We finally introduce an approximate correction for the conservation of angular momentum $g(W)$:

$$n = g(W)n' \quad (3)$$

for both K mesons and pions, where $g(W)$ decreases from 0.67 at the threshold for pion production to 0.51 at extremely high energies.[1] Within these limits the correction is somewhat arbitrary.

III. RESULTS AND DISCUSSION

The results of these calculations for pn collisions are given in Table II. They are only slightly different for pp collisions. For the mass of the K meson, $m=0.69$ has been used. The values at high energies are taken from our previous paper,[2] and the experimental values were found by the Bristol group.[5]

For the comparison of our results with experiment, one should keep in mind that, although the calculated values are for nucleon-nucleon collisions, the observed values are for nucleon-nucleus collisions. It is difficult to state the effect of the nucleus quantitatively, but, qualitatively, it should reduce the ratio for two reasons (a) the energy available in secondary collisions inside the nucleus is only a fraction of the primary energy, hence the production of K mesons becomes less probable; and (b) there is a chance for a K meson to be reabsorbed in the same nucleus in which it was created, and a pion may even be emitted on this occasion.[6] We would expect this process to occur more frequently at low energies where the K particle is slow than at high energies. In view of these arguments the agreement with experiment does not look unreasonable. One should also note that the whole theory contains only one parameter, the characteristic volume.

TABLE II. Comparison with experiment.

Energy in Mc^2 in lab. system	Calculated			Observed	Kothari (calculated)
	n_K	n_π	n_K/n_π	n_K/n_π	n_K/n_π
4	0.08	0.93	0.09		0.52
6	0.17	1.14	0.14	0.09±0.03	0.040
10	0.30	1.40	0.22		0.036
15	0.47	1.62	0.29	0.20±0.02	0.034
20	0.63	1.78	0.35		
50	1.17	2.34	0.50		
100	1.64	2.85	0.57		
200	2.25	3.43	0.66	0.5±0.2	

[5] D. H. Perkins, Rochester Conference, Dec. 1952 (Interscience Publishers, New York, 1953) and private communication.

[6] B. Peters, Cosmic Ray Conference, Bagnères-de-Bigorre, 1953 (unpublished).

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Renormalization

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A theoretical justification for the infinite subtractions, which have to be made in the renormalization of the S matrix, is given along the lines suggested by Gupta and developed by Takeda. It is shown that this is equivalent to working with the renormalized field variables of Dyson, and that the method deals very simply with overlapping divergences and the "wave function" renormalization associated with external lines. It also gives directly Ward's identities and brings out their essential dependence on gauge invariance. The method is applied to free and bound electrons in electrodynamics and all renormalizable meson theories.

In the later sections the new method is related to the original method of Dyson; the Bethe-Salpeter equation is renormalized and closed forms are derived for the renormalization constants.

INTRODUCTION

THE general proof of the renormalization of the charge expansion of the S matrix of interacting fields[1] falls into three distinct parts. Firstly the number of types of infinity (primitive divergents) in the theory is determined. (If this number is finite the theory is renormalizable.) The second step is to define a subtraction procedure which removes these infinities. The third step is to provide a theoretical justification for these subtractions. A general outline of this proof, applied to electrodynamics and various meson theories, has already been given by us.[2] The purpose of the present review is to assume the results of the first two parts of the proof for any renormalizable theory, and to give, in detail, a treatment of the third part, which has previously presented the greatest difficulty. The central idea is to treat all divergences of the theory by means of infinite counter terms, as has always been done for the mass renormalization. This was first suggested by

[1] F. J. Dyson, Phys. Rev. 75, 1736 (1949).

[2] P. T. Matthews and Abdus Salam, Revs. Modern Phys. 23, 311 (1951).

Gupta[3] and the idea was developed along the lines presented here by Takeda.[4] The derivation of the required counter terms was given by Dyson.[5]

The great advantage of the present method is that it reduces to trivialities the two main problems of Dyson's original approach,[1] concerning the treatment of overlapping divergencies and of external lines.

This program occupies the first five sections. In Secs. VI and VII the present approach is related to the point of view adopted originally by Dyson[1] and followed in our general review.[2]

In the final sections the theory of the renormalization of the Bethe-Salpeter equation is given and closed forms are derived for the renormalization constants.

We would like to stress that this paper is intended mainly as a review and contributes little that is essentially new to the general theory. We feel that the subject matter is sufficiently important to justify its repetition in a complete and relatively simple form.

