Intermediate Coupling Method for Meson-Nucleon Scattering*

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It is shown that for the meson-nucleon scattering processes an intermediate-coupling method can be applied which joins smoothly the results obtained from the weak- and strong-coupling limits. The method is illustrated by a detailed study of the charged scalar meson field with fixed nucleon.

I. INTRODUCTION

AN intermediate-coupling approximation that is applicable to the bound states of a nucleon-meson system, i.e., a "physical" nucleon and its possible isobars, has been given by Tomonaga.[1] This method is limited to a nonrelativistic treatment of the nucleon but does afford some insight into the problem when the coupling constant is not sufficiently small or large that either the usual weak- or strong-coupling treatment is trustworthy. Furthermore, in the limit of very small or very large values of the coupling constant the result of the intermediate-coupling treatment agrees with that of the correct weak- and strong-coupling calculation.

This intermediate-coupling method is a simple Hartree-Fock approximation in which the total number of mesons is not limited, but all mesons of the same type are assumed to be only in a finite number of orbital states. These orbital state functions, together with the probability amplitude for finding any number of virtual mesons in the nucleon state, are determined by minimizing the total energy. Calculations so far have been limited to determining the self-energies of the stable state of a nucleon with charged scalar mesons neglecting recoil[1] and obtaining the interaction between a nucleon with neutral scalar mesons including recoil.[2]

The purpose of the present note is to extend Tomonaga's method to the scattering of a meson by a nucleon.[3] The charged scalar meson field with fixed nucleon is chosen to illustrate the essential features. The extension of this method to the pseudoscalar meson with pseudovector coupling will be discussed in a subsequent paper.

For the scattering system of a free meson plus a "physical" nucleon we first assume that we have a wave function in the Fock space representing the "physical" nucleon. This wave function is multiplied by a scattering function of the meson which represents an incident and scattered wave. This product function is then symmetrized with respect to all the mesons, and the best functional form of the scattering function is determined by the usual variational technique. As such, the approximation allows for a distortion of the meson wave but does not allow for a "polarization" of the "physical" nucleon.

In the weak-coupling limit the results for the cross section agree with those of the perturbation method. In the strong-coupling limit the results differ from the rigorous strong-coupling results by numerical factors close to unity (depending slightly upon the source size and incident meson energy).

For intermediate values of the coupling constant the procedure is to calculate (numerically) the Tomonaga wave function for the bound states from which one determines the scattering function and consequently the cross section by solving the variational problem. As the coupling constant varies from small to large values it is found that resonance features may occur. It is interesting to note that in the scattering calculation the effect of self-energy and renormalization of coupling constant appear in a very natural way so that they can be identified and eliminated in an unambiguous manner.

II. BOUND STATE

We consider a charged scalar meson field with the Hamiltonian

$$H=\int \omega_k(\alpha_k^\dagger\alpha_k+\beta_k^\dagger\beta_k)d^3k \\ -g\int u(k)[\tau_+(\alpha_k+\beta_k^\dagger)+\tau_-(\alpha_k^\dagger+\beta_k)]d^3k, \quad (1)$$

where $\alpha_k^\dagger$, α_k, and $\beta_k^\dagger$, β_k are the creation and annihilation operators of the positive and negative mesons of momentum k, and

$$u(k)=(4\pi^2\omega_k)^{-\frac{1}{2}}\int U(r)e^{ik\cdot r}d^3r, \quad (2)$$

where $U(r)$ is the normalized source function. ω_k is the total energy of the meson, $\omega_k=(k^2+\mu^2)^{\frac{1}{2}}$.

In order to introduce our notation and have at hand some results already found by Tomonaga, we shall summarize his method for the bound state.

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[1] S. Tomonaga, Prog. Theoret. Phys. 2, 6 (1947). F. Harlow and B. Jacobson, Phys. Rev. 93, 333 (1954).

[2] T. D. Lee and D. Pines, Phys. Rev. 92, 883 (1953).

[3] During the completion of this paper, a similar extension of the intermediate-coupling method to include scattering problem has been worked out by S. Tomonaga, Abstract Book, International Conference of Theoretical Physics, Kyoto and Tokyo, 1953 (unpublished).

TABLE I. Various quantities for the ground state of the nucleon, as functions of g.

g^2	E_0/μ	$\mathfrak{N}_+^0$	$\mathfrak{N}_-^0$	$\mathfrak{M}_+^0$	$\mathfrak{M}_-^0$	λ_+^0/μ	λ_-^0/μ
$\ll 1$	$-g^2\int\frac{u^2}{\omega}d^3k$	$g^2\int\frac{u^2}{\omega^2}d^3k$	$O(g^4)$	$g\left(\int\frac{u^2}{\omega^2}d^3k\right)^{\frac{1}{2}}$	$O(g^3)$	0	$\sim M/\mu$
2	-1.75	0.385	0.05	0.465	0.12	0.2	4
6	-5.45	0.77	0.33	0.555	0.32	0.2	1.8
$\gg 1$	$-\frac{g^2}{2}\int\frac{u^2}{\omega}d^3k^{\text{a}}$	$\frac{g^2}{4}\int\frac{u^2}{\omega^2}d^3k$	$\frac{g^2}{4}\int\frac{u^2}{\omega^2}d^3k$	$\frac{g}{4}\left(\int\frac{u^2}{\omega^2}d^3k\right)^{\frac{1}{2}}$	$\frac{g}{4}\left(\int\frac{u^2}{\omega^2}d^3k\right)^{\frac{1}{2}}$	$\frac{1.22^{\text{b}}}{g^2}$	$\frac{10.8^{\text{b}}}{g^2}$

[a] See Eq. (3.38), reference 1, for the complete expression.
[b] See Eq. (3.37), reference 1, for the complete expression.

