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Nuclear Events at High Energies

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The statistical model, introduced by Fermi, is used to calculate the probabilities for various nuclear events at high energies. A general method for evaluating the phase integrals is proposed and used to find to some extent the consequences of neglect of pion momentum conservation. The effect of pion indistinguishability is studied and tables giving the probabilities for various processes are presented together with the energy spectra of the corresponding final nucleons.

I. INTRODUCTION

A STATISTICAL model has been introduced by Fermi[1] to study multiple processes occurring in high-energy nucleon encounters. The method is based on the assumption that in such an encounter there is a localization of energy in a small spatial volume which decays into the various possible final states, compatible with the constants of motion, with relative *a priori* probabilities proportional to their statistical weights. In view of the mathematical and physical uncertainties involved in field-theoretic approaches to such problems, the method may provide a simple means of providing a much needed background for discussion of nuclear events at the moderately high energies now accessible to laboratory study by means of the various high-energy particle accelerators. Fermi has explored the consequences of the model most fully for cases of extremely high energy where the statistical model may be treated by using thermodynamics. When using the more detailed approach, he has treated nucleons as being nonrelativistic, pions as being extremely relativistic; furthermore, the momentum of the pions is neglected.

In this paper a general approach to the calculation of phase volumes is proposed. This method is used to study the consequences of including momentum conservation in the simple case of extremely relativistic particles. Tables are presented giving the probabilities for pion and heavy meson production using the Fermi approximations. The effect of indistinguishability of the pions is also treated.

II. METHOD

Following Fermi, the statistical weight of a state leading to n particles of masses $M_1, \cdots, M_n$ is computed according to the following rule:

$$S_n=\frac{\mathcal{S}}{\mathcal{G}}\frac{\Omega^{n-1}}{(2\pi)^{3(n-1)}}\int\prod_{i=1}^{n}d^3\mathbf{p}_i. \qquad (1)$$

(We are using units such that $\hbar=c=1$.) The integration extends over that region of momentum space compatible with the constraints on the system. The configurational space volume Ω is taken to depend on the energy E in the center-of-mass system:

$$\Omega=(2M/E)\Omega_0;\quad \Omega_0=(4\pi/3)R^3. \qquad (2)$$

The factor $2M/E$ represents the Lorentz contraction of the pion cloud surrounding a nucleon whose radius, in its proper coordinate system, is R. Although R is expected to be about $\hbar/m_\pi c$, it may be regarded as an adjustable parameter. The symbol $\mathcal{S}$ represents the weight factors due to conservation of spin and isotopic spin. The factor[2] $\mathcal{G}$ converts the specific phase space volume into a generic one appropriate to indistinguishable particles.

The constraints on the momentum integration are those due to conservation of momentum, energy, and angular momentum. To simplify the calculations in what follows, angular momentum conservation is disregarded. There are other statistical factors which enter into the computation of reaction probabilities: conservation of spin, isotopic spin, and the effect of indistinguishability of emitted particles. Also some rule must be accepted regarding conservation of nucleons. If one believes in the possibility of nucleon pair production, this means that the difference between the numbers of nucleons and antinucleons is conserved:

$$N-N'=\text{constant}. \qquad (3)$$

The requirements of momentum and energy conservation may be imposed on the statistical weight by inserting discontinuous factors in the integrand of Eq. (1) and then allowing the integrations to go over the entire momentum space:

$$S_n=\frac{\mathcal{S}}{\mathcal{G}}\frac{\Omega^{n-1}}{(2\pi)^{3n+1}}\frac{d}{d(iE)}\int_{-\infty-i\epsilon}^{\infty-i\epsilon}\frac{d\alpha}{\alpha}\int_{-\infty}^{\infty}d^3\boldsymbol{\lambda}\prod_i d^3\mathbf{p}_i$$
$$\times\exp\{i[\boldsymbol{\lambda}\cdot\textstyle\sum_i \mathbf{p}_i+\alpha(E-\sum_i(p_i^2+M_i^2)^{\frac{1}{2}})]\}, \qquad (4)$$

where ϵ is a small positive number which insures that S_n is well defined. It is taken to be zero after the integrations are performed.