I. ELECTRODYNAMICS

The usual treatment starts with the Lagrangian:

$$\mathbf{L}=\mathbf{L}_0+\mathbf{L}_I, \tag{1.1}$$

$$\mathbf{L}_0(x)=-\tfrac{1}{4}\mathbf{F}_{\mu\nu}(x)\mathbf{F}_{\mu\nu}(x)-\tfrac{1}{2}[\nabla_\mu\mathbf{A}_\mu(x)]^2 -\boldsymbol{\psi}^*(x)[\gamma_\mu\nabla_\mu+\kappa]\boldsymbol{\psi}(x), \tag{1.2}$$

$$\mathbf{L}_I(x)=-ie\mathbf{A}_\mu(x)\boldsymbol{\psi}^*(x)\gamma_\mu\boldsymbol{\psi}(x)+\delta\kappa\boldsymbol{\psi}^*(x)\boldsymbol{\psi}(x). \tag{1.3}$$

Here κ is the observed mass and is related to the "bare" (or mechanical) mass κ_0 by

$$\kappa=\kappa_0+\delta\kappa. \tag{1.4}$$

However, e is the "bare" charge.[6] The Hamiltonian corresponding to this Lagrangian is

$$\mathbf{H}=\mathbf{H}_0+\mathbf{H}_I.$$

If one transform to the interaction representation (IR) by applying the transformation $U=\exp[i\mathbf{H}_0(t-t')]$ to the Schrödinger representation, defined at some fixed time t', the new equation of motion is

$$i\frac{\partial}{\partial t}\Psi\rangle=\int H_I(\psi^*,\psi,A,e)d^3x\Psi\rangle, \tag{1.5}$$

where ψ^*, ψ, and A are interaction representation operators which annihilate and create bare particles with the observed mass. If one uses the subtraction procedure defined by Dyson, as extended by Salam to deal with overlapping divergences,[6a] one obtains finite matrix elements for the S matrix defined by this equation. The finite part of the theory is defined in such a way that it is both Lorentz and gauge invariant.

It is necessary finally to justify the apparently arbitrary dropping of the divergent terms. This is the part of the proof which will be given here in detail.

The subtraction procedure is defined in terms of infinite constants[7] $A(e)$ and $B(e)$, $C(e)$, and $L(e)$, the sum of the true divergences[6a] from electron, and photon, self-energy parts and vertex parts, respectively. We now take the essential step in the proof.[4] Define renormalized Heisenberg variables and observed charge by the equations

$$\boldsymbol{\psi}=Z_2^{\frac{1}{2}}\boldsymbol{\psi}_1,\quad \boldsymbol{\psi}^*=Z_2^{\frac{1}{2}}\boldsymbol{\psi}_1^*, \tag{1.6}$$

$$\mathbf{A}=Z_3^{\frac{1}{2}}\mathbf{A}_1,$$

$$e=Z_1Z_2^{-1}Z_3^{-\frac{1}{2}}e_1, \tag{1.7}$$

where

$$Z_1=1-L(e_1),\quad Z_2=1+B(e_1),\quad Z_3=1+C(e_1). \tag{1.8}$$

$\boldsymbol{\psi}_1$, $\boldsymbol{\psi}_1^*$, and $\mathbf{A}_1$ are identical with the renormalized Heisenberg variables which have been discussed at length by Dyson,[5] but for our purposes they can be taken as defined by (1.6), (1.7), and (1.8). Expressing $\mathbf{L}$ in terms of renormalized Heisenberg variables, one obtains

$$\mathbf{L}\equiv\mathbf{L}_1\equiv\mathbf{L}_{10}+\mathbf{L}_{1I}, \tag{1.9}$$

where

$$\mathbf{L}_{10}=\mathbf{L}_0(\boldsymbol{\psi}_1,\boldsymbol{\psi}_1^*,\mathbf{A}_1) \tag{1.10}$$

is the same function of the new variables (provided we redefine the ∇A term which has no physical effect) and (see reference 24)

$$\mathbf{L}_{1I}=-ie_1[1-L(e_1)]\mathbf{A}_{1\mu}\boldsymbol{\psi}_1^*\gamma_\mu\boldsymbol{\psi}_1 -B(e_1)\boldsymbol{\psi}_1^*[\gamma_\mu\nabla_\mu+\kappa]\boldsymbol{\psi}_1+Z_2\delta\kappa\boldsymbol{\psi}_1^*\boldsymbol{\psi}_1 -\tfrac{1}{4}C(e_1)\mathbf{F}_{1\mu\nu}\mathbf{F}_{1\mu\nu}. \tag{1.11}$$

The corresponding Hamiltonian is

$$\mathbf{H}_1=\mathbf{H}_{10}+\mathbf{H}_{1I}.$$

Now transform to the renormalized interaction representation[8] (R.I.R.) by the transformation

$$U_1=\exp[i\mathbf{H}_{10}(t-t')]$$

[3] S. N. Gupta, Proc. Phys. Soc. (London) **A64**, 426 (1951).

[4] G. Takeda, Progr. Theoret. Phys. Japan **7**, 359 (1952).

[5] F. J. Dyson, Phys. Rev. **83**, 608 (1951).

[6] We denote by $\psi^*(\mathbf{\Psi}^*)$ the quantity which appears elsewhere in the literature as Ψ. $\psi^*=\psi\dagger\gamma_4$. [See J. Schwinger, Phys. Rev. **74**, 1439 (1948)].

[6a] Reference 1, Secs. I–VI; Abdus Salam, Phys. Rev. **82**, 217 (1951), Secs. II and III; and Phys. Rev. **84**, 426 (1951).

[7] We use Dyson's notation, except that factors 2π (or $2\pi i$) have been absorbed in A, B, and C.