The values of the various integrals occurring in the table are

$$\int\frac{u^2(k)}{\omega_k}d^3k=1.34;\quad \int\frac{u^2(k)}{\omega_k^2}d^3k=0.410;\quad \int\frac{u^2(k)}{\omega_k^3}d^3k=0.167.$$

A brief derivation of the form $\mathfrak{N}_+^0$ and $\mathfrak{M}_+^0$ is given in Appendix I.

We denote the probability amplitude for finding n positive and m negative mesons with wave numbers $k_1^+\cdots k_n^+$; $k_1^-\cdots k_m^-$ in the ground state of a single nucleon by

$$\psi_{n,m}^0\equiv\langle k_1^+,\cdots k_n^+;k_1^-,\cdots k_m^-|\Psi^0\rangle. \tag{3}$$

For definiteness we suppose Ψ^0 to be a proton state (the superscript 0 indicates a ground state); thus, the difference $n-m$ can only be 0 or 1, depending upon the charge of the "bare" nucleon.

Following Tomonaga we apply the Ritz variational principle,

$$\delta\langle\Psi^0|H-E_0|\Psi^0\rangle=0,$$

with trial functions of the form

$$\begin{aligned}\psi_{n,n}^0&=C_{n,n}^0\prod_{i=1}^{n}f_+^0(k_i^+)\prod_{j=1}^{n}f_-^0(k_j^-),\\ \psi_{n,n-1}^0&=C_{n,n-1}^0\prod_{i=1}^{n}f_+^0(k_i^+)\prod_{j=1}^{n-1}f_-^0(k_j^-),\end{aligned} \tag{4}$$

where $f_+^0(k)$ and $f_-^0(k)$ are normalized but otherwise arbitrary functions to be determined, together with the numerical constants $C_{n,m}^0$, by the application of the Ritz principle. It is easy to carry out the variation with respect to $f_\pm^0$ and show that the best forms of f_+^0 and f_-^0 are

$$f_\pm^0(k)=N_\pm^0u(k)/(\omega_k+\lambda_\pm^0), \tag{5}$$

where λ_+^0 and λ_-^0 are numerical constants depending on the coupling constant and $N_\pm^0$ are normalization constants. The result of carrying out the variation with respect to the constants $C_{n,m}^0$, λ_+^0, and λ_-^0 leads to a set of difference equations[4] in $C_{n,m}^0$, which can be evaluated analytically in the weak- and strong-coupling limits and give *identical* results for the binding energy with the rigorous treatments in these two limits.[5] For the intermediate value of coupling constant these equations can only be solved numerically.

As shall be shown later, the determination of the quantities,

$$\begin{aligned}\mathfrak{N}_+^0&\equiv\sum_{n=1}^{\infty}\sum_{m=n-1}^{n}|C_{n,m}^0|^2n,\\ \mathfrak{N}_-^0&\equiv\sum_{n=1}^{\infty}\sum_{m=n-1}^{n}|C_{n,m}^0|^2m,\\ \mathfrak{M}_+^0&\equiv\sum_{n=0}^{\infty}C_{n+1,n}^0C_{n,n}^0(n+1)^{\frac{1}{2}},\\ \mathfrak{M}_-^0&\equiv\sum_{n=0}^{\infty}C_{n+1,n+1}^0C_{n+1,n}^0(n+1)^{\frac{1}{2}},\end{aligned} \tag{6}$$

and $\lambda_\pm^0$ will be necessary for the evaluation of the meson-nucleon scattering cross section. With these definitions the normalization $N_\pm^0$, may be written as

$$N_\pm^0=g\mathfrak{M}_\pm^0/\mathfrak{N}_\pm^0. \tag{7}$$

The analytical forms of these quantities in the strong- and weak-coupling limits, together with some of their numerical values for the intermediate region, are listed in Table I and shown graphically in Figs. 1

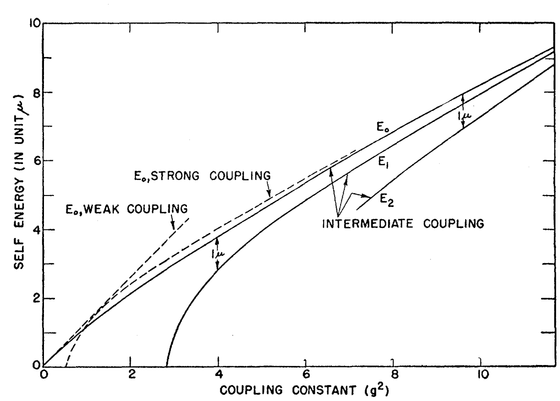

FIG. 1. The self-energy as calculated with the intermediate coupling approximation for the lowest state, E_0; 1st excited state with $m=\pm\frac{3}{2}$, E_1; and 2nd excited state with $m=\pm 5/2$, E_2.

[4] See reference 1, Eq. (3.15).

[5] The weak-coupling result is well known. The strong-coupling result is discussed by: G. Wentzel, Helv. Phys. Acta **13**, 269 (1940); **14**, 633 (1941); R. Serber and S. Dancoff, Phys. Rev. **63**, 143 (1943); S. Tomonaga, Prog. Theoret. Phys. **1**, 109 (1946).

