$$U_{\text{electric}}=\int\frac{E^2}{8\pi}dv=\tfrac{1}{2}C\left[\int_c \mathbf{E}\cdot d\mathbf{l}\right]^2. \tag{28}$$

The value of C obtained by Eq. (28) can be used in the noise formulas to calculate $\langle V^2\rangle_{\text{Av}}=[\int_c \mathbf{E}\cdot d\mathbf{l}]^2$, over the same contour which appears in expression (28). For the quantities G/C and R/L which appear in (19a) of I and (12A) of this paper we substitute the ratio,

$$\frac{\omega}{2\pi}\times\frac{\text{Energy dissipated per cycle}}{\text{Maximum stored energy per cycle}},$$

and compute this ratio classically.

CONCLUSION

Expressions (21) and (27) are the same as expressions (28) and (34) of I. We see that for a circuit damped by a radiation resistance the vacuum fluctuations are observable in electromotive force measurements at low temperatures. The available power tends to zero as the temperature approaches zero. The radiation resistance is seen to have the same classical and quantum effects as regards damping and noise, as an ordinary resistance. To first order there are no additional terms in the radiation resistance formula due to quantum effects.

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Vacuum Fluctuation Noise*†

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An electron stream interacting with a damped oscillator is considered. The vacuum fluctuations and the thermal fluctuations can be observed in the noise. It is shown that an electron stream provides a means for precisely measuring the mean squared electromotive force for certain modes.

INTRODUCTION

In previous papers,[1,2] to be referred to as I and II, we have shown that for a damped oscillator the results of precise measurements of the mean squared electromotive force are given by

$$\langle V^2\rangle_{\text{Av}}=\frac{1}{C}\left[\frac{\hbar\omega}{2}+\frac{\hbar\omega}{\exp(\hbar\omega/kT)-1}\right]. \tag{1}$$

In Eq. (1), C is the capacity, and ω is the natural (angular) frequency of the oscillator. The first term of Eq. (1) represents the effect of the vacuum fluctuations, and the second term represents the effect of the thermal fluctuations. We consider here the possibility of observing these small fluctuations by allowing an electron stream to interact with the oscillator and observing the resultant electron stream noise.

It is well known[3] that the use of electrons as test charges does not in general lead to precise field measurements. We will show, however, that in the experiments to be discussed a measurement of the mean squared electromotive force can give precisely the value above.

INTERACTION OF AN ELECTRON STREAM WITH A DAMPED ELECTRICAL OSCILLATOR

Consider an electron stream which interacts with a damped electrical oscillator (Fig. 1).

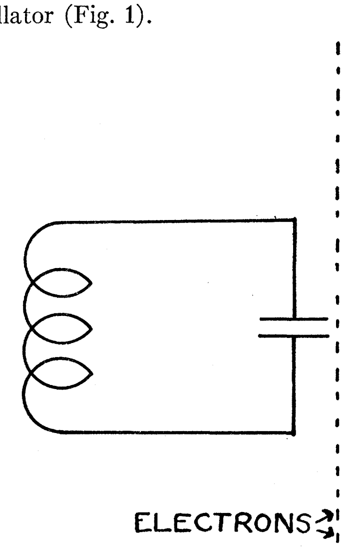

Fig. 1. An electron stream interacting with a damped oscillator.

The interaction can be imagined to take place by sending the stream of electrons near the condenser plates, through holes in the condenser plates, or through

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[1] J. Weber, Phys. Rev. **90**, 977 (1953).
[2] J. Weber, preceding paper, Phys. Rev. **94**, 211 (1954).
[3] W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, London, 1944), p. 78.

a cavity resonator. The circuit is assumed to be in thermal equilibrium with the conductance which is maintained at temperature T. If the damping is small, the effect of the conductance is mainly to determine the average electromagnetic field energy of the circuit. The wave-functions for the quantized fields of the circuit will not be significantly different from those of the undamped oscillator. Under these conditions we can employ the following Hamiltonian to discuss the interaction between an electron and the fields associated with the circuit, or a single mode of a cavity resonator:

$$\mathcal{H}=P^2/2m+\tfrac{1}{2}(p^2+\omega^2q^2)-(e/mc)\mathbf{A}\cdot\mathbf{P}. \quad (2)$$

$\mathbf{A}$ is the magnetic vector potential, and we let $\mathbf{A}=\mathbf{A}_0(r)q(t)$. The variable p is canonically conjugate to q. $\mathbf{P}$ is the operator corresponding to the electron momentum. It is unnecessary to include terms in (2) representing the conductance because there is no direct interaction between the electron and the conductance.

We assume that the circuit is in an eigenstate of its unperturbed Hamiltonian before the interaction begins. For the perturbed wave functions of the system we assume the expression

$$\Psi=\sum_{ij} a_i\psi_i b_j\phi_j \exp[-(i/\hbar)(E_i+E_j)t], \quad (3)$$

where ψ_i is an unperturbed wave function for the fields of the circuit, ϕ_j is an unperturbed wave function for the electron.

