]\int_0^k dk'k'^2F(k'). \tag{40}$$

The condition of stationary transfer implies that no energy is accumulated in a given wave number interval and hence that S_k is *independent of* k and equal to S_0. With (38) we thus get the following equation for $F(k)$:

$$S_0=2\left\{\rho\int_k^\infty dk'[F(k')/k'^3]^{\frac{1}{2}}+\eta\right\}\int_0^k dk''k''^2F(k''). \tag{41}$$

The solution of this equation is found to be[11]

$$F(k)=(8S_0/9\rho)^{2/3}k^{-5/3}(1+(k/k_S)^4)^{-4/3}, \tag{42}$$

with

$$k_S=(3S_0\rho^2/8\eta^3)^{1/4}.$$

As we see, for $k\ll k_S$, we find again our result (37) with a slight—and for our purpose, immaterial—change of the normalization constant. Also the value k_S is essentially identical with $1/L_S$ as given in (33). For $k\gg k_S$, $F(k)$ behaves like k^{-7}:

$$F(k)\cong 8^{-2/3}(S_0\rho/\eta)^2k^{-7}. \tag{43}$$

It might be interesting to compare here k_S with $k_0=1/L_0$ and $k_M=1/L_M$, L_M being the *mean free path* of the molecules of the gas: From simple gas kinetics, we have

$$(\eta/\rho)=(2\pi/3)v_M/k_M,$$

and

$$R=v_0L_0/v_ML_M \quad \text{(Reynolds number)}.$$

From (33) we then obtain:

$$k_S=R^{4/3}k_0, \tag{44a}$$

[see also (9)] and

$$k_M=(S_M/S_0)k_S, \tag{44b}$$

[11] The solution of Eq. (41) is due to J. Bass, Compt. rend. **228**, 22 (1949).

where we have introduced $S_M=\rho v_M{}^3k_M$ in analogy to S_0.

Equation (44a) shows that $k_S\gg k_0$ if only the Reynolds number is large enough.

Equation (44b) decides the more subtle question whether hydrodynamic concepts apply at all in the discussion of the dissipation process. $k_M\gg k_S$ is a necessary condition for it, and it holds only provided the power supply S_0 is not too large. We shall discuss the numerical aspect of these conditions in Sec. C.

Our analysis of the velocity spectrum is now essentially completed, and we shall presently return to a discussion of the density fluctuations as given by Eq. (29).

The correlation function $C(r)$ introduced in (24) may be expressed in terms of the ρ_k ($\rho_k=\Delta\rho_k$ for $k\neq 0$):

$$\langle\Delta\rho^2\rangle_{\mathrm{Av}}C(\mathbf{r})=\langle\Delta\rho(\mathbf{R})\Delta\rho(\mathbf{R}+\mathbf{r})\rangle_{\mathrm{Av}}$$

$$=\frac{1}{V}\int_V d\mathbf{R}\Delta\rho(\mathbf{R})\Delta\rho(\mathbf{R}+\mathbf{r})=\sum\nolimits_{\mathbf{k}}\langle\rho_{\mathbf{k}}\rho_{-\mathbf{k}}\rangle_{\mathrm{Av}}e^{i\mathbf{k}\cdot\mathbf{r}}.$$

[Note: The remaining averages—once the space averages are carried out—are time (phase) averages.] With (29) and $\langle\rho^2\rangle_{\mathrm{Av}}\cong\rho_0{}^2$, we get

$$C(\mathbf{r})=\left(\frac{\rho_0}{\gamma p_0}\right)^2\sum_{\mathbf{k}}k^{-4}e^{i\mathbf{k}\cdot\mathbf{r}}\sum_{\mathbf{k}'\mathbf{k}''}\langle(\mathbf{k}\cdot\mathbf{v}_{k'})(\mathbf{k}\cdot\mathbf{v}_{k-k'})\times(\mathbf{k}\cdot\mathbf{v}_{k''})(\mathbf{k}\cdot\mathbf{v}_{k''-k})\rangle_{\mathrm{Av}}.$$