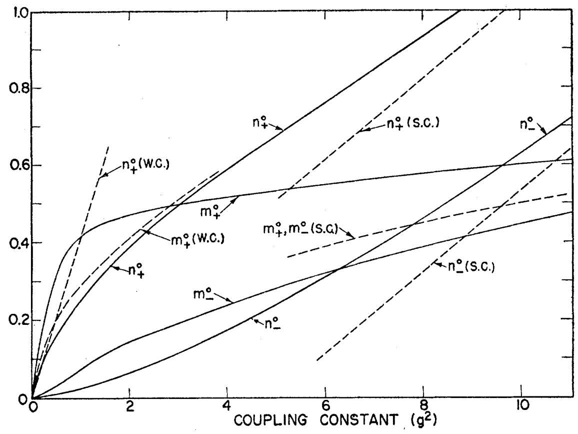

FIG. 2. Various quantities arising in meson-nucleon scattering as determined by the intermediate coupling approximation (solid lines).

and 2. For the numerical calculations the source function $U(r)=[M^2\exp(-M/r)]/4\pi r$, was used where $\mu/M=1/7$.

III. ELASTIC SCATTERING

To illustrate the extension of Tomonaga's method to meson-nucleon scattering, we shall consider first only elastic scattering and restrict ourselves to energies less than 2μ and to values of the coupling constant g such that no *stable* isobars exist. The method used can be easily extended to inelastic scattering and will be discussed in the next section.

The functional form of the state $\mathbf{\Psi}$, representing the scattering process $\pi^++p\to\pi^++p$, is assumed to be[6]:

$$\psi_{n+1,m}\equiv\langle k_1^+,\cdots k_n^+, k_{n+1}^+; k_1^-,\cdots k_m^-|\mathbf{\Psi}\rangle$$
$$=(n+1)^{-\frac{1}{2}}\mathcal{S}_+[\psi_{n,m}{}^0\chi(k_{n+1}{}^+)],\tag{8}$$

where $\mathcal{S}_+$ is the symmetrization operator with respect to positive meson, given by

$$\begin{aligned}\mathcal{S}_+[\psi_{n,m}{}^0\chi(k_{n+1}{}^+)]&\\ =\psi_{n,m}{}^0(k_1^+,\cdots k_n^+;k_1^-,\cdots k_m^-)\chi(k_{n+1}{}^+)&\\ +\psi_{n,m}{}^0(k_1^+,\cdots k_{n-1}^+,k_{n+1}^+;k_1^-,\cdots k_m^-)\chi(k_n{}^+)&\\ +\cdots&\\ +\psi_{n,m}{}^0(k_2^+,\cdots k_{n+1}^+;k_1^-,\cdots k_m^-)\chi(k_1{}^+).&\end{aligned}$$

$\psi_{n,m}{}^0$ is the probability amplitude for the ground state of the nucleon given by (3), and $\chi(k)$ represents the wave function of the free meson. To determine the functional form of $\chi(k)$, we use the variational principle for scattering, which may be *formally* written[7] as

$$J\equiv\langle\mathbf{\Psi}|H-E_0-\omega_0|\mathbf{\Psi}\rangle,$$
$$\delta J/\delta\chi=0,\tag{9}$$

where ω_0 is the energy of the incident meson and E_0 is the self-energy of the proton.

For computational purposes it is convenient to write Eq. (8) in the equivalent form,

$$\mathbf{\Psi}=\int d^3k\chi(k)\alpha_k{}^\dagger\Psi^0.\tag{10}$$

J may then be easily calculated by commuting $\alpha_k{}^\dagger$ in Eq. (10) with the Hamiltonian, H, and noting that Ψ^0 is an eigenfunction of H with eigenvalues E_0. Thus

$$\begin{aligned}2J=&\left\langle\Psi^0\middle|\int d^3k'\chi^*(k')\alpha_{k'}\int d^3k(\omega_k-\omega_0)\chi(k)\alpha_k{}^\dagger\middle|\Psi^0\right\rangle\\ &-g\left\langle\Psi^0\middle|\int d^3k'\chi^*(k')\alpha_{k'}\int d^3k\tau_+u(k)\chi(k)\middle|\Psi^0\right\rangle\\ &+\text{c.c.}\end{aligned}\tag{11}$$

Upon varying J with respect to $\chi(k)$, we find the integral equation

$$(\omega_k-\omega_0)\chi(k)=\int K(k,k')\chi(k')d^3k',\tag{12}$$

where

$$\begin{aligned}2K(k,k')=&g\langle\Psi^0|[\alpha_{k'}u(k)+\alpha_ku(k')]\tau_+|\Psi^0\rangle\\ &-\langle\Psi^0|\alpha_k{}^\dagger(\omega_{k'}-\omega_0)\alpha_{k'}|\Psi^0\rangle\\ &-\langle\Psi^0|\alpha_{k'}{}^\dagger(\omega_k-\omega_0)\alpha_k|\Psi^0\rangle.\end{aligned}\tag{12a}$$

By using the Tomonaga approximate form for Ψ^0, Eq. (4), and the expressions for $\langle\alpha^\dagger\alpha\rangle_{\text{Av}}$ and $\langle\alpha\tau_+\rangle_{\text{Av}}$ given by Eqs. (A.3) and (6), Eq. (12a) becomes

$$\begin{aligned}2K(k,k')=&g\mathfrak{N}_+{}^0[f_+{}^0(k)u(k')+u(k)f_+{}^0(k')]\\ &-\mathfrak{N}_+{}^0[\omega_k+\omega_{k'}-2\omega_0]f_+{}^0(k)f_+{}^0(k').\end{aligned}$$

Finally upon using Eqs. (5) and (7) we find,

$$K(k,k')=\mathfrak{N}_+{}^0(\omega_0+\lambda_+{}^0)f_+{}^0(k)f_+{}^0(k').\tag{13}$$

The solution for the scattering problem of Eq. (12) is

$$\chi(k)=\delta^3(k-k_0)+\frac{\mathfrak{N}_+{}^0(\omega_0+\lambda_+{}^0)f_+{}^0(k)f_+{}^0(k_0)}{(\omega_k-\omega_0-i\epsilon)\left[1-\mathfrak{N}_+{}^0(\omega_0+\lambda_+{}^0)\int d^3k'|f_+{}^0(k')|^2/(\omega_{k'}-\omega_0-i\epsilon)\right]},\tag{14}$$

[6] For π^-+p, an additional term corresponding to the formation of a neutron must be included. See Appendix II for a detailed discussion.