At the time interaction begins: $a_n=1$, $b_m=1$, $a_i=0$, $i\neq n$, $b_j=0$, and $j\neq m$.

It can be shown[4] that at any time t during the interaction time,

$$|a_K(t)b_l(t)|^2 = \frac{4|H_{KlNM}'|^2\sin^2\{[(E_K+E_l-E_N-E_M)/2\hbar]t\}}{(E_K+E_l-E_N-E_M)^2}, \quad (4)$$

where

$$H_{KlNM}'=\frac{e}{mc}\int\psi_K^*\phi_l^*(\mathbf{A}\cdot\mathbf{P})\psi_N\phi_M d\tau_\psi d\tau_\varphi. \quad (5)$$

Referring to (4), it is much easier to calculate a_K than to calculate b_l, and we use a method and approximations first suggested by Smith.[5] We consider the electron to be localized, so that $\mathbf{A}\cdot\mathbf{P}$ is a very slowly varying function of position over the region of the electron and the electron energy is not known during the interaction time. Following Smith[5] we write

$$\int\phi_l^*(\mathbf{A}\cdot\mathbf{P})\phi_M d\tau_\varphi\approx(\mathbf{A}\cdot\mathbf{P})_{\text{Av}}\delta_{lM},$$

and expression (5) becomes

$$H_{KlNM}'\approx H_{KN}'P_{\text{Av}}\delta_{lM}=A_0P_{\text{Av}}\delta_{lM}\langle E_{FK}|q|E_{FN}\rangle\frac{e}{mc}. \quad (5\text{A})$$

In (5A), $\langle E_{FK}|q|E_{FN}\rangle$ is the matrix element of q between the quantum states of the field with quantum numbers K and N. P_{Av} is the average value of the electron momentum over the initial and final quantum states of the electron. We assume that there are electrodes in the system beyond the interaction region which limit the electrons observed to those with velocities fairly close to the original electron velocity, and that the electron velocities are ultimately observed when the electrons have travelled a considerable distance beyond the circuit interaction gap. Making use of (5A) and the initial condition $b_m(t)=1$, we can write (4) in the form

$$|a_K(t)|^2=\frac{4e^2A_0^2P_{\text{Av}}^2|\langle E_{FK}|q|E_{FN}\rangle|^2\sin^2(\tfrac{1}{2}\omega_{KN}t)}{m^2c^2\hbar^2\omega_{KN}^2}. \quad (6)$$

In (6) ω_{KN} is the natural frequency of the oscillator, and (6) gives us the probability that the field has gained or lost a quantum at time t. It is interesting to note that (6) can also be obtained directly from the Hamiltonian (2), if we quantize the field but treat the electron classically, by not regarding $\mathbf{P}$ as an operator.

We assume that the interaction gap is small and of length l. The interaction time τ will be given approximately by

$$\tau=lm/|P_{\text{Av}}|. \quad (7)$$

From Eq. (5) of I we have

$$\langle E^2\rangle=(1/c^2)p^2\langle A_0^2\rangle,\quad p^2=CV^2=C(El)^2. \quad (8)$$

For a resonator, (8) defines C.

Making use of 6, 7, and 8, we can write Eq. (4) at time τ in the form

$$|a_K(\tau)|^2=\frac{e^2}{\hbar^2C}|\langle E_{FK}|q|E_{FN}\rangle|^2\left(\frac{\sin(\tfrac{1}{2}\omega_{KN}\tau)}{\tfrac{1}{2}\omega_{KN}\tau}\right)^2. \quad (9)$$

We let $\omega_{KN}\tau=\theta$, where θ is the electron transit angle. Making use of the matrix elements for the quantum states of the field we obtain for (9)

$$|a_K(\tau)|^2=\frac{e^2}{\hbar\omega C}\left[\frac{\sin\frac{1}{2}\theta}{\frac{1}{2}\theta}\right]^2\left(\frac{N+1}{2}\right),\quad \text{for } K=N+1, \quad (10)$$

$$|a_K(\tau)|^2=\frac{e^2}{\hbar\omega C}\left[\frac{\sin\frac{1}{2}\theta}{\frac{1}{2}\theta}\right]^2\left(\frac{N}{2}\right),\quad \text{for } K=N-1, \quad (11)$$

$$|a_K(\tau)|^2=0,\quad \text{for } K\neq N\pm 1. \quad (12)$$

Expression (12) states that the exchange of energy takes place in one-quantum steps. Equation (10) is the probability that the field will gain a quantum from

[4] W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, London, 1944), p. 88.

[5] L. P. Smith, Phys. Rev. **69**, 195 (1946).

the electron, and (11) is the probability that the field will lose a quantum to the electron. We assume that the experiment is so arranged that (9) is small and therefore only single-quantum exchanges are significant.

