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Quantum Theory of a Damped Electrical Oscillator and Noise. II. The Radiation Resistance*

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The results of a previous paper are extended to include damping and noise due to a radiation resistance. The average electromagnetic field energy of an oscillator of natural frequency ω, with inductance L, coupled to a radiation resistance R, as a function of time t is given by

$$U=\left[\tfrac{1}{2}\hbar\omega+\frac{\hbar\omega}{\exp(\hbar\omega/kT)-1}\right][1-e^{-(R/L)t}]+U_0e^{-(R/L)t}.$$

Using first-order perturbation theory and quantum statistics, an expression is derived for the radiation resistance of a circuit. This agrees exactly with the classical value.

For an oscillator damped by a radiation resistance the vacuum fluctuation noise, and available noise power are shown to be the same as for an oscillator damped by a lumped resistance. All of the previous results are shown to be applicable to cavity resonators.

INTRODUCTION

In a previous paper,[1] hereafter denoted by I, we discussed some quantum effects in damping and noise. The vacuum fluctuations were shown to be directly observable in noise experiments. The assumption was made that the circuit did not radiate. In a subsequent paper it will be shown that the vacuum fluctuations can be observed as noise induced in an electron stream which interacts with a circuit. To do such an experiment coupling holes would have to be provided, and a radiation resistance would be thereby introduced. For this reason the results of I are extended to include the radiation resistance. It will be shown that the radiation resistance affects the system in essentially the same way as an ordinary resistance. Also for some purposes a cavity resonator is more suitable than a circuit composed of lumped elements. It will therefore be shown that the results already obtained are applicable to cavity resonators also.

ELECTRICAL OSCILLATOR DAMPED BY RADIATION RESISTANCE

We assume that the radiation resistance is a series element, as in Fig. 1. For the Hamiltonian of Fig. 1, we have

$$\mathcal{H}=\tfrac{1}{2}(p_F^2+\omega^2q_F^2)+H_{FR}+H_\epsilon' +\textstyle\sum_\lambda 1/2m_\lambda(\mathbf{p}_\lambda-e_\lambda\mathbf{A}/c)^2. \quad (1)$$

p_F and q_F are the field variables employed in I, and the first two terms of (1) are the Hamiltonian of the dissipationless oscillator. The term H_{FR} is the Hamiltonian of the unperturbed radiation fields, and the term H_ϵ' represents part of the unperturbed Hamil-

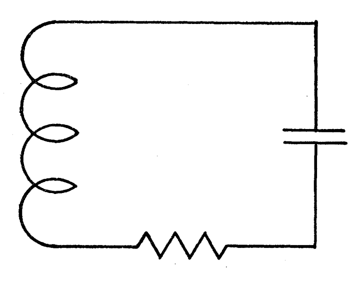

FIG. 1. Electrical oscillator damped by a radiation resistance.

* Supported by the U. S. Office of Naval Research.

[1] J. Weber, Phys. Rev. **90**, 977 (1953). We have used the notation of the previous paper except that in this paper the variable q of the circuit is written as q_F.

tonian of the electrons.† The last term is a summation over all of the electrons, and represents the interaction between the electrons and the circuit and radiation fields. m_λ is the mass, $\mathbf{p}_\lambda$ is the momentum, and e_λ is the charge for the λth electron, and $\mathbf{A}$ is the magnetic vector potential of the radiation fields. We assume that no resistance other than radiation resistance is present, for simplicity. If both kinds of resistance are present additional terms are added to the Hamiltonian and the effects can be shown to take place independently.