[7] A rigorous formulation of the variational principle is given in Appendix III.

corresponding to a differential cross section given by (see Appendix III):

$$\frac{d\sigma}{d\Omega}=\frac{[4\pi^2\omega_0 u^2(k_0)Rg^2]^2}{[4\pi^2\omega_0 k_0 u^2(k_0)Rg^2]^2+(\omega_0+\lambda_+^0)^2\left[1-Rg^2(\omega_0+\lambda_+^0)\mathrm{P}\int d^3k u^2/(\omega+\lambda_+^0)^2(\omega-\omega_0)\right]^2}, \tag{15}$$

where P indicates that the principal value of the integral is to be taken and $R=(\mathfrak{M}_+^0)^2/\mathfrak{N}_+^0$.

It may be emphasized that the self-energy E_0 has been explicitly eliminated in the kernel $K(k,k')$, Eq. (12), and that in (15), R is always multiplied by g^2 and plays the role of a charge renormalization factor. The result (15) is then completely convergent even for a delta function source.

Weak-Coupling Limit

For a sufficiently small coupling constant, $R=1$ and $\lambda_+^0=0$ so that with a delta function source the cross section becomes

$$d\sigma/d\Omega=g^4/\omega_0^2,$$

which agrees exactly with the usual weak-coupling calculation.

IV. INELASTIC SCATTERING

For sufficiently large value of the coupling constant a *stable* isobaric state of the nucleon with charge 3 is possible and the scattering will be inelastic; that is, the reaction

$$\text{(A)}\quad \pi^+ + p \rightarrow \pi^- + p^{+++}$$

competes with the scattering process

$$\text{(B)}\quad \pi^+ + p \rightarrow \pi^+ + p.$$

Therefore, instead of Eq. (9) we use the trial function[8]

$$\psi_{n+1,m}=(n+1)^{-\frac{1}{2}}\mathcal{S}_+[\psi_{n,m}{}^0\chi_+(k_{n+1}{}^+)] +m^{-\frac{1}{2}}\mathcal{S}_-[\psi_{n+1,m-1}{}^1\chi_-(k_m{}^-)], \tag{16}$$

where $\psi_{n+1,m-1}{}^1$ is the probability amplitude of the triply charged stable isobaric state p^{+++} for finding $n+1$ positive mesons and $m-1$ negative mesons. The procedure for the determination of the isobaric state wave function is identical with that used for the ground state. We shall in this paper use superscript 0 for quantities connected with the ground state and the superscript 1 for the corresponding quantities associated with the isobaric state p^{+++}. $\chi_-(k)$ represents the outgoing wave of a negative meson and $\chi_+(k)$ is defined as before.

Applying the variational principle (9) to the trial function (16) with independent variations of $\chi_+(k)$ and $\chi_-(k)$ we have

$$\begin{aligned}(\omega_k-\omega_0)\chi_+(k)&=\int K_{++}(k,k')\chi_+(k')d^3k' +\int K_{+-}(k,k')\chi_-(k')d^3k',\\ (\omega_k-\omega_1)\chi_-(k)&=\int K_{-+}(k,k')\chi_+(k')d^3k' +\int K_{--}(k,k')\chi_-(k')d^3k',\end{aligned} \tag{17}$$

where ω_0 and ω_1 are the total energy of the free positive meson and the outgoing negative meson, respectively. The kernels are

$$2K_{++}(k,k')=\mathfrak{M}_+^0 g[f_+^0(k)u(k')+u(k)f_+^0(k')] -\mathfrak{N}_+^0(\omega_k+\omega_k{}'-2\omega_0)f_+^0(k)f_+^0(k'),$$

$$2K_{+-}(k,k')=2K_{-+}(k',k) =g\mathcal{O}f_+^1(k)u(k')+g\mathcal{P}u(k)f_-^0(k') -\mathcal{Q}(\omega_k+\omega_{k'}-\omega_0-\omega_1)f_+^1(k)f_-^0(k'),$$

and

$$2K_{--}(k,k')=\mathfrak{M}_+^1 g[f_-^1(k)u(k')+u(k)f_-^1(k')] -\mathfrak{N}_+^1(\omega_k+\omega_{k'}-2\omega_1)f_-^1(k)f_-^1(k'),$$

where $\mathfrak{M}_+^0$, $\mathfrak{N}_+^0$ are defined by Eq. (6) and $\mathfrak{M}_+^1$, $\mathfrak{N}_+^1$ are the corresponding quantities for the state p^{+++}. The constants $\mathcal{O}$, $\mathcal{P}$, $\mathcal{Q}$, are

$$\begin{aligned}\mathcal{O}&\equiv\sum_{n=1}^{\infty}C_{n,n-1}{}^0C_{n+1,n-1}{}^1(n+1)^{\frac{1}{2}}\left[\int f_+^1(k)f_+^0(k)d^3k\right]^n \times\left[\int f_-^1(k)f_-^0(k)d^3k\right]^{n-1},\\ \mathcal{P}&\equiv\sum_{n=1}^{\infty}C_{n+1,n-1}{}^1C_{n+1,n}{}^0 n^{\frac{1}{2}}\left[\int f_+^0(k)f_+^1(k)d^3k\right]^{n+1} \times\left[\int f_-^0(k)f_-^1(k)d^3k\right]^{n-1},\end{aligned} \tag{18}$$

and

$$\mathcal{Q}\equiv\sum_{n=1}^{\infty}\sum_{m=n-1}^{n}C_{n,m}{}^0C_{n+1,m-1}{}^1[(n+1)m]^{\frac{1}{2}} \times\left[\int f_+^1(k)f_+^0(k)d^3k\right]^n\left[\int f_-^1(k)f_-^0(k)d^3k\right]^{m-1},$$