The sum $\sum_{\mathbf{k}'\mathbf{k}''}$ can be collapsed if we notice that only the combinations

$$\text{(a) } \mathbf{k}'=\mathbf{k}'',\quad \mathbf{k}-\mathbf{k}'=\mathbf{k}-\mathbf{k}'',$$
$$\text{(b) } \mathbf{k}'=\mathbf{k}-\mathbf{k}'',\quad \mathbf{k}-\mathbf{k}'=\mathbf{k}''$$

give nonzero contributions in the average. Thus $\sum_{\mathbf{k}'\mathbf{k}''}$ reduces to

$$2\sum\nolimits_{\mathbf{k}'}\langle(\mathbf{k}\cdot\mathbf{v}_{k'})(\mathbf{k}\cdot\mathbf{v}_{-k'})\rangle_{\mathrm{Av}}\times\langle(\mathbf{k}\cdot\mathbf{v}_{k-k'})(\mathbf{k}\cdot\mathbf{v}_{k'-k})\rangle_{\mathrm{Av}}.$$

Furthermore, since the incompressibility implies $(\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}})=0$, we have:

$$\langle(\mathbf{k}\cdot\mathbf{v}_{k'})(\mathbf{k}\cdot\mathbf{v}_{-k'})\rangle_{\mathrm{Av}}=\langle(\mathbf{k}_\perp\cdot\mathbf{v}_{k'})(\mathbf{k}_\perp\cdot\mathbf{v}_{-k'})\rangle_{\mathrm{Av}}=\tfrac{1}{2}|\mathbf{v}_{k'}|^2k_\perp{}^2,$$

where

$$\mathbf{k}_\perp=\mathbf{k}-\mathbf{n}'(\mathbf{k}\cdot\mathbf{n}'),\quad \mathbf{n}'=\mathbf{k}'/k',$$

and, therefore,

$$k_\perp{}^2=[k^2k'^2-(\mathbf{k}\cdot\mathbf{k}')^2]/k'^2.$$

This gives

$$(\langle\Delta\rho^2\rangle_{\mathrm{Av}}/\rho^2)C(\mathbf{r})=2(\rho_0/\gamma p_0)^2\sum_k k^{-4}e^{i\mathbf{k}\cdot\mathbf{r}}\sum_{\mathbf{k}'}\langle|\mathbf{v}_{k'}|^2\rangle_{\mathrm{Av}}\langle|\mathbf{v}_{k-k'}|^2\rangle_{\mathrm{Av}}\times\frac{[k^2k'^2-(\mathbf{k}\cdot\mathbf{k}')]^2}{4k^2(\mathbf{k}-\mathbf{k}')^2}.$$

Since we need only the Fourier-transformed $C(\mathbf{k})$ of $C(\mathbf{r})$, [see (25) and (26)], we write the $\mathbf{k}$ sums in terms of Fourier integrals with the aid of the substitution

$$\sum\nolimits_{\mathbf{k}}\rightarrow V(2\pi)^{-3}\int d\mathbf{k}.$$

Introducing at the same time $F(k)$ by (34a), we get

$$(\langle\Delta\rho^2\rangle_{\mathrm{Av}}/\rho^2)C(\mathbf{r})=\pi(\rho_0/\gamma p_0)^2(2\pi)^{-3}\int d\mathbf{k}k^{-4}e^{i\mathbf{k}\cdot\mathbf{r}}\times\int d\mathbf{k}'F(k')F(|\mathbf{k}-\mathbf{k}'|)\left(\frac{k^2k'^2-(\mathbf{k}\cdot\mathbf{k}')^2}{k^2(\mathbf{k}-\mathbf{k}')^2}\right)^2,$$

or, in view of (26):

$$\langle\Delta\rho^2\rangle_{\mathrm{Av}}/\rho^2C(\mathbf{k})=\pi(\rho_0/\gamma p_0)^2k^{-4}\int d\mathbf{k}'F(k')\times F(|\mathbf{k}-\mathbf{k}'|)\left(\frac{k^2k'^2-(\mathbf{k}\cdot\mathbf{k}')^2}{k^2(\mathbf{k}-\mathbf{k}')^2}\right)^2. \quad (45)$$