[8] Some care is required in deriving the Hamiltonian and the commutation relations in the renormalized interaction representation (R.I.R.). Firstly, all velocities $d\boldsymbol{\phi}/dt$ must be eliminated from the Hamiltonian, which must be expressed in terms of the field variables and their canonical momenta, $\boldsymbol{\phi}_1$ and $\boldsymbol{\pi}_1$, say, where

$$\boldsymbol{\pi}_1=\frac{\partial L_1(\boldsymbol{\phi}_1)}{\partial(d\boldsymbol{\phi}_1/dt)}.$$

After transforming to the R.I.R. the interaction Hamiltonian is the same function of the new canonical variables $H_{1I}(\phi_1,\pi_1)$, where

$$\phi_1=U_1^{-1}\boldsymbol{\phi}_{1s}U_1,$$

$$\pi_1=U_1^{-1}\boldsymbol{\pi}_{1s}U_1,$$

where the suffix s denotes the Schrödinger representation. This shows that π_1 and ϕ_1 satisfy the equations of motion determined by H_{10}. Therefore π_1 is determined by the free Lagrangian only, on the Schrödinger representation. The resulting equation of motion for the state vector is

$$i\frac{\partial}{\partial t}\Psi\rangle=\int H_{1I}(\psi_1^*,\psi_1,A_1,e_1)d^3x\Psi\rangle, \qquad (1.12)$$

where H_{1I} is essentially equal to $-L_{1I}$ apart from 'surface-dependent' terms which can safely be neglected since they cancel exactly with similar terms from the P brackets in the S matrix.[9]

The new field variables ψ_1^*, ψ_1, and A_1 are operators which annihilate and create a new type of bare particles which have not only the observed mass but also the observed charge. They will be referred to as renormalized bare particles. We now show that the new Hamiltonian contains just the right counter terms to cancel all divergences, provided $\delta\kappa$ is chosen so that

$$Z_2\delta\kappa=A(e_1), \qquad (1.13)$$

and that it leads directly to the finite results of the subtraction procedure.

It should be noted, firstly, that in this formulation no difficulty arises from self-energy insertions in external lines. If one works with ordinary bare particles in the interaction representation (I.R.) based on (1.3) these give rise to terms of the form [reference 1, Eq. (37)].

$$\begin{aligned}\psi'(p)&=\psi(p)+S_F(p)\Sigma(p)\psi'(p)\\&=\psi(p)+S_F(p)B(e_1)S_F^{-1}(p)\psi(p),\end{aligned} \qquad (1.14)$$

where, to obtain the second equality, the linear divergence of $\Sigma^*(p)$ has to be canceled by $\delta\kappa$ and the rest vanishes because it operates on an external particle. The remaining term is ambiguous because

$$S_F(p)\times[S_F^{-1}(p)\psi(p)]=0, \qquad (1.15)$$

but

$$[S_F(p)S_F^{-1}(p)]\times\psi(p)=\psi(p). \qquad (1.16)$$

The ambiguity is only removed if an adiabatic switching on and off of the charge is introduced to make (1.14) determinate.[10] However, working in the R.I.R. The counter terms are defined so that *all* the radiative corrections to an external line vanish identically. The analog of (1.14) is unambiguously

$$\psi_1'(p)=\psi_1(p). \qquad (1.17)$$

No adiabatic switching is required. The renormalized bare particles entering or leaving a scattering process can be identified with true free particles. All graphs containing insertions in external lines can be ignored.

namely

$$\pi_1=\frac{\partial L_{10}(\phi_1)}{\partial\dot{\phi}_1}.$$

Hence, ψ_1^*, ψ_1, and A_1 satisfy the ordinary free field commutation relations (with no Z factor), and are correctly normalized to annihilate and create particles. The last equation can be used to express H_{1I} in terms of velocities in the R.I.R., to give (1.12).

[9] P. T. Matthews, Phys. Rev. **76**, 684 (1949).

[10] F. J. Dyson, Phys. Rev. **83**, 608 (1951), Sec. X. See particularly the remark following Eq. (178).

Now consider the separation of internal divergences. When no overlaps occur, all graphs can be built up unambiguously by insertions of vertex and self-energy parts into all the vertices and lines of irreducible graphs. Whenever vertex parts or proper self-energy parts are inserted, one may also insert the appropriate new counter term of H_{1I} which, using (1.13) automatically cancels the infinities and carries out correctly Dyson's[1] subtraction procedure.

When overlaps occur, one must use Salam's prescription.[6] This is equivalent to Dyson's when no overlaps occur. Since this case has already been dealt with, Salam's prescription need only be applied explicitly to graphs involving overlaps, that is to say, in electrodynamics graphs.

Consider any self-energy graph—to be referred to as "the original" graph—of any degree of complexity, but built up of vertices at which only the main interaction term, $ieA_\mu\psi^*\gamma_\mu\psi$, is operating. If there exists a subpart of this graph, which is divergent, Salam's prescription is that one must subtract the infinite part (true divergence) of this subgraph, multiplied by the rest of the graph (reduced integral) obtained by shrinking to a single vertex the divergent subpart in the original graph (to be referred to as the "shrunk" vertex). If the graph has m nonoverlapping divergent subparts, with true divergences $D_1,\cdots,D_m$, then the true divergence of all the subparts together is defined to be $(-1)^mD_1D_2\cdots D_m$, and the reduced integral is the graph obtained by shrinking each of the m divergent subparts to a single vertex. A subtraction must be made for all possible ways of splitting the graph into divergent subparts and a reduced integral, including the case in which the subpart is the whole original graph. Finally, all self-energy graphs and their subtractions (up to some given order in the charge) must be summed.