We consider our oscillator to be weakly damped, and employ first-order perturbation theory. The term in A^2 in (1) can be neglected. The Hamiltonian becomes

$$\mathcal{H}=\tfrac{1}{2}(p_F^2+\omega^2q_F^2)+H_{FR}+H_\epsilon-\sum_\lambda \frac{e_\lambda \mathbf{p}_\lambda\cdot\mathbf{A}}{m_\lambda c}. \quad (2)$$

We have included the term $\sum_\lambda(p_\lambda^2/2m_\lambda)$ in the term H_ϵ. We expand the radiation fields in normal modes, $\mathbf{A}=\sum_{\omega i}\mathbf{A}_{\omega i}(\mathbf{r})q_{\omega i}(t)$. The last term in (2) can then be written

$$\sum\frac{e_\lambda\mathbf{p}_\lambda\cdot\mathbf{A}}{m_\lambda c}=\sum_{\omega i}\sum_\lambda\frac{e_\lambda\mathbf{p}_\lambda\cdot\mathbf{A}_{\omega i}q_{\omega i}}{m_\lambda c}. \quad (3)$$

For a circuit this becomes

$$\sum_{\omega i}\sum_\lambda\frac{e_\lambda\mathbf{p}_\lambda\cdot\mathbf{A}_{\omega i}q_{\omega i}}{m_\lambda c}=\int_c\frac{I(\mathbf{r})}{c}\sum_{\omega i}\mathbf{A}_{\omega i}q_{\omega i}\cdot d\mathbf{l}. \quad (4)$$

We can write the current $I(\mathbf{r})$ as $I(\mathbf{r})=I_0f(\mathbf{r})$, where $f(r)$ is the current distribution function and I_0 is the current at some reference point. We define $F_{\omega i}$ by

$$F_{\omega i}=\left|\int_c f(\mathbf{r})\mathbf{A}_{\omega i}\cdot d\mathbf{l}\right|,$$

and we define the inductance L by the relation $LI_0^2=q_F^2\omega^2$. By utilizing these relations, Eq. (4) can be written:

$$\sum_{\omega i}\sum_\lambda\frac{e_\lambda\mathbf{p}_\lambda\cdot\mathbf{A}_{\omega i}q_{\omega i}}{m_\lambda c}=\frac{\omega q_F}{c\sqrt{L}}\sum_{\omega i}q_{\omega i}F_{\omega i}. \quad (5)$$

The term (5) is an interaction term which will cause transitions with exchange of energy between the circuit and the radiation fields through the coupling furnished by the electrons. The transition probability can be shown to be

$$W_r=\frac{2\pi\omega^2}{c^2\hbar}(F_\omega^2)_{\text{Av}}\Big[\rho(E_{FR}+\hbar\omega)\langle E_{FR}|q_\omega|E_{FR}+\hbar\omega\rangle^2 \times\left\langle E_F\left|\frac{q_F}{\sqrt{L}}\right|E_F-\hbar\omega\right\rangle^2+\rho(E_{FR}-\hbar\omega) \times\langle E_{FR}|q_\omega|E_{FR}-\hbar\omega\rangle^2\left\langle E_F\left|\frac{q_F}{\sqrt{L}}\right|E_F+\hbar\omega\right\rangle\Big]^2. \quad (6)$$

† Our circuit consists of perfect conductors in series with a radiation resistance. The electrons referred to are in the radiation resistance.

In Eq. (6), $(F_\omega^2)_{\text{Av}}$ is an average over all directions of $(F_\omega)^2$, the symbol $\langle E_{FR}|q_\omega|E_{FR}+\hbar\omega\rangle$ indicates the matrix element of the operator q between the quantum states of the radiation fields with eigenvalues E_{FR} and $E_{FR}+\hbar\omega$. $\langle E_F|q_F|E_F+\hbar\omega\rangle$ has the corresponding meaning for the fields of the circuit. $\rho(E_{FR}+\hbar\omega)$ and $\rho(E_{FR}-\hbar\omega)$ are the density in energy of the quantum states of the radiation fields in the vicinity of $(E_{FR}+\hbar\omega)$ and $(E_{FR}-\hbar\omega)$, respectively. We assume that initially the circuit is in an eigenstate, but that only the temperature of the radiation fields is known. We therefore need to average Eq. (6) over an ensemble of similar systems. Now the density in energy of the quantum states of the radiation fields depends only on the frequency and not on the energy, so that $\rho(E_{FR}+\hbar\omega)=\rho(E_{FR}-\hbar\omega)=\omega^2/2\pi^2\hbar c^3$. By employing the harmonic oscillator matrix elements and the relation $E_{FR}=(m+\frac{1}{2})\hbar\omega$, the average of the squared matrix element $\langle E_{FR}|q_\omega|E_{FR}+\hbar\omega\rangle^2$ can be shown to be