We shall first evaluate (45) for a case

$$k_0\ll k\ll k_S. \quad (46)$$

In that case, we shall use the spectrum $F(k)$ as given by (37) and use a straight cutoff at $k\geqslant k_S$, as well as for $k\leqslant k_0$. We shall see, however, that with the assumptions of (46), the k' integral is essentially independent of the two limits. The k' integral in (45)—let us call it $J(k)$—is most easily evaluated in bipolar coordinates:

$$k'=kr_1,\quad |\mathbf{k}-\mathbf{k}'|=kr_2,\quad d\mathbf{k}'=2\pi k^3r_1r_2dr_1dr_2.$$

This gives for $J(k)$:

$$J(k)=(8S_0/9\rho)^{4/3}(2\pi k^{-1/3})\int\int dr_1dr_2(r_1r_2)^{-14/3}\times[(r_1{}^2-r_2{}^2)^2-2(r_1{}^2+r_2{}^2)+1]^2/16$$
$$=(8S_0/9\rho)^{4/3}(2\pi k^{-1/3})H(k_0/k;k_S/k). \quad (47)$$

H is a dimensionless integral and with (46) approximately an absolute constant. So we have

$$(\langle\Delta\rho^2\rangle_{\mathrm{Av}}/\rho^2)C(\mathbf{k})\cong 2\pi^2(\rho_0/\gamma p_0)^2(S_0/\rho)^{4/3}k^{-13/3}H. \quad (48)$$

An approximate integration gives for H, assuming (46),

$$H=1.2+1.6(k/2k_S)^{7/3}+\cdots. \quad (49)$$

(In all practical cases, k is probably very much larger than k_0 and not so far from k_S. We, therefore, did not evaluate corrections $\propto k_0/k$ in H.)

In the limit of very high wave numbers k,

$$k\gg k_S, \quad (50)$$

the main contribution to $J(k)$ will arise from the shaded regions of Fig. 1. So we may write

$$J(k)=\int d\mathbf{k}'F(k')F(|\mathbf{k}-\mathbf{k}'|)\sin^4\alpha$$

$$\cong 2\int_{k'<k_S} d\mathbf{k}'F(k')F(k)\sin^4\alpha.$$

Introducing again dimensionless variables $\mathbf{z}$: $\mathbf{k}'=k\mathbf{z}$, we get

$$J(k)\cong 2(8S_0/g\rho)^{4/3}k^{-1/3}(k_S/k)^{16/3}\int_{z\leq k_S/k} d\mathbf{z}z^{-5/3}\sin^4\alpha,$$

with

$$k_S\gg k \quad (50), \quad \sin^2\alpha\cong\sin^2\varphi \text{ (see Fig.1)},$$

and hence

$$\int d\mathbf{z}z^{-5/3}\sin^4\alpha\cong 2\pi(3/4)(k_S/k)^{4/3}\times(16/15),$$

and

$$J(k)\cong(8S_0/9\rho)^{4/3}2\pi k^{-1/3}\times(8/5)(k_S/k)^{20/3}. \quad (51)$$

So $H(k_S/k)$ is $8/5(k_S/k)^{20/3}$, compared with 1.2 for the case $k\ll k_S$. We see that the two expressions (47) and (51) for $H(k_S/k)$ give almost identical values for $k=k_S$:

$H=1.2$ and 1.6, respectively.

We, therefore, conclude that the formulas hold up to (or down to) values rather close to that limit and that an interpolation should be easy.

Let us finally write down the cross-section formula corresponding to the two cases $k\ll k_S$ and $k\gg k_S$.

According to (48) and (16) we have

$$\sigma d\Omega=\frac{g^2}{8\lambda^4}(\rho_0/\gamma p_0)^2(S_0/\rho)^{4/3}K^{-13/3}H(K/k_S)\sin^2\chi d\Omega.$$

Now $(\rho_0/\gamma p_0)=(3/\gamma v_M{}^2)$, where $v_M{}^2$ is the mean square molecular velocity, and $\gamma=1.4$ for oxygen and nitrogen. Then finally, with $K=2k\sin\frac{1}{2}\theta$,

$$\sigma d\Omega\cong\tfrac{1}{2}g^2(S_0{}^{4/3}v_M{}^{-4})$$
$$\times\lambda^{1/3}(2\sin\tfrac{1}{2}\theta)^{-13/3}H\left(2\frac{k}{k_S}\sin\tfrac{1}{2}\theta\right)\sin^2\chi d\Omega.$$