[8] For simplicity we have restricted ourselves to energy less than 2μ. At greater energy, multiple production of mesons is possible which must be taken into account by adding additional outgoing waves in the trial function.

where $C_{n,m}^0$ and $C_{n,m}^1$ are the total probability amplitude for finding n positive mesons and m negative mesons in the ground state and the triply charged isobaric state of the nucleon, respectively.

Upon noticing that the kernels of the integral equation (17) can be written as a sum of product functions each having only one variable, the solution for χ_+ and χ_- must, therefore, be of the form

$$\chi_+(k)=\delta^3(k-k_0)+\alpha_+^0f_+^0(k)+\alpha_+^1f_+^1(k)+[\beta_+^0f_+^0(k)+\beta_+^1f_+^1(k)+\gamma_+u(k)]\times[\omega-\omega_0-i\epsilon]^{-1}, \quad (19)$$

and

$$\chi_-(k)=\alpha_-^0f_-^0(k)+\alpha_-^1f_-^1(k)+[\beta_-^0f_-^0(k)+\beta_-^1f_-^1(k)+\gamma_-u(k)][\omega-\omega_1-i\epsilon]^{-1},$$

where $\alpha_\pm^i$, $\beta_\pm^i$ $(i=0, 1)$, and $\gamma_\pm$ are numerical constants which can be determined by substituting Eq. (19) back into the integral equation (17).

Strong-Coupling Limit

It is of interest to examine the solution when the coupling constant is very large. In this case the energy separation $(\omega_0-\omega_1)$ of the isobaric state with the ground state is of the order g^{-2} and is neglected. Furthermore, the distribution functions of mesons around the nucleon becomes identical for the isobaric and the ground state as $g\rightarrow\infty$. We can then *in this limit*[9] set

$$\mathfrak{N}_+^0=\mathfrak{N}_+^1=\mathcal{Q},$$

$$\mathfrak{M}_+^0=\mathfrak{M}_+^1=\mathcal{O}=\mathcal{P},$$

and

$$f_+^1=f_+^0=f_-^1=f_-^0=(u(k)/\omega)\left(\int d^3ku^2/\omega^2\right)^{-\frac{1}{2}}.$$

Thus the kernels become

$$K_{++}=K_{+-}=K_{-+}=K_{--}=K(k,k'),$$

where K is identical with that in Eq. (12). By forming the linear combinations $\chi_+\pm\chi_-$ we readily find

$$(\omega-\omega_0)[\chi_+(k)-\chi_-(k)]=0, \quad (20)$$

and

$$(\omega-\omega_0)[\chi_+(k)+\chi_-(k)]=2\int K(k,k')[\chi_+(k')+\chi_-(k')]d^3k'.$$

The first equation shows that the difference between $\chi_+(k)$ and $\chi_-(k)$ can only be a $\delta^3(k-k_0)$ so that *in the strong-coupling limit the charge exchange and nonexchange cross sections are equal.* The second equation can now be solved by casting $\chi_+(k)+\chi_-(k)$ into the same form as $\chi(k)$ in Eq. (14). On using the limiting values for $\mathfrak{N}_\pm^0$ and $\mathfrak{M}_\pm^0$ as tabulated in Table I we find the differential cross section $d\sigma_{++}/d\Omega$ and $d\sigma_{+-}/d\Omega$ for the reaction (A) and (B) to be

$$d\sigma_{++}/d\Omega=d\sigma_{+-}/d\Omega=4\pi^4\frac{f^4(k_0)\omega_0^2}{m^{-2}+[4\pi^2f^2(k_0)k_0\omega_0]^2}+O(\mu/M), \quad (21)$$

where M^{-1} is the size of the source function and

$$m^{-1}=\mathrm{P}\int[f^2(k)/(\omega-\omega_0)]d^3k, \quad (22)$$

where P indicates that the principal value of the integral is to be taken. For a δ-function source (i.e., $M\rightarrow\infty$) we get for the total cross section when $\omega_0=\mu$

$$\sigma_{++}=\sigma_{+-}=\pi\mu^{-2}(2^{-1}+\pi^{-1})^{-2}, \quad (23)$$

while the rigorous treatment in the strong-coupling limit[4] yields

$$\sigma_{++}=\sigma_{+-}=\pi\mu^{-2}. \quad (24)$$

Hence, unlike Tomonaga's treatments on the bound states where the variational method leads to exact agreement with both the rigorous calculations for weak- and strong-coupling limits, our calculations for meson scattering give exact agreement only in the weak-coupling limit while in the strong-coupling limit the rigorous treatment differs from our result by the factor[10]

$$(\tfrac{1}{2}+1/\pi)^2$$

at zero energy. At other energy this factor is

$$\left[\left(\frac{1}{2}+\frac{1}{\pi}\right)+\frac{\mu-\omega_0}{2\omega_0}+\frac{(\omega_0^2-\mu^2)^{\frac{1}{2}}}{\pi\omega_0}\ln\frac{(\omega_0+\mu)^{\frac{1}{2}}-(\omega_0-\mu)^{\frac{1}{2}}}{(\omega_0+\mu)^{\frac{1}{2}}+(\omega_0-\mu)^{\frac{1}{2}}}\right]^2+\left[\frac{\omega_0^2-\mu^2}{\omega_0^2}\right],$$

which only varies from about 0.7 to 0.8 as ω_0 increases from μ to 2μ.

