Method of Approximation for the Meson-Nucleon Problem when the Interaction is Fixed and Extended*

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It is proposed that a consistent and satisfactory method for evaluating the fixed extended source meson problem is: (a) first to renormalize the calculation by the method of the preceding paper; (b) to evaluate the fundamental quantities L_r and Σ_r by the usual weak coupling method; (c) also to use standard perturbation theory for the irreducible diagrams except when poles occur in integrations over intermediate states. Sums of diagrams in which repeated poles occur should be evaluated exactly.

The proposed method is similar to but not identical with the Tamm-Dancoff approximation. The new method is applied to the pion-nucleon scattering problem.

I. INTRODUCTION

In the preceding paper, hereafter to be referred to as I, it has been shown that explicit mass and charge renormalization may be performed in advance even for the static limit of a meson theory, that is, for a theory with a fixed source. The purpose of the present paper is to outline a practical method of approximate calculation especially suitable to the solution of the renormalized fixed source problem.

It must be realized, first of all, that the static approximation can make sense only for energies substantially smaller than the nucleon rest energy, which is $\sim$1 Bev. Not only must the initial and final energies be small, however, but also the energies of all intermediate pions. That means that the Fourier transform of the source function, which is the cut-off factor in momentum space, must restrict pion energies to values well below 1 Bev. Under these conditions the Tamm-Dancoff[1] type of approach becomes promising. For example, Blair and Chew[2] have shown explicitly that after renormalization the dominant fourth-order contributions to pion-nucleon scattering are those whose intermediate states all involve less than three pions. This is only true for low values of the momentum cutoff, but fortunately a low cutoff seems required not only by the nonrelativistic nature of the theory but also apparently by experiment.[3]

A study of the results of Blair and Chew[2] suggests that, with the required large source and with charge renormalization, the effective coupling between pions and nucleons is by no means strong. In fact the dimensionless number which characterizes the rate of convergence of the perturbation expansion is $\sim$0.2. Why, then, are higher-order terms so important in the pion-nucleon scattering problem? The answer lies in those special intermediate states which can have an energy very close to the initial energy and whose magnitude consequently is anomalously large. Except for these states a perturbation expansion would converge fairly well. The validity of the above argument is substantiated by the calculations of Brueckner and Watson[4] and of Henley and Ruderman[5] on the nuclear force problem and by Friedman[6] on the nucleon magnetic moment problem.

Therefore we shall now propose a new criterion of approximation, similar to but not identical with the Tamm-Dancoff approximation.

(a) After renormalization, ordinary perturbation theory (i.e., an expansion in powers of the coupling constant) is to be applied, except when intermediate states occur which may have an energy coincident with the initial energy and which therefore give rise to poles in the integrations over intermediate states.

(b) For each pole the order of the corresponding diagram must be considered as *lowered* by one power of f^2. Thus, for example, the diagrams of Fig. 1 which contribute to pion-nucleon scattering, are all to be considered as of the same order in this sense, namely, of order f^2. (By chance, in this theory, the actual numerical order of magnitude of the "resonance" denominators, relative to "ordinary" denominators, turns out to be approximately equivalent to a factor f^2. Even if the resonance denominators were much smaller, however, our criterion would still make sense because we never treat any resonant state as weak.) For low-

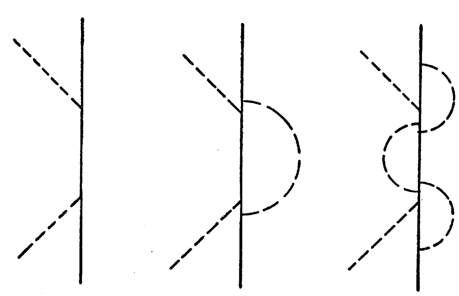

FIG. 1. Sample diagrams contributing to pion-nucleon scattering, all of which are of order f^2 in the sense of this paper.

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[1] S. M. Dancoff, Phys. Rev. **78**, 382 (1950); I. Tamm, J. Phys. U.S.S.R. **9**, 449 (1945).
[2] J. S. Blair and G. F. Chew, Phys. Rev. **90**, 1065 (1953).
[3] J. S. Blair and G. F. Chew, Ann. Rev. Nuc. Sci. **2**, 163 (1953); G. F. Chew, Phys. Rev. **89**, 591 (1953).
[4] K. Brueckner and K. Watson, Phys. Rev. **92**, 1023 (1953).
[5] E. Henley and M. Ruderman, Phys. Rev. **92**, 1036 (1953).
[6] M. H. Friedman (private communication).