$$\frac{\sum_i\langle E_{FRi}|q_\omega|E_{FRi}+\hbar\omega\rangle^2\exp(-E_{FRi}/kT)}{\sum_i\exp(-E_{FRi}/kT)} =\frac{\sum_0^\infty\hbar(m+1)\exp[-(m+\frac{1}{2})\hbar\omega/kT]}{2\omega\sum_0^\infty\exp[-(m+\frac{1}{2})\hbar\omega/kT]} =\frac{\hbar}{2\omega[1-\exp(-\hbar\omega/kT)]}. \quad (8)$$

We can calculate the ensemble average of the squared matrix element $\langle E_{FR}|q_\omega|E_{FR}-\hbar\omega\rangle^2$ in the same way and employ Eq. (8) and Eq. (7) to write Eq. (6) in the form

$$W_r=\frac{\omega^3}{2\pi\hbar c^5}(F_\omega^2)_{\text{Av}}\left[\frac{1}{1-\exp(-\hbar\omega/kT)}\right] \times\left[\left\langle E_F\left|\frac{q_F}{\sqrt{L}}\right|E_F-\hbar\omega\right\rangle^2 +\left\langle E_F\left|\frac{q_F}{\sqrt{L}}\right|E_F+\hbar\omega\right\rangle\exp(-\hbar\omega/kT)\right]. \quad (9)$$

By inserting the harmonic oscillator matrix elements for the circuit, Eq. (9) becomes

$$W_r=\frac{\omega^2}{4L\pi c^5}(F_\omega^2)_{\text{Av}}\left[\frac{1}{1-\exp(-\hbar\omega/kT)}\right] \times[n+(n+1)\exp(-\hbar\omega/kT)]. \quad (10)$$

In (10) the first term of the last factor represents the effect of a downward transition and the second term represents the effect of an upward transition. The rate of change of the field energy associated with the circuit is the difference of the two terms multiplied by $\hbar\omega$ and is

$$\frac{dU}{dt}=\frac{\hbar\omega^3}{4L\pi c^5}(F_\omega{}^2)_{\text{Av}}\left[\frac{1}{1-\exp(-\hbar\omega/kT)}\right]\times[(n+1)\exp(-\hbar\omega/kT)-n]. \quad (11)$$

For an ensemble, expression (11) holds for all time; we can integrate Eq. (11) to obtain

$$U=\left[\tfrac{1}{2}\hbar\omega+\frac{\hbar\omega}{\exp(\hbar\omega/kT)-1}\right]\times\left[1-\exp\left(-\frac{\omega^2}{4L\pi c^5}[(F_\omega{}^2)_{\text{Av}}]t\right)\right]+U_0\exp\left(-\frac{\omega^2}{4L\pi c^5}[(F_\omega{}^2)_{\text{Av}}]t\right). \quad (12)$$

For large energy (classical limit) the second term is the only significant one. Comparing (10) with the well-known classical result $U=U_0e^{-(R/L)t}$, we obtain

$$R=\frac{\omega^2}{4\pi c^5}(F_\omega{}^2)_{\text{Av}}=\frac{\omega^2}{4\pi^2c^5}\left[\left|\int_c f(\mathbf{r})\mathbf{A}_\omega\cdot\mathbf{dl}\right|^2\right]_{\text{Av}} \quad (13)$$

(13) is a formula for the radiation resistance, it agrees exactly with the classical[2] value. In terms of Eq. (13), Eq. (12) is

$$U=\left[\tfrac{1}{2}\hbar\omega+\frac{\hbar\omega}{\exp(\hbar\omega/kT)-1}\right][1-e^{-(R/L)t}]+U_0e^{-(R/L)t}. \quad (12\text{A})$$