V. INTERMEDIATE-COUPLING RANGE

In this section we shall present the results of the numerical calculations in the intermediate-coupling range. Figure 3 shows $k^2(d\sigma/d\Omega)$ as a function of energy for π^++p and π^-+p scattering. It is of interest to notice that even as $g^2=10$ the deviation in cross section from the strong-coupling limit is still substantial. Thus, at least throughout the range $1<g^2<10$, both the weak-

[9] The constants $\lambda_\pm^i$ in the strong-coupling limit are of the order g^{-2}. Thus, their influence on each of the overlapping integrals in (18) is of order g^{-4} which can be neglected in the present calculation. On the other hand, in the calculation of isobaric separation it is necessary to include $\lambda_\pm^i$ since the isobaric separation energy is itself of the order g^{-2}.

[10] This discrepancy presumably is due to the absence of polarization effect in our approximation. The inclusion of such effect, though desirable, tends to make the calculations at intermediate coupling range much more complicated.

and strong-coupling results are in very poor agreement with the intermediate-coupling results. In Fig. 4, $k\cot\delta$ is plotted against energy and shows that the scattering amplitudes for $\pi^+ + p$ and $\pi^- + p$ are very similar to *that of a short-range attractive potential and a short-range repulsive potential, respectively*, while for $\pi^+ + p$, a 90° phase shift is always possible at a suitable value of the energy, provided $g^2 \geq 4.0$. Yet the phase shift for $\pi^- + p$ would always be smaller than 90° except may be at the region of extremely large g values. Figure 5 shows the manner in which the scattering amplitude for zero incident kinetic energy ($\omega_0 = \mu$) changes smoothly from weak-coupling value to the strong-coupling value. It may be noted that for zero-energy $\pi^+ + p$ scattering, a phase shift of 90° will occur at a value of $g^2 = 4.0$. From Fig. 1 one sees that, by using the Tomonaga's technique for bound state, also at $g^2 = 4.0$, the first excited state p^{++} has a zero binding energy. Thus, at least in this case, the method for the bound state agrees with that for the scattering state.

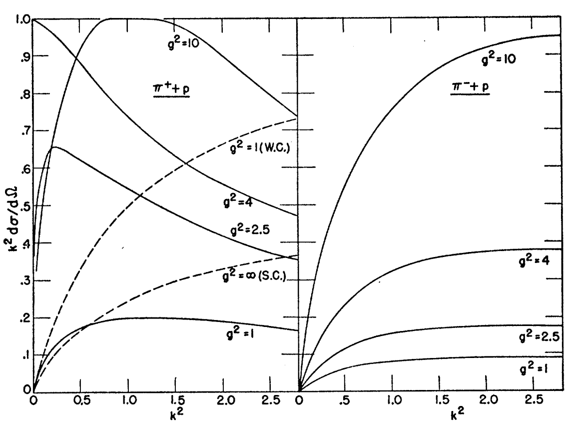

FIG. 3. $k^2 d\sigma/d\Omega$ *versus* k^2. Solid lines refer to intermediate coupling calculations.

While the detailed results are essentially of a numerical nature, some approximate expressions are found useful in understanding the over-all behavior of the cross sections. For $\pi^+ + p$ scattering it is found that with λ_+^0 in (15) taken equal to zero the phase shift δ may be written as[11]

$$\tan\delta = \frac{4\pi^2 (Rg^2) k_0 u^2(k_0)}{1-\Delta},$$

$$\Delta = Rg^2 \omega_0 \mathrm{P} \int [u^2/\omega^2(\omega-\omega_0)] d^3k. \tag{25}$$

Since the integral is in general positive (if $\omega_0 < 2\mu$), it is possible to find one resonance at any energy less than 2μ for some value of g. On the other hand, since the

[11] A similar form for the phase shifts is obtained by G. Chew for pseudoscalar meson theory, using the Dancoff-type wave functions [Phys. Rev. 89, 591 (1952)].

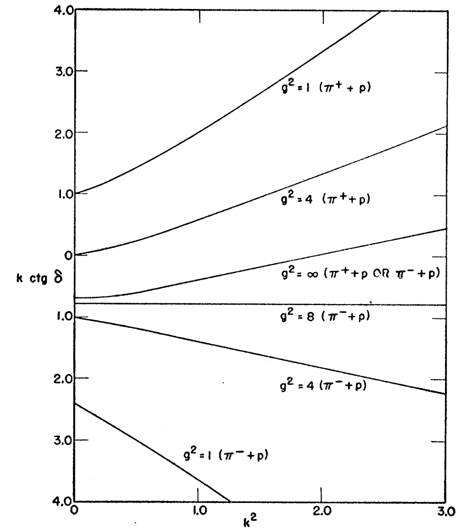

FIG. 4. $k\cot\delta$ *vs* k^2 for various g values.

integral is a decreasing function of ω_0, the phase shift will be less than $\pi/2$ if g is smaller than the value of g required to give a resonance at zero energy, $\omega_0 = \mu$.