Each momentum integral occurring in Eq. (4) has the form

$$I=\int d^3\mathbf{p}\,\exp\{i[\boldsymbol{\lambda}\cdot\mathbf{p}-\alpha(p^2+M^2)^{\frac{1}{2}}]\}. \qquad (5)$$

[1] Enrico Fermi, Progr. Theoret. Phys. (Japan) **5**, 570 (1950).

[2] For example, $\mathcal{G}=1/n!$ if the state is one of n identical pions.

The angular integration may be carried out and yields

$$I=-\frac{2\pi}{\lambda}\frac{d}{d\lambda}\int_{-\infty}^{\infty}dp\,\exp\{i[\lambda p-\alpha(p^2+M^2)^{\frac{1}{2}}]\}. \quad (6)$$

In this expression λ is the magnitude of the vector $\boldsymbol{\lambda}$.

In order to evaluate this integral, it is convenient to introduce a change of variable:

$$p=M\sinh\theta. \quad (7)$$

One finds from Eq. (6):

$$I=\frac{-2\pi M}{\lambda}\frac{d}{d\lambda}\int_{-\infty}^{\infty}d\theta\cosh\theta \times\exp[iM(\lambda\sinh\theta-\alpha\cosh\theta)]. \quad (8)$$

Depending on the relative values of λ and α, the argument of the exponential in (8) may be written in three different ways:

(i) $\alpha>\lambda>0$:

$$\lambda\sinh\theta-\alpha\cosh\theta=-(\alpha^2-\lambda^2)^{\frac{1}{2}}\cosh(\theta-\phi_1),$$
$$\cosh\phi_1=\alpha/(\alpha^2-\lambda^2)^{\frac{1}{2}};$$

(ii) $|\alpha|<\lambda>0$:

$$(\lambda\sinh\theta-\alpha\cosh\theta)=(\alpha^2-\lambda^2)^{\frac{1}{2}}\sinh(\theta-\phi_2),$$
$$\sinh\phi_2=\alpha/(\lambda^2-\alpha^2)^{\frac{1}{2}};$$

(iii) $-\alpha>\lambda>0$:

$$(\lambda\sinh\theta-\alpha\cosh\theta)=(\alpha^2-\lambda^2)^{\frac{1}{2}}\cosh(\theta+\phi_3),$$
$$\cosh\phi_3=-\alpha/(\alpha^2-\lambda^2)^{\frac{1}{2}}. \quad (9)$$

The integrals may be reduced to standard forms if the parameters lie in the first or the last region. For $\alpha>\lambda>0$,

$$I=\frac{-2\pi M}{\lambda}\frac{d}{d\lambda}\int_{-\infty}^{\infty}d\theta\cosh\theta \times\exp[-iM(\alpha^2-\lambda^2)^{\frac{1}{2}}\cosh(\theta-\phi_1)]. \quad (10)$$

If a change of variable $\theta'=\theta-\phi_1$ is made, one obtains

$$I=\frac{-2\pi M}{\lambda}\frac{d}{d\lambda}\frac{\alpha}{(\alpha^2-\lambda^2)^{\frac{1}{2}}}\int_{-\infty}^{\infty}d\theta'\cosh\theta' \times\exp[-i(\alpha^2-\lambda^2)^{\frac{1}{2}}\cosh\theta'].$$

Thus, I is expressible in terms of the Hankel functions,[3]

$$I=2\pi^2M^2\alpha\frac{H_2^{(2)}(M(\alpha^2-\lambda^2)^{\frac{1}{2}})}{\alpha^2-\lambda^2}. \quad (11)$$