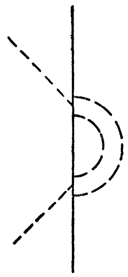


FIG. 2. A typical pion-nucleon scattering diagram which illustrates the difference between the Tamm-Dancoff approximation and that proposed here.

energy pion-nucleon scattering the criterion can be better stated as follows: In first approximation, only those virtual pions are considered which *at some time are alone* in the field. Pions which are never alone may be treated by a perturbation expansion. This criterion differs from that of Tamm and Dancoff who in lowest order would keep all diagrams in which no intermediate state involves more than two pions. For example, the diagram in Fig. 2 would be kept by the lowest-order Tamm-Dancoff approximation but omitted by that proposed here. After renormalization, we consider this diagram of order f^4 and therefore small compared to those of Fig. 1.

In the nuclear force problem, we would regard the diagrams Fig. 3a, b, c, as all being of the order f^2 but consider Fig. 3d, e, f, as being of order f^4.

Charge renormalization is enormously important in justifying our approximation. For example, Fig. 2 without charge renormalization would be much too large to neglect. The point is that having made charge renormalization, we have actually taken most of Fig. 2 into account and neglected only a part which is rather small.

Many diagrams do not involve poles and in such cases it is sufficient to expand the modified vertex and propagation functions in powers of the coupling constant. Let us begin, therefore, by examining the lowest-order modifications of these functions.

II. THE f^2 MODIFICATION OF THE NUCLEON PROPAGATION FUNCTION

The starting point for calculating the modified nucleon propagation function by the method of I is the quantity $\Gamma_0(E)$, the recipe for which is given by formula (I-29). (Since we shall always in this paper be dealing with the nucleon energy *minus its rest energy* the "prime" used in I will be dropped from E. Similarly we drop the subscript r from the coupling constant f since the unrenormalized coupling constant never appears.) To order f^2, we have only one fundamental diagram contributing to $\Lambda_{0r}(E)$, that shown in Fig. 2a of reference I. The value of $\Lambda_{0r}(E)$ to this order is therefore

$$\Lambda_{0r}(E)=f^2\sum_j V_j\frac{1}{E-\omega_j}\frac{1}{E-\omega_j}V_j^*, \tag{1}$$

since we replace L_r and S_r' by their zeroth-order values. From (I-29) we then get

$$\Gamma_{0r}(E)=1+f^2\sum_j V_jV_j^*\left[\frac{1}{(E-\omega_j)^2}-\frac{1}{\omega_j^2}\right], \tag{2}$$

which by (I-16) leads to

$$1/S_r'(E)=E-f^2\sum_j V_jV_j^*\left[\frac{1}{E-\omega_j}+\frac{1}{\omega_j}+\frac{E}{\omega_j^2}\right] \tag{3}$$

$$=E[1+f^2\Delta(E)],$$

where

$$\Delta(E)=\frac{3}{\pi}\int_\mu^\infty d\omega\frac{k^3}{\mu^2\omega^2}\frac{E}{\omega-E}v^2(k). \tag{4}$$

The function $\Delta(E)$ is plotted in Fig. 4 for the square cut-off function $v(k)$ with $\omega_{\max}=3.2\mu$, as determined by Chew[3] in fitting the P-wave pion-nucleon scattering. In evaluating $\Delta(E)$ in the "resonance" region, the principal value of the integral has been taken. This is correct if we are interested eventually in the reactance matrix. It is seen that only for $\mu<E<\omega_{\max}$ does the magnitude of $\Delta(E)$ become substantially greater than unity. (We may disregard the region for $E>\omega_{\max}$ because the theory should not be applied to external energies close to or greater than $\omega_{\max}$ and the internal nucleon energy can never be greater than the total external energy.) This corresponds to the fact that only in this region can the virtual pion "resonate" with the energy E. It seems, therefore, with $f^2\sim 0.2$ as determined by Chew,[3] that for E not in this region a weak coupling expansion is adequate.

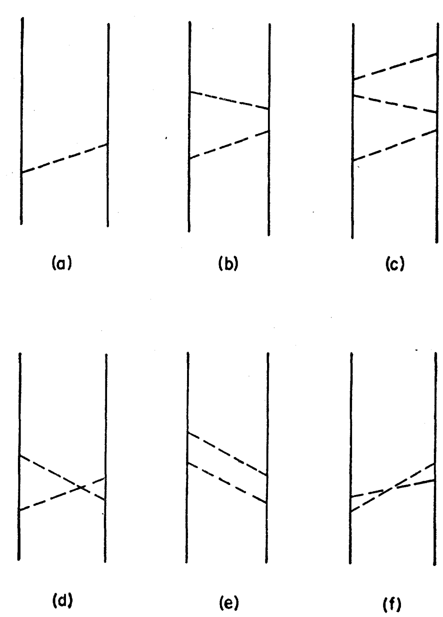


FIG. 3. Sample diagrams contributing in second and fourth order to the nuclear force problem.