EQUIVALENCE OF RADIATION RESISTANCE AND A NOISE GENERATOR

In I we showed the equivalence of a noise current generator and a conductance. In this section, we will show the equivalence of a noise voltage generator and a radiation resistance. We imagine that the radiation resistance is removed and replaced by a voltage generator. In order that the transition probability be proportional to time the generator needs to have a continuous spectrum in the vicinity of ω. The Hamiltonian of the system of Fig. 2 is

$$H=\tfrac{1}{2}(p_F{}^2+\omega^2q_F{}^2)-(p_FV(t)/\omega\sqrt{L}); \quad (14)$$

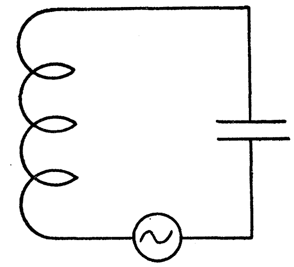

FIG. 2. Electrical oscillator and a voltage generator.

the transition probability is

$$W_G=\frac{\pi V^2(\omega)}{\omega^2\hbar^2L}[\langle E_F|p_F|E_F+\hbar\omega\rangle^2+\langle E_F|p_F|E_F-\hbar\omega\rangle^2]. \quad (15)$$

In Eq. (15) the mean-square value of the voltage over a range $d\omega$ is $V^2(\omega)d\omega$. Inserting the harmonic oscillator matrix elements, Eq. (15) becomes

$$W_G=\frac{\pi V^2(\omega)}{2\hbar\omega L}[(n+1)+n]. \quad (16)$$

In order to compare the transition probability induced by the voltage generator with that induced by the radiation resistance we employ Eq. (13) to write Eq. (10) in terms of the resistance. With this substitution, Eq. (10) becomes

$$W_r=\frac{R}{L}\left[\frac{(n+1)+n}{\exp(\hbar\omega/kT)-1}+n\right]. \quad (17)$$

Comparison of Eq. (16) and Eq. (17) shows that the transition probability will be the same insofar as the first term of Eq. (17) is concerned if

$$V^2(\omega)=\frac{2R\hbar\omega}{\pi[\exp(\hbar\omega/kT)-1]}. \quad (18)$$

Expression (18) is the Nyquist formula in the voltage representation, modified for quantum effects. There is still the last term in Eq. (17). This term is seen to be the transition probability at $T=0$, that is, the transition probability if the resistance is in its lowest state and the quantum number of the circuit is n. We conclude that for the radiation resistance the transitions required by Eq. (17) will be produced by a noise voltage generator described by Eq. (18), plus spontaneous emission, that is, plus the effect of the absorber in its lowest state.

We can also imagine the second term of Eq. (17) to be equivalent to a voltage generator which can only induce downward transitions; comparing the second term of Eq. (17) and Eq. (16) we see that the equivalent voltage for such a generator is

$$V^2(\omega)=\frac{2}{\pi}\hbar\omega R. \quad (19)$$

[2] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949), p. 251. In calculating the average of $(F_\omega{}^2)$ two polarizations need to be included for each direction of propagation $\mathbf{A}_\omega$.

This result is formally analogous to that of[3] Park and Epstein in their treatment of spontaneous emission.

MEAN SQUARED NOISE VOLTAGE AND AVAILABLE POWER

To calculate the observable noise voltage we obtain the equilibrium value of $[\int \mathbf{E}\cdot d\mathbf{l}]^2$ over the ensemble. Proceeding as in I, letting C be the capacity, one obtains the result that

$$\langle V^2\rangle_{\text{Av}}=\frac{\sum_n \left\langle E_{Fn}\left|\frac{p^2}{C}\right|E_{Fn}\right\rangle \exp[-(n+\frac{1}{2})(\hbar\omega/kT)]}{\sum_n \exp[-(n+\frac{1}{2})(\hbar\omega/kT)]}. \quad (20)$$

Equation (20) is identical with Eq. (27) of I, and the value of $\langle V^2\rangle_{\text{Av}}$ is

$$\langle V^2\rangle_{\text{Av}}=\frac{1}{C}\left[\frac{\hbar\omega}{2}+\frac{\hbar\omega}{\exp(\hbar\omega/kT)-1}\right]$$
$$=\omega^2 L\left[\frac{\hbar\omega}{2}+\frac{\hbar\omega}{\exp(\hbar\omega/kT)-1}\right]. \quad (21)$$