Suppose that M electrons interact with the circuit, and that a sufficiently long time elapses between interactions so that the circuit can return to a state of equilibrium with the conductance. This will be the case if the current in the electron stream is very small. Under these conditions we will have $|a_K(\tau)|^2{}_{K=N+1}M$ electrons lose a quantum, and $|a_K(\tau)|^2{}_{K=N-1}M$ electrons gain a quantum. If an electron neither gains nor loses energy we can say that the electromotive force of the circuit during that interaction time was zero. If an electron gains or loses a quantum we can say that the electromotive force during that interaction time was $V=\hbar\omega/e$. The mean squared noise voltage for a circuit whose quantum number is N is

$$\begin{aligned}\langle V^2\rangle(N)&=\frac{\sum_i V_i^2}{M}\\&=\frac{\hbar^2\omega^2}{e^2M}[|a_K(\tau)|^2{}_{K=N+1}+|a_K(\tau)|^2{}_{K=N-1}]M\\&=\frac{\hbar\omega}{C}(N+\tfrac{1}{2})\left(\frac{\sin\frac{1}{2}\theta}{\frac{1}{2}\theta}\right)^2.\end{aligned}\qquad(13)$$

If we consider an ensemble, the ergodic theorem guarantees that the measured value at temperature T will be the ensemble average. The ensemble average of N is

$$N_{\text{Av}}=\frac{\sum_N N\exp[-(N+\frac{1}{2})\hbar\omega/kT]}{\sum_N \exp[-(N+\frac{1}{2})\hbar\omega/kT]}=\frac{1}{\exp(\hbar\omega/kT)-1}.\qquad(14)$$

Inserting (14) into (13) we obtain

$$\langle V^2\rangle_{\text{Av}}=\frac{1}{C}\left[\frac{\hbar\omega}{2}+\frac{\hbar\omega}{\exp(\hbar\omega/kT)-1}\right]\left(\frac{\sin\frac{1}{2}\theta}{\frac{1}{2}\theta}\right)^2.\qquad(15)$$

If the transit angle θ is small, (15) becomes

$$\langle V^2\rangle_{\text{Av}}=\frac{1}{C}\left[\frac{\hbar\omega}{2}+\frac{\hbar\omega}{\exp(\hbar\omega/kT)-1}\right].\qquad(16)$$

A circuit, and a cavity, will have many modes. An experiment can be so arranged that the only electrons studied are those with energies between values $E_{\max}$ and $E_{\min}$. This limits the number of modes considered to a definite number, namely those of frequency lower than $\omega_{\max}=(E_{\max}-E_{\min})/\hbar$, because we are considering one-quantum processes. The modes of frequency higher than the lowest mode can be made to have a very small contribution because the transit angle will be large and the factor $(\sin\frac{1}{2}\theta/\frac{1}{2}\theta)^2$ appearing in (15) will be very small.

Accordingly we say that the contribution of all modes other than the principal mode can be made very small. We see that (16) agrees with (1) and that the electron stream provides a means for precisely measuring the mean squared electromotive force (including quantum effects) as given by (1), provided the transit angle is small.

CONCLUSION

We have studied the random changes of velocity of an electron stream interacting with a damped oscillator. The vacuum fluctuations are directly observable as noise in the electron stream if the circuit is at low temperature.

One might wonder why the vacuum (radiation) fields outside of the circuit do not also contribute to the zero point electron stream noise. The radiation fields outside do not contribute because a free electron cannot radiate. This is well known and results from the fact that the conditions of conservation of energy and conservation of momentum cannot be simultaneously satisfied. The electron can exchange energy with the fields of the circuit because during the interaction time the electron is not free. Its momentum is not precisely known during the interaction time, because its position is known to be localized in the interaction gap. The electron can therefore undergo spontaneous emission to the circuit, even if the circuit is in its lowest state. This loss of energy by the electron is a purely random process and therefore contributes to the noise. This is the origin of the zero point noise contribution in this case.

A factor which must be considered in doing a low-temperature noise experiment, is the noise (such as shot[6] noise), present in the beam before interaction with a resonator. The amount of this noise will depend on the manner in which the beam is prepared. The uncertainty principle does not preclude preparation of a low-current beam in which the electron velocities are closely grouped within a range substantially less than the velocity increment due to loss of a high-frequency microwave quantum. The vacuum fluctuation noise could be observed in the random changes of velocity of the electrons, after interaction. In any case, it is not essential that the noise in the electron beam before interaction, be less than the vacuum fluctuation noise. This is because the latter noise can be observed as an increment in the noise already present. Somewhat similar procedures are employed for temperature measurement by microwave radiometers.[7]

[6] W. Schottky, Ann. Physik **57**, 541 (1918).
[7] R. H. Dicke, Rev. Sci. Instr. **17**, 268 (1946).