Examination of the simplified form of the cross section indicates that essentially two distinct conditions must be satisfied to validate a perturbation calculation. These are (1) that the number of mesons about a nucleon be sufficiently small that the perturbation result agrees with the intermediate coupling result, namely

$$\mathfrak{N}_+^0 \cong g^2 \int [u^2(k)/\omega_k^2] d^3k < 1, \tag{26}$$

and (2) that the damping terms in the denominator be small compared with unity, i.e.,

$$\Delta < 1. \tag{27}$$

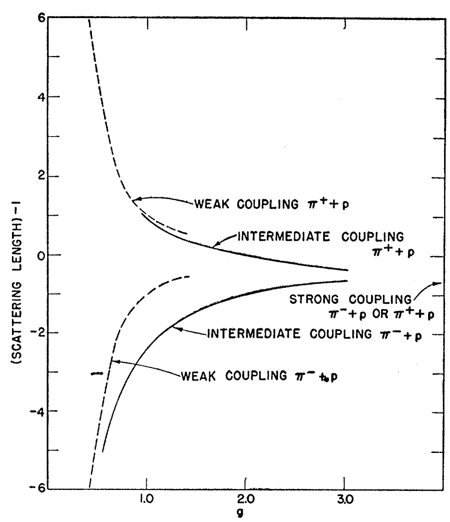

FIG. 5. (Scattering length)$^{-1}$ *vs* g for $\omega_0 = \mu$.

VI. CONCLUSIONS

It has been shown that for the scattering processes as well as the bound states the intermediate-coupling method may be applied, and that this treatment joins all results calculated for the weak- and strong-coupling limits smoothly. For the ground state it is found that $g^2 \lesssim 1$ for a 10 percent accuracy in the self-energy if the weak-coupling method is used and $g^2 \gtrsim 3$ if the strong-coupling method is used. More stringent limits are placed upon g^2 when the same accuracy is desired for more critical quantities such as average number of mesons, isobar separation, scattering, etc. For example, the calculated value of g^2 for a stable isobar of charge two is $g^2 \geq 4.0$ with the intermediate-coupling method and $g^2 \geq 5.8$ with the strong-coupling method. By referring to Figs. 3 and 5, it is seen that for scattering processes $g^2 < 1.0$ and $g^2 \gg 10$ for the applicability of the weak- and strong-coupling methods, respectively.

It has also been shown that $\pi^+ + p$ scattering is similar to scattering by a short-range attractive potential and that $\pi^- + p$ is similar to scattering by a repulsive potential.

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APPENDIX I

To derive the explicit forms of $\mathfrak{N}_\pm{}^0$ and $\mathfrak{M}_\pm{}^0$ we consider an alternative formulation of the Tomonaga method. In this formulation one approximates the annihilation and creation operators of the mesons by

$$\begin{aligned} \alpha_k &= f_+(k)\alpha; \quad \alpha_k{}^\dagger = f_+(k)\alpha^\dagger, \\ \beta_k &= f_-(k)\beta; \quad \beta_k{}^\dagger = f_-(k)\beta^\dagger, \end{aligned} \tag{A.1}$$

where α, β, and $\alpha^\dagger$, $\beta^\dagger$ are annihilation and creation operators independent of k. (In this section we omit the superscript indicating the state of the total charge as all equations are identical in form for any isobaric state.) The Hamiltonian (1) is then reduced to

$$\begin{aligned} H = \omega_+\alpha^\dagger\alpha + \omega_-\beta^\dagger\beta - \tau_+[g_+\alpha + g_-\beta^\dagger] \\ - \tau_-[g_+\alpha^\dagger + g_-\beta], \end{aligned} \tag{A.2}$$

where

$$\omega_\pm = \int \omega_k f_\pm{}^2(k) d^3k,$$

$$g_\pm = g \int u(k) f_\pm(k) d^3k.$$

It can then be shown that

$$\mathfrak{N}_+ = \sum |C_{n,m}|^2 n = \langle \alpha^\dagger\alpha \rangle_{\mathrm{Av}},$$

and

$$\mathfrak{M}_+ = \sum C_{n+1,n} C_{n,n} (n+1)^{\frac{1}{2}} = \langle \alpha\tau_+ \rangle_{\mathrm{Av}}, \tag{A.3}$$

where $\langle\ \rangle_{\mathrm{Av}}$ means the diagonal matrix element. These averages can be obtained explicitly by converting α, β into real canonical conjugate variables, and the problem of determining the eigenstates of H [Eq. (A.2)] reduces to the problem of two coupled harmonic oscillators. In the strong-coupling limit, one may neglect higher orders in g^{-1} and set $f_+(k) = f_-(k)$. The results are listed in Table I.

APPENDIX II

The trial function $\mathbf{\Psi}$ representing the process,

$$\pi^- + p \rightarrow \pi^- + p,$$

differs slightly from that for π^+ scattering, Eq. (10), due to the possibility of finding a neutron with no mesons. The assumed form is now a combination of

$$\mathbf{\Psi} = \int \chi_-(k)\beta_k{}^\dagger \Psi_P d^3k + c\Psi_N, \tag{A.4}$$

where Ψ_P and Ψ_N are the state vectors representing a "physical" proton and a "physical" neutron, respectively. c is a numerical constant. Upon varying

$$\langle \mathbf{\Psi} | H - E_0 - \omega_0 | \mathbf{\Psi} \rangle$$

with respect to $\chi_-(k)$ and c, one finds

$$c = -\mathfrak{R} \int f_+{}^0(k)\chi_-(k) d^3k, \tag{A.5}$$

and

$$(\omega - \omega_0)\chi_-(k) = \int K(k,k')\chi_-(k') d^3k', \tag{A.6}$$

where

$$\mathfrak{R} = \sum_{n=0}^{\infty} \sum_{m=n-1}^{n} C_{n,m}{}^0 C_{m+1,n}{}^0 (m+1)^{\frac{1}{2}} \times \left(\int f_+{}^0 f_-{}^0 d^3k \right)^{n+m}, \tag{A.7}$$

and

$$K(k,k') = \mathfrak{N}_-{}^0(\omega_0 + \lambda_-{}^0) f_-{}^0(k) f_-{}^0(k') - \mathfrak{R}^2 \omega_0 f_+{}^0(k) f_+{}^0(k'). \tag{A.8}$$

Equation (A.6) can then be solved by assuming a form for $\chi_-(k)$ similar to that of Eq. (14). In the weak-coupling limit it yields the rigorous scattering amplitude.