In the region $-\alpha>\lambda>0$, the integral may be evaluated in the same manner:

$$I=-2\pi^2M^2\alpha\frac{H_2^{(1)}(M(\alpha^2-\lambda^2)^{\frac{1}{2}})}{\alpha^2-\lambda^2}. \quad (12)$$

In the region $\lambda>|\alpha|>0$, we may infer the value of the integral by observing that I is an analytic function of α and λ and by appealing to the principle of continuation. That choice of phases of the square root $(\alpha^2-\lambda^2)^{\frac{1}{2}}$ must be made which joins the first and last regions through the other. A careful examination shows that I may be compactly represented in any region as

$$I=2\pi^2M^2\alpha\frac{H_2^{(2)}(M(\alpha^2-\lambda^2)^{\frac{1}{2}})}{\alpha^2-\lambda^2}, \quad (13)$$

if it is understood that the phase of the square root is chosen according to the following scheme:

$$\begin{array}{ll} (\alpha^2-\lambda^2)^{\frac{1}{2}}, & \alpha>\lambda>0 \\ e^{-i\pi/2}(\lambda^2-\alpha^2)^{\frac{1}{2}}, & \lambda>|\alpha|>0 \\ e^{-i\pi}(\alpha^2-\lambda^2)^{\frac{1}{2}}, & -\alpha>\lambda>0. \end{array} \quad (14)$$

If Eq. (13) is combined with Eq. (4), one finds for the density of states:

$$S_n=\frac{\mathcal{S}}{\mathcal{G}}\frac{\Omega^{n-1}}{2^{2n-1}}\frac{\prod_i M_i^2}{\pi^n}\int_{-\infty-i\epsilon}^{\infty-i\epsilon}d\alpha\cdot\alpha^n e^{i\alpha E} \times\int_0^{\infty}\frac{d\lambda\cdot\lambda^2}{(\alpha^2-\lambda^2)^n}\prod_i H_2^{(2)}(M_i(\alpha^2-\lambda^2)^{\frac{1}{2}}). \quad (15)$$

S_n may be evaluated readily only when the masses of all particles are small compared to E. In this case $H_2^{(2)}$ may be replaced by[4]

$$H_2^{(2)}(z)\cong 4i/\pi z^2. \quad (16)$$

One finds

$$S_n=\frac{\mathcal{S}}{\mathcal{G}}\frac{2\Omega^{n-1}i^n}{\pi^{2n}}\int_{-\infty-i\epsilon}^{\infty-i\epsilon}d\alpha\cdot\alpha^n e^{i\alpha E}\int_0^{\infty}\frac{d\lambda\cdot\lambda^2}{(\alpha^2-\lambda^2)^{2n}}. \quad (17)$$

This may be evaluated by applying the method of residues. The integral over λ may be evaluated by closing the contour above the real axis. There are poles of order $2n$ at $\lambda=\pm\alpha$ but since $\epsilon>0$ only the pole at $\lambda=-\alpha$ will be encircled. One finds

$$\int\frac{d\lambda\cdot\lambda^2}{(\lambda^2-\alpha^2)^{2n}}=\frac{-\pi i(4n-3)!}{2^{4n-3}(2n-1)!(2n-2)!\alpha^{4n-3}}. \quad (18)$$

The remaining integral over α in the expression for S_n may be similarly evaluated by closing the contour

[3] G. N. Watson, *Bessel Functions* (MacMillan Company, New York, 1948), p. 180, Eq. (11); p. 74, Eq. (10).