Consider a particular way of splitting the original graph into a divergent subpart and a reduced integral. When the summation over all graphs is made, there occur a set of graphs all of which have the same reduced integral and the same type of subpart (vertex or self-energy), but in the set this part appears in all possible forms. Adding together the subtractions which must be made for this particular splitting of this set of graphs, one gets the reduced integral multiplied by the sum of all the true divergences (again up to the same given order in the charge) of all possible forms of the divergent subpart. But this is exactly the term which will arise from the Hamiltonian H_{1I} of (1.12) if the appropriate counter term is operating at the "shrunk" vertex of this set of graphs. This argument is readily extended to the case when the splitting of the original graph involves two or more nonoverlapping subparts, and leads to a subtraction term with two or more "shrunk" vertices at which the appropriate counter terms of H_{1I} are operating. Summing over all possible graphs given by H_{1I}, one sees that also when overlaps occur the subtraction procedure is automatically carried out by the Hamiltonian H.

The "subtraction procedure" is thus not a mutilation of the original theory by the arbitrary dropping of infinite terms, but a simple reinterpretation brought about by a change of normalization—(infinite) change of scale—of the charge and field variables. This is its justification, and any 'subtractions' which can be interpreted in this way will be referred to as renormalizations.

II. THE BOUND INTERACTION REPRESENTATION

For many problems it is convenient to split the electromagnetic field into the radiation field and an external part, due to external sources, which is assumed to be given classical function. For this purpose we replace A above by $A^{ex}+A$, where A now means the quantized radiation field. Following through exactly the same argument as before one obtains

$$\mathbf{L}_{10}=\mathbf{L}_0(\psi_1{}^*,\psi_1,\mathbf{A}_1{}^{ex}+\mathbf{A}_1) \tag{2.1}$$

and additional terms in the 'interaction.'

$$\mathbf{L}_{1I}{}^{ex}=-ie_1[1-B(e_1)]A_{1\mu}{}^{ex}\psi_1{}^*\gamma_\mu\psi_1 \\ -\tfrac{1}{4}C(e_1)[2F_{1\mu\nu}{}^{ex}\mathbf{F}_{1\mu\nu}+F_{1\mu\nu}{}^{ex}F_{1\mu\nu}{}^{ex}]. \tag{2.2}$$

These will be referred to as the external interaction term plus three external counter terms. If the external field is weak, so that an expansion in terms of it is suitable, the renormalization argument goes through just as before. The three external counter terms being just those required to cancel the infinities, which involve the external interaction term. The last one makes finite the vacuum-to-vacuum expectation value of the S matrix which appears as an essential factor in the matrix element of any process if the external field can create pairs.[11]

Of more interest is the case of a strong external field, the outstanding example of which is of course the theory of the Lamb shift in which the electron under consideration is bound in the potential of the nucleus. For these problems we include the external interaction term in H_{10}, and put only the external counter terms in H_{1I}. With this splitting of the Hamiltonian one can pass to the renormalized bound interaction representation (R.B.I.R.). This is similar to that introduced by Furry[12] in that the electron field operators annihilate and create electrons in bound states and the electron propagator $S_F{}^{ex}$ is now the Feynman sum over bound states. It differs from Furry's bound interaction representation in that it contains counter terms with constants defined by the free I.R. Now any divergent graph (self-energy or vertex parts) in the R.B.I.R. will involve $S_F{}^{ex}(p)$ at least once. This can be replaced by the *exact* expression,

$$S_F{}^{ex}(p)=S_F(p)+S_F(p)\gamma A^{ex}(p)S_F(p) \\ +S_F(p)\gamma A^{ex}S_F{}^{ex}(p)\gamma A^{ex}S_F(p). \tag{2.3}$$

In this way any divergent expression involving $S_F{}^{ex}$ may be expressed as a divergent expression involving S_F plus a finite expression involving $S_F{}^{ex}$ [the terms in $S_F{}^{ex}$ are finite since at least two extra external lines have been introduced, corresponding to the two factors A^{ex} in the last term of (2.3)]. But since the first two terms in the right-hand side of (2.3) are the beginning of the expansion of $S_F{}^{ex}$ in terms of the external field, the internal lines of all the divergent parts obtained in this way, are identical with those obtained in the renormalized free interaction representation. Since the infinities were canceled there by the counter terms, they will be canceled here too by the same counter terms to any order in the charge. Thus the R.B.I.R. gives directly finite results for any cross section or energy shift. This is the theoretical basis of the subtraction procedure adopted by Baranger[13] and by Kroll and Pollock.[14]