APPENDIX III

In the case of elastic scattering (i.e., $\omega < 2\mu$ and no stable isobar of the nucleon exists), the rigorous Schrödinger wave function for the scattering of a positive meson with incident wave number k_0 by a proton can be written as

$$\psi_{n,m} = n^{-\frac{1}{2}} \mathcal{S}_+ \{ \psi_{n-1,m}{}^0 (k_1{}^+, \cdots, k_{i-1}{}^+, k_{i+1}{}^+, \cdots, k_n{}^+; k_1{}^-, \cdots, k_m{}^-) \times [\delta^3(k_i - k_0) + B_{n,m}{}^i (\omega_k{}^+ - \omega_0 - i\epsilon)^{-1}] \}, \tag{A.8}$$

where $B_{n,m}{}^i$ is a regular function of $k_1{}^+\cdots k_n{}^+$ and $k_1{}^-\cdots k_m{}^-$. The value of $B_{n,m}{}^i$ when $k_i{}^+=k_0$ represents the scattering amplitude of the ith meson; consequently

$$B_{n,m}{}^i|_{k_i{}^+=k_0}=B(k_0)$$

depends on k_0 only. These conditions are physically necessary for any elastic scattering problem and can be readily verified by a direct substitution into the Schrödinger equation. The differential cross section $d\sigma/d\Omega$ and the phase shift η are related to $B(k_0)$ by the relations:

$$\begin{aligned} d\sigma/d\Omega &= (4\pi^2)^2|B(k_0)|^2\omega_0{}^2, \\ B(k_0) &= (e^{2i\eta}-1)/8i\pi^2k_0\omega_0. \end{aligned} \tag{A.9}$$

Following Kohn's[12] variational treatment for scattering problem in momentum space, we find for the correct solution

$$\begin{aligned} J &= \langle \Psi | H-E_0-\omega | \Psi \rangle \\ &= B(k_0)-4i\pi^2k_0\omega_0|B(k_0)|^2 \\ &= \sin 2\eta/8\pi^2k_0\omega_0, \end{aligned} \tag{A.10}$$

and the variation of J in the neighborhood of the correct Ψ is

$$\begin{aligned} \delta J &= \delta B(k_0)+\delta B^*(k_0) \\ &\quad +4i\pi^2k_0\omega_0[-B^*(k_0)\delta B(k_0)+B(k_0)\delta B^*(k_0)] \\ &= (4\pi^2k_0\omega_0)^{-1}(1+\cos 2\eta)\delta\eta. \end{aligned} \tag{A.11}$$

Equations (A.10) and (A.11) then give the variational calculation for η. It can be easily seen that if one assumes Ψ to be of the form (8) a formal variation with respect to χ [Eq. (9)] gives the same result as (A.10) and (A.11).

[12] W. Kohn, Phys. Rev. 84, 495 (1951).

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Solutions of Heitler's Integral Equation by Iteration Method

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An iterative procedure analogous to Wagner's method for the numerical evaluation of the Fredholm integral equation has been proposed for the solution of Heitler's integral equation on radiation damping. It has been applied to the scattering of mesons by nucleons. The solution for the scattering of π^+ mesons by neutrons agrees with that of Hsueh and Ma, and Goldberger. For the scattering of π^+ mesons by protons, the solution has been taken up to the first approximation; the energy dependence and the angular distribution of the scattering cross section are shown in the accompanying figures.

1. INTRODUCTION

THE consideration of radiation reaction in quantum-mechanical collisional problems is absolutely necessary to remove the divergence difficulties inherent in collisions. The effect of radiation reaction is taken into account by an integral equation which cannot be solved exactly except in a very few cases; the difficulty of solving such an integral equation is due to the complicated nature of the kernel.

In view of this difficulty we are led to consider an approximate method for the solution of the integral equation. One approximate method that is often used for the treatment of scattering processes is the variational principle. But a serious objection to the variational techniques is that no definite mathematical statement can be made as to the error involved in the solution. On the other hand, the ordinary iterative procedure (Neumann sequence) is very slowly convergent and it suffers from the difficulty that the higher-order terms in the integral equation for the scattering process are very involved for computational purposes and in most cases the resulting series cannot even be summed. Hence there arises the necessity of considering a modified sequence for the iterational procedure so that it may converge rapidly and that only a few iterations should suffice.

The semivariational technique of Hsüeh and Ma[1] has been applied to the scattering of neutrons by protons[2] and mesons by nucleons.[1,3] Their solution shows the influence of radiation reaction only in the energy dependence of the total cross section but it fails to give any effect on the angular distribution of the scattered particle.

Another improved type of variational technique has been formulated by Goldberger.[4] The superiority of this method over that of Hsüeh and Ma has been shown by the fact that the solution obtained by Goldberger's variational procedure for the scattering of negative mesons by protons agrees with the exact solution of the

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[2] D. Basu, Proc. Roy. Irish Acad. 53, 31 (1950); D. Basu, Indian J. Phys. 25, 246 (1951).
[3] S. N. Biswas, Indian J. Phys. 26, 617 (1952).
[4] M. L. Goldberger, Phys. Rev. 84, 99 (1951).