The first term of Eq. (21) represents the effect of the vacuum fluctuations and the second term represents the effect of the thermal fluctuations. In I we proved that the first term of Eq. (21) cannot be removed by making formal changes in the Hamiltonian which remove the zero point energy, and the same proof applies here. This is because such changes in the Hamiltonian do not affect the wave functions and therefore do not affect the summations in Eq. (20).

In order to calculate the available power, we consider an arrangement similar to Fig. 1, but with two series resistances R_1 at temperature T_1 and R_2 at temperature T_2. The Hamiltonian is

$$H=\tfrac{1}{2}(p_F^2+\omega^2 q_F^2)+H_{FR}+H_\epsilon$$
$$+\frac{\omega q_F}{\sqrt{L}}\left[\sum_i q_{\omega i}[F_{1\omega i}+F_{2\omega i}]\right]. \quad (22)$$

The rate of change of field energy can be obtained in the same manner as Eq. (11) was obtained, the result is

$$\frac{dU}{dt}=\frac{\hbar\omega^3}{4L\pi c^5}\left[(F_{1\omega}^2)_{\text{Av}}\left[\frac{1}{1-\exp(-\hbar\omega/kT_1)}\right]\right.$$
$$\times[(n+1)\exp(-\hbar\omega/kT_1)-n]$$
$$+(F_{2\omega}^2)_{\text{Av}}\left[\frac{1}{1-\exp(-\hbar\omega/kT_2)}\right]$$
$$\left.\times[(n+1)\exp(-\hbar\omega/kT_2)-n]\right]. \quad (23)$$

By making use of Eq. (13), Eq. (23) becomes

$$\frac{dU}{dt}=\frac{\hbar\omega}{L}\left[\left(\frac{R_1}{1-\exp(-\hbar\omega/kT_1)}\right)\right.$$
$$\times((n+1)\exp(-\hbar\omega/kT_1)-n)$$
$$+\left(\frac{R_2}{1-\exp(-\hbar\omega/kT_2)}\right)$$
$$\left.\times((n+1)\exp(-\hbar\omega/kT_2)-n)\right]. \quad (24)$$

In order to calculate the noise power transferred to the system by R_1 we first determine the stationary value of n by setting Eq. (24) equal to zero, the value is

$$n=\left(\frac{R_1}{\exp(\hbar\omega/kT_1)-1}+\frac{R_2}{\exp(\hbar\omega/kT_2)-1}\right)\Big/(R_1+R_2). \quad (25)$$

This is the most probable value of n a long time after R_1 and R_2 (at temperatures T_1 and T_2) have been coupled to the circuit. We calculate the power transferred to the system by R_1 by inserting the value of n given by Eq. (25) into expression (11) for the power transferred by R_1 when the quantum number of the circuit is n. The result is

$$P_{\text{transferred by } R_1}=\frac{R_1\hbar\omega}{L}\left[\frac{1}{\exp(\hbar\omega/kT_1)-1}\right.$$
$$-\left(\frac{R_1}{\exp(\hbar\omega/kT_1)-1}+\frac{R_2}{\exp(\hbar\omega/kT_2)-1}\right)\Big/$$
$$\left.(R_1+R_2)\right]. \quad (26)$$

Equation (26) will approach a maximum if $R_2/R_1\rightarrow\infty$, $T_2\rightarrow 0$; the maximum value which is approached is

$$P_{\max}\rightarrow\frac{R_1\hbar\omega}{L}\left[\frac{1}{\exp(\hbar\omega/kT_1)-1}\right]=\frac{\hbar\omega\Delta\omega}{\exp(\hbar\omega/kT_1)-1}, \quad (27)$$

and this is the same as expression (34) of I.