III. WARD'S IDENTITY

It has been shown above that a theory with divergences of the type which appear in electrodynamics can be renormalized even if no relation exists between the 'true divergences' of self-energy and vertex parts. However, the infinite constants have in fact been defined in such a way that the finite S-matrix elements given by the subtraction procedure are gauge invariant. This implies that if these finite elements can be derived from a new Lagrangian, this new Lagrangian must also be gauge invariant. Therefore the terms involving ψ_1 and either A_1 or ∇ in L_1 must combine to give an expression involving only

$$\nabla+ie_1A_1. \tag{3.1}$$

Since the original Lagrangian was gauge invariant, this is equivalent to the condition

$$e_1A_1=eA. \tag{3.2}$$

It follows that

$$Z_1=Z_2 \tag{3.3}$$

and hence that

$$L(e_1)=-B(e_1). \tag{3.4}$$

This is Ward's identity.[15] The above derivation brings out very clearly its direct dependence on the gauge invarance of the theory. This argument is due to Takeda.[4]

IV. MESON INTERACTIONS

It is clear that exactly the same argument can be applied to any renormalizable meson-nucleon interaction. Consider, for example, the pseudoscalar inter-

[11] R. P. Feynman, Phys. Rev. **76**, 749 (1949). Abdus Salam, and P. T. Matthews, Phys. Rev. **90**, 690 (1953).
[12] W. H. Furry, Phys. Rev. **81**, 115 (1951).
[13] M. Baranger, Phys. Rev. **84**, 866 (1951).
[14] N. M. Kroll and F. Pollock, Phys. Rev. **86**, 876 (1952).
[15] J. C. Ward, Phys. Rev. **78**, 182 (1950).

action of pseudoscalar mesons. All that is required is to replace the free photon terms in the Lagrangian by

$$\mathbf{L}_m = -\tfrac{1}{2}[(\nabla\phi)^2 + \mu^2\phi^2], \tag{4.1}$$

and

$$\mathbf{L}_I = -ig\phi\psi^*\gamma_s\psi + \tfrac{1}{2}\delta\mu^2\phi^2 + \delta\lambda\phi^4 + \delta\kappa\psi^*\psi, \tag{4.2}$$

where the masses are already renormalized and the necessary contact interaction $\delta\lambda\phi^4$ has been introduced. Changing to renormalized variables as in the previous section, produces a Lagrangian which leads directly to the finite theory.

An even greater simplification is brought about by this approach in the interaction of spin-zero mesons with the electromagnetic field, where the overlapping of divergences can be very complicated. Here L consists of the free meson and photon terms, and

$$\mathbf{L}_I = ie'\mathbf{A}_\mu(\phi\nabla_\mu\phi^* - \nabla_\mu\phi\cdot\phi^*) - e^2\mathbf{A}^2\phi\phi^* + \delta\mu^2\phi\phi^* + \delta\lambda\phi^{*2}\phi^2. \tag{4.3}$$

e is supposed equal to e' (so that L is gauge invariant) but we distinguish, for the moment, between charges in the linear and bilinear terms. The S matrix based on this interaction is infinite, but can be made finite by subtractions which are defined in terms of infinite constants L, B, C, R given by Salam.[16] The next step is to define renormalized field variables, which introduce the required counter terms into the theory. It is not hard to see that we must take

$$\phi = Z_2^{\frac{1}{2}}\phi_1 \quad \mathbf{A} = Z_3^{\frac{1}{2}}\mathbf{A}_1 \tag{4.4}$$

where

$$\begin{aligned} Z_2 &= 1 + B(e_1), \\ Z_3 &= 1 + C(e_1). \end{aligned} \tag{4.5}$$

This gives the required self-energy counter terms. To cancel these in the main interaction terms and introduce the required factors in L and R, we must define

$$\begin{aligned} e' &= Z_3^{-\frac{1}{2}}Z_1Z_2^{-1}e_1, \\ e^2 &= Z_3^{-1}Z_2^{-1}Z_4e_1^2, \end{aligned} \tag{4.6}$$

where

$$Z_1 = 1 - L(e_1), \quad Z_4 = 1 - R(e_1). \tag{4.7}$$

Then

$$\begin{aligned} L_{1I} = {} & ie_1[1 - L(e_1)]A_{1\mu}(\phi_1\nabla_\mu\phi_1^* - \nabla_\mu\phi_1\cdot\phi_1^*) \\ & - e_1^2[1 - R(e_1)]A_1^2\phi_1\phi_1^* + \delta\mu^2Z_2\phi_1\phi_1^* \\ & + \delta\lambda Z_1^2\phi_1^{*2}\phi_1^2 - \tfrac{1}{4}C(e_1)F_{1\mu\nu}F_{1\mu\nu} \\ & - B(e_1)(\nabla_\mu\phi_1\nabla_\mu\phi_1^* + \mu^2\phi_1\phi_1^*). \end{aligned} \tag{4.8}$$

The subtractions defined by Salam,[6] whether overlapping divergences are involved or not, take place automatically when the S matrix is based on this Lagrangian, which thus provides the theoretical basis of the finite theory.

By this approach we have avoided all the complicated argument required when the Z factors are introduced and justified by the method of Salam[16] (following Dyson), or the ingenious, but far from perspicuous manipulations involved in the proof due to Ward.[17] Also, as above, no ambiguity arises from external lines.