III. THE f^2 MODIFICATION OF THE VERTEX FUNCTION

The lowest-order correction to the vertex function is obtained from Eq. (I-30) in the same manner as we obtained Γ_{0r} from (I-29). The difference is that the failure of the external vertex operator to commute with the internal vertex operators leads to a numerical factor of 1/9,

$$L_r(E_2,E_1)=1+(f^2/9)\sum_j V_jV_j^*\left[\frac{1}{E_2-\omega_j}\frac{1}{E_1-\omega_j}-\frac{1}{\omega_j^2}\right]$$

$$=1+(f^2/9)\Delta(E_2,E_1), \qquad (5)$$

where

$$\Delta(E_2,E_1)=\frac{3}{\pi}\int_\mu^\infty d\omega\frac{k^3}{\mu^2\omega^2}\frac{(E_2+E_1)\omega-E_2E_1}{(\omega-E_2)(\omega-E_1)}v^2(k). \qquad (6)$$

It is easy to verify that $\Delta(E_2,0)=\Delta(E_2)$ and $\Delta(0,E_1)=\Delta(E_1)$, so that for the correction to the first or last vertex in a diagram, the discussion given above for the propagation function modification is certainly valid here. The discussion for internal vertices is more complicated but leads to the same conclusions. The quantity $\Delta(E_2,E_1)$ can be large only when one or the other of the energies lies in the range between μ and $\omega_{\max}$. In addition, however, the numerical factor 1/9 in formula (5) should not be overlooked. Whenever pion lines cross, as they do in the vertex modification, the effective

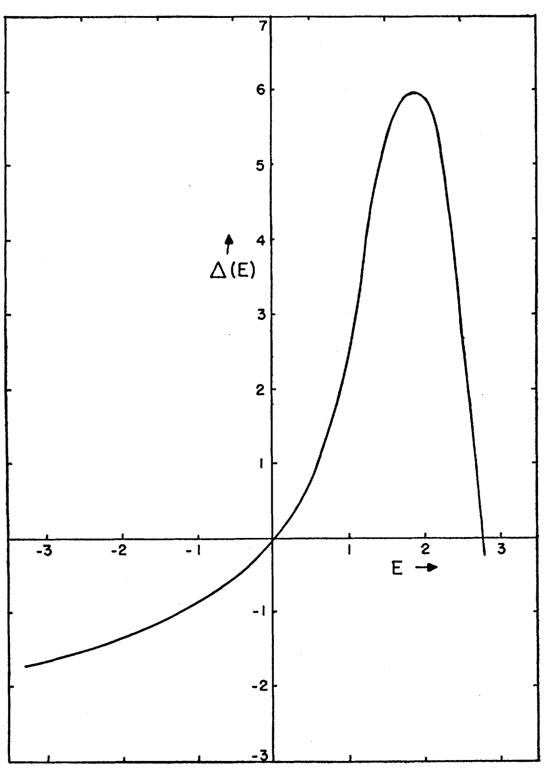


FIG. 4. The function $\Delta(E)$, defined in Eq. (4), for a "square" cutoff with $\omega_{\max}=3.2\mu$.

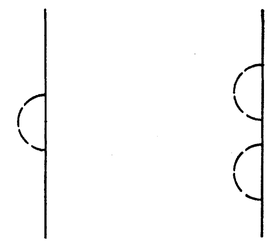
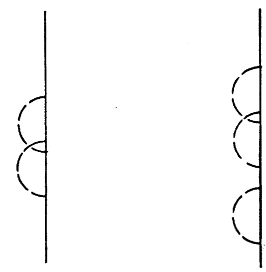


FIG. 5. Sample diagrams contributing to S_r'.



coupling strength is reduced because the nucleon may not be able to reabsorb every pion which it emits. For example, suppose a proton emits a positive pion, becoming a neutron. It can obviously reabsorb the original pion if nothing else happens; but, if in the meantime it emits a negative pion or absorbs a positive, it can no longer reabsorb the original positive pion. A similar situation prevails with respect to angular momentum, so that actually only 1/9 as many pions can cross a vertex as can contribute to propagation modification in lowest order. This extra factor of smallness means that we can always treat vertex modifications by perturbation theory, regardless of the energy situation.[7]

IV. PROPAGATION FUNCTION MODIFICATION FOR $\mu<E<\omega_{\max}$

We now turn to the problem of the nucleon propagation modification in the "resonance" region, when $\mu<E<\omega_{\max}$. According to our original criterion we must, in lowest order, take account of the series of diagrams indicated in Fig. 5. This means keeping in $\sum_r$ a series of terms corresponding to the diagrams of Fig. 6.