CAVITY RESONATORS

The results of I and this paper, may be extended to apply to a single mode of a resonant cavity in the following way. For the dissipationless cavity the Hamiltonian for a single mode is identical with that for a circuit, and the interaction terms which bring in the dissipation will be the same in form. A study of the theory developed here shows that for any mode we can define an equivalent capacity C by the relation

[3] D. Park and H. T. Epstein, Am. J. Phys. **17**, 301 (1949).

$$U_{\text{electric}}=\int\frac{E^2}{8\pi}dv=\tfrac{1}{2}C\left[\int_c \mathbf{E}\cdot d\mathbf{l}\right]^2. \quad (28)$$

The value of C obtained by Eq. (28) can be used in the noise formulas to calculate $\langle V^2\rangle_{\text{Av}}=[\int_c \mathbf{E}\cdot d\mathbf{l}]^2$, over the same contour which appears in expression (28). For the quantities G/C and R/L which appear in (19a) of I and (12A) of this paper we substitute the ratio,

$$\frac{\omega}{2\pi}\times\frac{\text{Energy dissipated per cycle}}{\text{Maximum stored energy per cycle}},$$

and compute this ratio classically.

CONCLUSION

Expressions (21) and (27) are the same as expressions (28) and (34) of I. We see that for a circuit damped by a radiation resistance the vacuum fluctuations are observable in electromotive force measurements at low temperatures. The available power tends to zero as the temperature approaches zero. The radiation resistance is seen to have the same classical and quantum effects as regards damping and noise, as an ordinary resistance. To first order there are no additional terms in the radiation resistance formula due to quantum effects.

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Vacuum Fluctuation Noise*†

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An electron stream interacting with a damped oscillator is considered. The vacuum fluctuations and the thermal fluctuations can be observed in the noise. It is shown that an electron stream provides a means for precisely measuring the mean squared electromotive force for certain modes.

INTRODUCTION

In previous papers,[1,2] to be referred to as I and II, we have shown that for a damped oscillator the results of precise measurements of the mean squared electromotive force are given by

$$\langle V^2\rangle_{\text{Av}}=\frac{1}{C}\left[\frac{\hbar\omega}{2}+\frac{\hbar\omega}{\exp(\hbar\omega/kT)-1}\right]. \quad (1)$$

In Eq. (1), C is the capacity, and ω is the natural (angular) frequency of the oscillator. The first term of Eq. (1) represents the effect of the vacuum fluctuations, and the second term represents the effect of the thermal fluctuations. We consider here the possibility of observing these small fluctuations by allowing an electron stream to interact with the oscillator and observing the resultant electron stream noise.

It is well known[3] that the use of electrons as test charges does not in general lead to precise field measurements. We will show, however, that in the experiments to be discussed a measurement of the mean squared electromotive force can give precisely the value above.

* A brief report of this work was given at the June, 1953, Rochester Meeting of the American Physical Society.

† Supported by the U. S. Office of Naval Research.

[1] J. Weber, Phys. Rev. **90**, 977 (1953).

[2] J. Weber, preceding paper, Phys. Rev. **94**, 211 (1954).

[3] W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, London, 1944), p. 78.

INTERACTION OF AN ELECTRON STREAM WITH A DAMPED ELECTRICAL OSCILLATOR

Consider an electron stream which interacts with a damped electrical oscillator (Fig. 1).

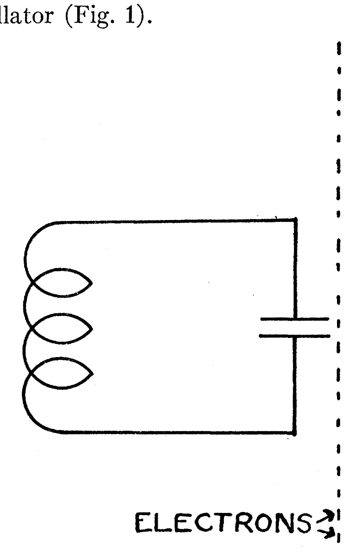

Fig. 1. An electron stream interacting with a damped oscillator.

The interaction can be imagined to take place by sending the stream of electrons near the condenser plates, through holes in the condenser plates, or through