Further, by the argument given in Sec. III, the renormalized Lagrangian is gauge invariant. Therefore

$$eA = e'A = e_1A_1. \tag{4.9}$$

Hence

$$Z_1 = Z_2 = Z_4. \tag{4.10}$$

It is clear that these considerations can be extended immediately to the combined interaction of three fields.

This completes our main purpose of providing, in a simple way, a theoretical justification for the infinite subtractions which occur in the renormalization of a divergent S matrix.

V. ALTERNATIVE FORMULATIONS

We have shown that the Lagrangian (1.9) leads to an S matrix in electrodynamics which is finite, term by term, in the charge expansion. The result of course does not depend on the particular split between the "free" and interaction parts. We may, alternatively, write exactly the same Lagrangian in the form

$$\mathbf{L} \equiv \mathbf{L}\dagger = \mathbf{L}_0\dagger + \mathbf{L}_I\dagger, \tag{5.1}$$

where

$$\mathbf{L}_0\dagger = -\tfrac{1}{4}Z_3\mathbf{F}_{1\mu\nu}\mathbf{F}_{1\mu\nu} - \tfrac{1}{2}Z_3(\nabla_\mu\mathbf{A}_{1\mu})^2 - Z_2\psi^*(\gamma_\mu\nabla_\mu + \kappa)\psi, \tag{5.2}$$

$$\mathbf{L}_I\dagger = -ie_1Z_1\mathbf{A}_{1\mu}\psi_1^*\gamma_\mu\psi_1 + Z_2\delta\kappa\psi_1\psi_1, \tag{5.3}$$

and define a new interaction representation[18] in terms of $L_0\dagger$ in which the variables are denoted by †. In this representation the complete electron and photon propagators are infinite, that is S_{F1}', not S_F', is generated by making all possible self-energy insertions in a simple line. However, the simple propagator contains infinities coming from the Z factors in $L_0\dagger$. Thus $S_F\dagger(x) = Z_2^{-1}S_F(x)$, $D_F\dagger(x) = Z_3^{-1}D_F(x)$. Again the Z factors in $L_0\dagger$ gives rise to Z factors in the commutation relations of $\psi_1^*\dagger$, $\psi_1\dagger$, and $A_1\dagger$ so they are not correctly normalized for annihilators and creators of bare particles. To avoid this and to get graphs with simple propagators S_F rather than $S_F\dagger$ we express the interaction in terms of $Z_3^{\frac{1}{2}}A_1\dagger$, $Z_2^{\frac{1}{2}}\psi_1^*\dagger$ and $Z_3^{\frac{1}{2}}\psi_1\dagger$. Or equivalently we continue to work with unrenormalized field variables and introduce only the charge renormalization (1.7). This gives the original Lagrangian (1.1) but with the interaction expressed in terms of the renormalized charge

$$L_I = -ie_1Z_3^{-\frac{1}{2}}Z_1Z_2^{-1}A_\mu\psi^*\gamma_\mu\psi + \delta\kappa\psi^*\psi, \tag{5.4}$$

which again must lead to a finite S matrix when expressed as a power series in e_1. This is the formulation

[16] Abdus Salam, Phys. Rev. **86**, 731 (1952).

[17] J. C. Ward, Phys. Rev. **84**, 897 (1951).

[18] S. Kamefuchi and H. Umezawa, Progr. Theoret. Phys. Japan **7**, 399 (1952).

given by Dyson,[1] but established quite differently by him.

VI. DYSON'S RELATIONS

For completeness we now derive from our definitions, the relations by which the Z factors were defined by Dyson.[1]

By definition

$$-\tfrac{1}{2}S_F(y-y^1)=\langle T(\psi(y),\psi^*(y^1))\rangle_0, \quad (6.1)$$

where T is the chronological product as defined by Wick.[19] It follows immediately from the graphical definition of S_F', as the electron propagator modified by all possible self-energy insertions, that

$$-\tfrac{1}{2}S_F'(y-y^1)=\langle T(S,\psi(y),\psi^*(y^1))\rangle_0/\langle S\rangle_0, \quad (6.2)$$

where $\langle\cdots\rangle_0$ is the bare vacuum expectation value and

$$T(S,\psi(y),\psi^*(y^1)) =\sum_n \frac{(-i)^n}{n!}\int\cdots\int dz_1\cdots dz_n T(H(z_1)\cdots \times H(z_n),\psi(y),\psi^*(y^1)). \quad (6.3)$$

By the argument given by Low and Gell-Mann[20] and using their definition of the true vacuum $\Psi_0\rangle$, it follows that

$$-\tfrac{1}{2}S_F'(y-y^1)=\langle\Psi_0 T(\psi(y),\psi^*(y^1))\Psi_0\rangle. \quad (6.4)$$

By an exactly similar argument, only replacing all operators by the corresponding renormalized operators, the finite propagator is

$$-\tfrac{1}{2}S_{F1}'(y-y^1)=\langle\Psi_0 T(\psi_1(y),\psi_1^*(y^1))\Psi_0\rangle. \quad (6.5)$$