However, since each new pion line crosses one vertex, we are assured that the coupling constant is effectively $(1/9)f^2$, not f^2, so that we actually need keep only the lowest-order diagram, where there are no crossings. Thus formula (3) is still correct, if no attempt is made to treat $f^2\Delta(E)$ as small. Notice that with $f^2=0.2$, no new singularities are introduced in the range of E for

[7] This numerical factor of 1/9 also means that Z_2/Z_1 is substantially smaller than 1, in contrast to electrodynamics where gauge invariance insures that $Z_2=Z_1$. The consequence that the renormalized coupling constant here is smaller than the original coupling constant is one of the main practical reasons for renormalizing the static approximation.

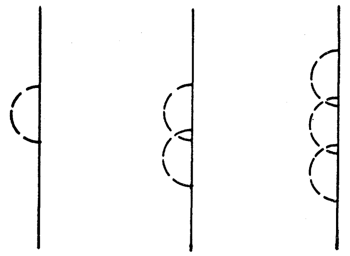


FIG. 6. Sample diagrams contributing to Σ_r.

which the theory may be sensible, i.e., $E<\omega_{\max}$. This is a reassuring feature of the theory and of our method of approximation.

V. APPLICATION TO THE PION-NUCLEON SCATTERING PROBLEM

With a fixed source and linear coupling, pion-nucleon scattering can occur only in P-states; and using the conservation of total angular momentum as well as isotopic spin it is well known that the problem can be expressed in terms of four eigenstates, which have been labeled with the notation, 33, 31, 13, 11. The first number is twice the total isotopic spin and the second is twice the total angular momentum. Now the *bare nucleon* is a 11 state, so that from the conservation laws it follows that diagrams containing at any point a bare nucleon line can contribute only to the 11 scattering. A little thought then suffices to show that by the criteria we have set up one needs to keep *only* the sequence of diagrams shown in Fig. 7 for the 33, 31, and 13 states; no propagation or vertex modifications are needed. If we denote the sum of all such diagrams by $(f|K|0)$, then an integral equation for $(f|K|0)$ may be written down

$$(f|K|0)=(f|U_0|0)+\sum_n (f|U_0|n)\frac{1}{\omega_0-\omega_n}(n|K|0), \quad (7)$$

where

$$(m|U_0|n)=f^2(V_nV_m^*)/(\omega_0-\omega_n-\omega_m).$$

The first step in solving (7) is to use the conservation laws to break the single equation in many variables into four separate equations each depending only on the magnitude of the momentum. The result is as follows:

$$(k_f|K_{0\alpha}|k_0)=(k_f|U_{0\alpha}|k_0)+\frac{1}{2\pi^2}\times\int_0^\infty dkk^2\frac{(k_f|U_{0\alpha}|k)(k|K_{0\alpha}|k_0)}{\omega_0-\omega_k}, \quad (8)$$

where

$$(k_n|U_{0\alpha}|k_m)=C_\alpha 2\pi\frac{f^2}{\mu^2}\frac{k_nk_m}{(\omega_n\omega_m)^{\frac{1}{2}}}\frac{v(k_n)v(k_m)}{\omega_0-\omega_n-\omega_m}, \quad (9)$$

with $C_{33}=4/3$, $C_{31}=-2/3$, $C_{13}=-2/3$, and $C_{11}=1/3$. If the principal value of the integral in (8) is taken, then the relation between all phase shifts, except δ_{11}, and the matrix K_0 is as follows:[8]

$$\tan\delta_\alpha=-(k_0\omega_0/2\pi)(k_0|K_{0\alpha}|k_0). \quad (10)$$

To calculate δ_{11} we have to add more diagrams which involve internal bare nucleon lines, but before proceeding to this problem, let us write down an approximate solution of the integral equation (8).