So, for the two cases, by using the extreme values (49) and (51) for H, and substituting S_0 from (31), we finally get

$$\sigma\cong 0.6g^2(v_0/v_M)^4\lambda^{1/3}L_0{}^{-4/3}[2\sin(\theta/2)]^{-13/3}\sin^2\chi,$$
$$[2k\sin(\theta/2)\ll k_S]; \quad (52)$$

$$\sigma\cong 0.8g^2(v_0/v_M)^9\lambda^7L_0{}^{-3}L_M{}^{-5}[2\sin(\theta/2)]^{-11}\sin^2\chi,$$
$$2k\sin(\theta/2)\gg k_S. \quad (53)$$

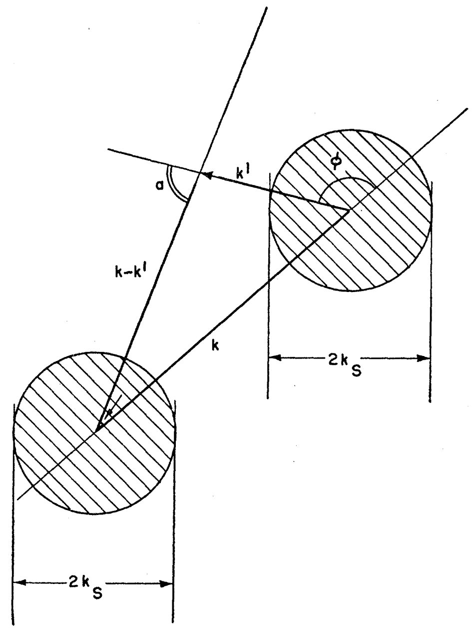

FIG. 1. The integral $J(k)$.

C. APPLICATION TO SPECIAL CASES

1. Ionospheric Scattering

We apply our scattering formula (21) to the experiment of Bailey *et al.*,[12] in which radiation of 50 Mc was scattered by the E layer. By using the same geometry as Bailey *et al.*[12] we obtain for the ratio P_r/P_t of the power received to the power transmitted:

$$\frac{P_r}{P_t}=4\sigma\frac{bA}{\sin(\theta/2)D^2},$$

where b is the thickness of the scattering layer and A is the aperture of the receiving antenna. We use the following numbers which correspond to the setup in their experiments:

$$\lambda=10^2\text{ cm},\quad A=3\times10^6\text{ cm}^2,\quad D=1.2\times10^8\text{ cm},$$
$$b=5\times10^5\text{ cm},\quad \theta=24°.$$

For g^2 we use (5) with $\omega_N=1.5$ Mc and the turbulence constants as given in (13a).

Substituting these values, we get

$$P_r/P_t=1.0\times10^{-18}\text{ [theoretical from (21)]},$$

whereas the experiments have given

$$P_r/P_t=0.36\times10^{-18}\text{ (experimental)}.$$

This is a remarkably good agreement. We note that $L=(2k\sin\frac{1}{2}\theta)^{-1}=250$ cm, which, according to (13b), is just equal to L_S. Hence, we actually should expect a somewhat smaller result than (21) and a steeper falling-off than (21) when going to larger angles or smaller wavelengths.

[12] D. K. Bailey *et al.*, Phys. Rev. **86**, 141 (1952).

2. Tropospheric Scattering

We have compared our results with the data of the experiments of (a) Round Hill, Massachusetts;[13] (b) Cheyenne Mountain, Colorado;[14] (c) the North Sea;[15] and (d) the Caribbean Sea.[16]

In experiment (a), we have (see Fig. 2): $D=3\times10^5$ m; $\lambda=1.5$ cm; $\alpha=$ antenna lobe width $=0.6°$; $\theta=$ scattering angle $=2.4°$; and $H=$ minimum height of intersection $=1.8$ km. The useful scattering volume is $V\cong(\frac{1}{2}D\sin\frac{1}{2}\alpha)^3/\sin\theta$.