Using the definitions of ψ_1 and ψ_1^*, it follows that

$$S_F'(e)=Z_2S_{F1}'(e_1). \quad (6.6)$$

Similarly,

$$D_F'(e)=Z_1D_{F1}'(e_1). \quad (6.7)$$

Let $\Lambda_1(e_1)$ be the sum over all vertex parts obtained from H_{1I} in the R.I.R. Consider any vertex part which does not contain any self-energy part. The effect of all self-energy insertions, including their counter terms, is given by writing $S_{F1}'(e_1)$ and $D_{F1}'(e_1)$ for the lines. The vertex counter term is included by multiplying by $1-L(e_1)(=Z_1)$ at each vertex. If the part has $2n+1$ vertices, there are $2n$ electron lines and n photon lines. Using relations (1.7), (6.6), and (6.7), and summing over all such vertex parts, gives

$$Z_1^{-1}\Lambda_1(e_1)=\Lambda(e), \quad (6.8)$$

where $\Lambda(e)$ is the sum over all vertex parts given by H_I in the I.R. Now

$$\begin{aligned}\Gamma(e)&=\gamma+\Lambda(e)\\ &=Z_1^{-1}[Z_1\gamma+\Lambda_1(e_1)]\\ &=Z_1^{-1}\Gamma_1(e_1).\end{aligned} \quad (6.9)$$

(6.7), (6.8), and (6.9) are the relations which Dyson[1] uses to *define* the Z factors. It should be noted that they are not required for the general proof of renormalization as presented here.

VII. THE BETHE-SALPETER EQUATION

The derivation of the Bethe-Salpeter[21] equation for pseudoscalar coupling given by Low and Gell-Mann[20] can be repeated step by step using renormalized quantities throughout, to derive the renormalized equation, which contains no infinities.

Alternatively, one can start from the unrenormalized form, which can be written symbolically as[22]

$$K=S_F'S_F'-\int S_F'S_F'GK. \quad (7.1)$$

Define the renormalized two-particle propagator

$$K_1(x_1,x_2,x_3,x_4) =\langle\Psi_0 T(\psi_1(x_1),\psi_1(x_2),\psi_1^*(x_3),\psi_1^*(x_4))\Psi_0\rangle. \quad (7.2)$$

Then, immediately from the definitions of ψ_1, etc.,

$$K=Z_2^2K_1. \quad (7.3)$$

The interaction function G can be derived from the sum of all irreducible graphs with just four external nucleon lines, by replacing the lines and vertices by S_F', Δ_F' and Γ_5. A graph with $2n$ vertices has n meson lines and $2n-2$ nucleon lines. Therefore, by (6.7), (6.8), and (6.9),

$$G(e)=Z_2^{-2}G_1(e_1). \quad (7.4)$$

Substituting into (7.1), all the Z factors cancel and

$$K_1=S_{F1}'S_{F1}'-\int S_F'S_F'G_1K_1,$$

which is the finite equation for the renormalized propagator. Similarly, the renormalized "wave function" is

$$\begin{aligned}\chi_{n1}(x_1,x_2)&=\langle\Psi_0T(\psi_1(x_1),\psi_1^*(x_2))\Psi_n\rangle\\ &=Z_2^{-1}\chi_n(x_1,x_2),\end{aligned}$$

and again, on substitution into the B-S equation,[23] all Z factors cancel, leaving the renormalized equation:

$$\chi_1=-\int S_F'S_F'G_1\chi_1.$$

It is important for this argument that no divergences are produced by the iteration of the equation for K.

VIII. CLOSED FORM OF THE THEORY

In this final section the infinite constants introduced in (1.6)–(1.8) are expressed in terms of renormalized field variables.

[19] G. C. Wick, Phys. Rev. **80**, 268 (1950).
[20] M. Gell-Mann and F. E. Low, Phys. Rev. **84**, 350 (1951).
[21] H. A. Bethe and E. E. Salpeter, Phys. Rev. **84**, 1232 (1951).
[22] Reference 20, Eq. (23).
[23] Reference 20, Eq. (37).

Returning to S_F', we have the general relation:

$$S_F'(y-y^1)=S_F(y-y^1)+\int\int dzdz^1 S_F(y-z)\Sigma(z-z^1)S_F(z^1-y^1). \quad (8.1)$$

Compare this with (6.1) and (6.2). The first term on the right-hand side of (8.1) is given by putting $n=0$. The second term is got by combining $\psi(y)$ with any $\psi^*(z)$ and $\psi^*(y^1)$ with any $\psi(z^1)$ $(z^1\neq z)$ in the T bracket, which can be done in $n(n-1)$ ways in the nth term. There are additional nonvanishing terms in which $\psi(y)$ and $\psi^*(y^1)$ are combined with the $\psi^*(z)$ and $\psi(z)$ of a single $\delta\kappa$ term, which may be done in n ways in the nth term. One thus obtains an expression for $\Sigma(z-z^1)$, namely

$$\Sigma(z-z^1)=\langle T(S,f(z),f^*(z^1))\rangle_0/\langle S\rangle_0-\tfrac{1}{2}i\delta\kappa\delta(z-z^1) \quad (8.2)$$

$$=\tfrac{1}{2}[\langle\Psi_0 T(\mathbf{f}(z),\mathbf{f}^*(z^1))\Psi_0\rangle-i\delta\kappa\delta(z-z^1)], \quad (8.3)$$

where

$$\mathbf{f}=ig\phi\gamma_5\psi-\delta\kappa\psi,\qquad \mathbf{f}^*=ig\phi\psi^*\gamma_5-\delta\kappa\psi^*, \quad (8.4)$$

which, for pseudoscalar interaction, is the analog of the current in the equations of motion for ψ and ψ^*, respectively. To obtain (7.3) we have again used the argument of Low and Gell-Mann.[20]