This solution is based on an application of one of Schwinger's variational principles which has recently been discussed in detail by Altshuler[9] and by Chew.[10] Sufficient conditions for the validity of the approximation appear to be: (1) that no more than one bound state shall be possible; (2) that the potential function U shall be of a single sign and have important components only over a logarithmically small range of momenta. All of these conditions are well satisfied by our potentials $U_{0\alpha}$, although the last would not be if the momentum cutoff were, say, 10μ rather than 3.2μ. The result of using this approximation is

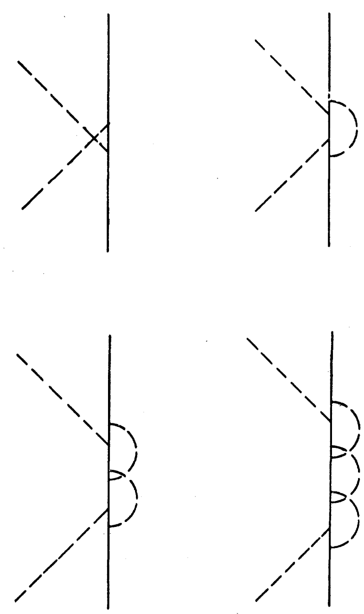


FIG. 7. The series of diagrams summed up in Eq. (7).

$$(k_0|K_{0\alpha}|k_0)=\frac{(k_0|U_{0\alpha}|k_0)}{1-(k_0|U_{0\alpha}|k_0)^{-1}\frac{1}{2\pi^2}\int_0^\infty dkk^2\frac{(k|U_{0\alpha}|k_0)^2}{\omega_0-\omega_k}}, \quad (11)$$

which then yields immediately by the relation (10) the results already published[3] for δ_{33}, δ_{31}, and δ_{13}.

In order to obtain δ_{11} we must add other diagrams to the set given by the above. These diagrams all involve

[8] B. A. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950).
[9] S. Altshuler, Phys. Rev. **89**, 1278 (1953).
[10] G. F. Chew, Phys. Rev. **93**, 341 (1954).

a bare nucleon line at some internal point and thus they are all reducible to the diagram shown in Fig. 8, whose value is given by

$$f^2 V_f{}^* L_r(0,\omega_0) S_r'(\omega_0) V_0 L_r(\omega_0,0). \tag{12}$$

Now it has been shown that our zeroth-order approximation amounts to replacing L_r by 1 and $S_r'(E)$ by formula (5). Thus (12) becomes

$$\frac{f^2 V_f{}^* V_0}{\omega_0[1+f^2\Delta(\omega_0)]}, \tag{13}$$

the projection of which onto the 11 state is

$$2\pi\left(\frac{3f^2}{\mu^2}\right)\frac{k_f k_0}{(\omega_f\omega_0)^{\frac{1}{2}}}\frac{v(k_f)v(k_0)}{\omega_0[1+f^2\Delta(\omega_0)]}. \tag{14}$$

The result already published[3] for δ_{11} is a rough approximation to the sum of (14) and (11), motivated by the fact that (14) carries a numerical factor 9 times as great as that for (11). Note that the relation between the function $\Delta-$ of reference 3 and the function Δ of the present paper is as follows: $f^2\Delta(E) = +(9/2)\Delta-(E)$.

SUMMARY AND CONCLUSION

An approximation procedure has been formulated which allows a relatively simple evaluation of the fixed source meson problem with an accuracy of $\sim$20 percent. The procedure is as follows: (1) Disregard all reducible diagrams. (2) Keep only the lowest-order irreducible diagrams, counting each intermediate state which can "resonate" with the initial state as equivalent to $1/f^2$ in order of magnitude. 3 When the energy of an internal nucleon lies between μ and $\omega_{\max}$, modify the propagation function by formula (5); otherwise do not modify either propagation or vertex functions.

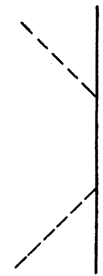


FIG. 8. The main diagram for scattering in the 11 state.

This procedure has here been applied to the pion-nucleon scattering problem and in separate papers will be applied to the problems of photo pion production, anomalous nucleon magnetic moments, neutron-electron interaction, and Compton scattering by nucleons. Higher-order corrections will eventually be calculated by standard perturbation methods.

It should be remembered that in all of these problems the fixed-source model cannot possibly make sense for energies greater than the cut-off energy $\omega_{\max}$ and probably should be restricted to energies well below $\omega_{\max}$. At higher energies it will certainly be necessary to take account of nucleon recoil as well as the existence of heavier mesons. It is also possible that even at low energies an S-wave pion-nucleon interaction cannot consistently be ignored. An S-state interaction can be included in the fixed source theory via a term quadratic in the pion field, but the charge dependence is not determined by general principles, nor is the amplitude relative to the linear term. For reasons of simplicity, therefore, we have not here considered a quadratic interaction term.

Note added in proof.—Recent developments incline the author now to use a larger cutoff ($\omega_{\max}=5.6\ \mu$) and a smaller coupling constant ($f^2=0.058$). The only aspect of this paper to be seriously affected by this change is the accuracy of the variational result (11).