Throughout, in these four experiments, we shall compare the actual received (through scattering) power, P_{sc}, to the power received in free space over the same distance P_f. Thus the antenna characteristics are eliminated, and we get

$$P_{sc}/P_f=4(4\sigma V/D^2). \tag{54}$$

Using (54) with (53), and with v_0, L_0 as given in (12), and $g=0.45\rho$, corresponding to the dry-air term of (6a), we get

$$P_{sc}/P_f\cong10^{-12}.$$

The experimental data indicate a ratio of order 10^{-8}, that is, a factor 10^4 stronger.

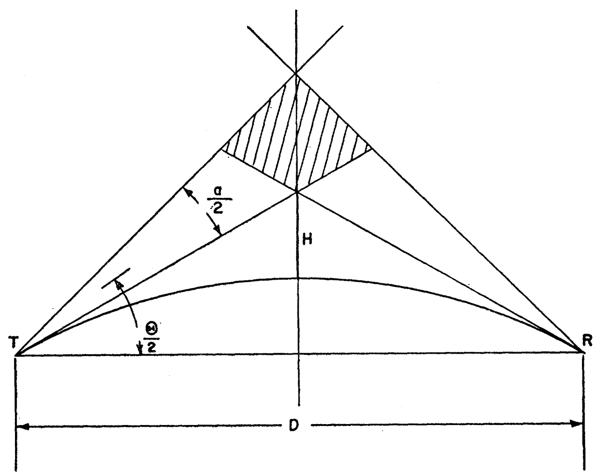

FIG. 2. Geometry for tropospheric scattering (narrow beams).

In experiment (b) (and the following ones), we have a slightly different geometry: the beams are wide, and it is the form-factor of the scattering cross section rather than the lobe width which determines the useful scattering volume. We take, as a representative case in (*b*), a result obtained with $D=3.6\times10^5$ m, $\lambda=10$ cm, $\theta=$ minimum scattering angle $=3°14'$, and $H=$ minimum height of intersection $=2.54$ km, and get $P_{sc}/P_f=10^{-11}$, compared to an observed 10^{-7}. Again, we are a factor 10^{-4} off.

That the Cheyenne mountain experiments are quite typical is seen from the data from (c) and (d). The very weak λ dependence of the cross section ($\propto\lambda^{1/3}$) makes comparison easy, as far as orders of magnitudes go.

D. DISCUSSION

Further ionospheric experiments would be necessary to tell us whether the proposed picture of the dynamics of turbulence represents a reasonable approximation. Since in the troposphere the Reynolds numbers are generally much larger than in the ionosphere, we expect in general that our model is even better suited for tropospheric application. But there, in terms of dry-air fluctuations, an increase of the turbulent power output by a factor 10^3 would be necessary to account for the observed data, and this seems entirely inconceivable.

A natural explanation seems then to hold fluctuations in humidity responsible for the effect. If the assumptions made at the end of Sec. 2 are valid, we infer from (6a) and (14a) that, with say $\rho'=10^{-2}\rho$, $T\cong300°$, only a very small value for the large-scale humidity fluctuations $(\Delta\rho')_0$ is necessary to explain the observed scattering: $(\Delta\rho)_0^2/\rho'^2\cong2\times10^{-4}$. No measurements of $(\Delta\rho')_0^2$ and the humidity fluctuation correlations are available yet. Measured large-scale humidity fluctuations in and outside cumulus clouds[17] give variations in $\Delta\rho'/\rho'$ of the order of 10 percent over distances of 500 m. Fluctuations in a cloudless atmosphere are probably much smaller but might still be sufficient (1 percent) to explain the effect.

A detailed investigation of the humidity effect (and concurrently of the temperature effect) appears to be most important, and is under way.

We wish to thank Professor Henry Booker (Cornell) and Dr. J. de Bettencourt (M.I.T.) for many stimulating discussions.

[13] Private communication.

[14] Chambers, Herbstreit, and Norton, Nat. Bur. Standards, Report No. 1826, 1952 (unpublished).

[15] E. C. S. Megaw, Nature **166**, 1100 (1950); Proc. Inst. Elec. Engrs. III, **100**, 1 (1953).

[16] For these and similar experiments, see T. J. Carroll, Nat. Bur. Standards, Report No. 1416, 1952 (unpublished).

[17] J. S. Malkus, Sci. American **189**, 31 (1953).