Alternatively, working with renormalized variables

$$\Sigma_1(z-z^1)=\tfrac{1}{2}[F_1(z-z^1)+iZ_2\delta\kappa-B(e_1)(\gamma_\mu\nabla_\mu+\kappa))\delta(z-z^1)], \quad (8.5)$$

where

$$F_1(z-z^1)=\langle\Psi_0 T(\mathbf{f}_1(z),\mathbf{f}_1^*(z^1))\Psi_0\rangle, \quad (8.6)$$

and

$$\mathbf{f}_1=igZ_1\phi_1\gamma_5\psi_1+B(e_1)(\gamma_\mu\nabla_\mu+\kappa)\psi_1-Z_2\delta\kappa\psi_1. \quad (8.7)$$

Expressing (8.5) in momentum space,

$$\Sigma_1(p)=\tfrac{1}{2}[F_1(p)+iZ_2\delta\kappa+B(e_1)(p-i\kappa)] \quad (8.8)$$

where

$$p=p_\mu\gamma_\mu. \quad (8.9)$$

But the infinite constants B and $\delta\kappa$ have been defined so that

$$\Sigma_1(i\kappa)=0, \quad (8.10)$$

and

$$[\partial\Sigma_1/\partial p]_{i\kappa}=0. \quad (8.11)$$

Therefore

$$Z_2\delta\kappa=-iF_1(i\kappa), \quad (8.12)$$

and

$$B(e_1)=-[\partial F_1/\partial p]_{i\kappa}. \quad (8.13)$$

Similar expressions can be obtained for $Z_3\delta\mu^2$ and $C(e_1)$ by considering Δ_{F1}'.

Again, from a consideration of the graphs, it can be seen that the finite part of all vertex parts is

$$\Lambda_1(x,y,z)=\langle T(S_1,f_1^1(x)f_1^{*1}(y),\times j_1^1(z))\rangle_0/\langle S_1\rangle_0-\gamma L \quad (8.14)$$

where

$$f_1^1=ig_1Z_1\phi_1\gamma_5\psi_1, \quad (8.15)$$

$$j_1^1=Z_1\psi_1^*\gamma_5\psi_1. \quad (8.16)$$

Thus, adding the term from the single vertex,

$$\Gamma_1(x,y,z)=G_1(x,y,z)+\gamma_5(1-L), \quad (8.17)$$

where

$$G_1(x,y,z)=\langle\Psi_0 T(\mathbf{f}_1^1(x),\mathbf{f}_1^{*1}(y),\mathbf{j}_1^1(z))\Psi_0\rangle.$$

In momentum space,

$$\Gamma_1(p,p^1)=G(p,p^1)+\gamma_5(1-L). \quad (8.18)$$

Also, since

$$\Gamma_1(i\kappa,i\kappa)=\gamma_5, \quad (8.19)$$

$$\gamma_5 L=G_1(i\kappa,i\kappa). \quad (8.20)$$

Relations similar to (8.12) and (8.13) were first obtained by Kallen.[24] With the use of these expressions it is possible to state the renormalized theory, without mention of power series expansions. Thus the Lagrangian is (1.9)–(1.11). The infinite constants appearing in the Lagrangian are defined by the implicit relations (8.12), (8.13), (8.20), etc.

We do not wish to suggest that this is necessarily a finite theory independent of charge expansion. Dyson's analysis[25] of the divergences into primitive divergents depends essentially on the charge expansion. If this expansion is not absolutely convergent, there is no reason to suppose that the S-matrix elements, obtained by some method other than a charge expansion, would have divergences restricted in the same way and removable by the same renormalizations.[26]

One of us (A. S.) would like to thank Prof. R. E. Peierls for the hospitality of Birmingham University extended to him during the summer of 1953. We also acknowledge helpful conversations with Dr. J. G. Valatin.

[24] G. Kallen, Helv. Phys. Acta 25, 417 (1952). If Σ_1 is defined by the relation,

$$\langle\Psi_0 T(\psi,\psi^*)\Psi_0\rangle=-\tfrac{1}{2}(S_F+S_F\Sigma_1 S_F),$$

and the infinite constants B and $\delta\kappa$ are defined by (8.10) and (8.11), then Eqs. (8.12) and (8.13) can be derived without the use of power series by methods employed by P. T. Matthews and Abdus Salam, Proc. Roy. Soc. (London) **A221**, 128, Sec. III.

[25] Reference 1, Sec. V.

[26] See, for example, G. Feldman, Proc. Roy. Soc. (London) (to be published).