[4] See reference 3, p. 84.

above the real axis,

$$\oint d\alpha \frac{e^{i\alpha E}}{\alpha^{3n-3}} = \frac{2\pi i (iE)^{3n-4}}{(3n-4)!}. \qquad (19)$$

The expression for the density of states in the extremely relativistic limit is therefore

$$S_n = \frac{\mathcal{S}\Omega^{n-1}(4n-3)!E^{3n-4}}{\mathcal{G}\pi^{2n-2}2^{4n-4}(2n-1)!(2n-2)!(3n-4)!}. \qquad (20)$$

This is to be compared with the corresponding formula of Fermi's paper[5] which neglects momentum conservation:

$$S_n = \frac{\Omega^{n-1}}{\pi^{2n}} \frac{E^{3n-1}}{(3n-1)!}. \qquad (21)$$

For large values of n, Stirling's approximation may be applied to evaluate Eq. (20).

$$S_n = \frac{\mathcal{S}\Omega^{n-1}}{\mathcal{G}\pi^{2n}} \left(\frac{2\pi^3}{n}\right)^{\frac{1}{2}} \frac{E^{3n-4}}{(3n-4)!}. \qquad (22)$$

III. APPROXIMATE EVALUATION OF THE STATISTICAL WEIGHT

It is not clear how to evaluate the expression for S_n when the masses of the particles cannot be neglected. In order to obtain some numerical results, Fermi has suggested that one divide particles into two classes: (1) pions and (2) nucleons and heavy mesons. He treats the pions as being extremely relativistic and neglects the momentum carried away by them; the other particles are taken as nonrelativistic and it is assumed that they carry most of the momentum.

If these approximations are made in the expression for I, Eq. (5), one finds

$$\left.\begin{aligned} I_{ER} &= 4\pi \int_0^\infty dp p^2 \exp(-i\alpha|p|) \quad \text{(extremely relativistic)} \\ I_{NR} &= \frac{-2\pi}{\lambda} e^{-i\alpha M} \frac{d}{d\lambda} \int_{-\infty}^\infty dp \exp[i(\lambda p - \alpha p^2/2M)] \quad \text{(nonrelativistic)} \end{aligned}\right\}. \qquad (23)$$

One finds

$$\begin{aligned} I_{ER} &= 8\pi i/\alpha^3, \\ I_{NR} &= -\pi^{\frac{3}{2}} e^{i\pi/4}(2M/\alpha)^{\frac{3}{2}} e^{-i\alpha M} \exp(iM\lambda^2/2\alpha). \end{aligned} \qquad (24)$$

By using these approximations, the expression for S_n may be evaluated by the method of the preceding section. One finds the result given by Fermi

$$S_n = \frac{\mathcal{S}}{\mathcal{G}} \Omega^{n-1} \times \frac{\prod_i M_i^{3/2}(E - \sum_i M_i)^{3s/2+3m-5/2}}{(\sum_i M_i)^{3/2} 2^{3s/2-1/2} \pi^{3s/2+2m-3/2} \Gamma(3s/2+3m-3/2)}, \qquad (25)$$

where Γ is the ordinary Γ function, s is the number of heavy particles, and m the number of light ones.

It is of some interest to derive an expression for the energy spectrum of inelastically scattered nucleons (see Fig. 1). Since nucleon pair formation and heavy meson production presumably contribute inappreciably to the totality of the observed processes at moderate energies, one may write down the expression for the case when two nucleons and m pions are formed. Using the Fermi approximations, one finds

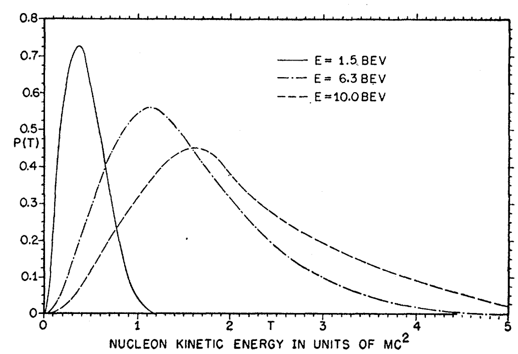

FIG. 1. Graph showing energy spectrum of inelastic nucleons resulting from a high-energy collision. The curves are normalized so that their areas represent the total probability at the indicated energy of all inelastic processes.

$$S_{m+2} = \frac{\mathcal{S}M^{3/2}\Omega^{m+1}}{\mathcal{G}2^{7/2}\pi^{2m+5}(3n-1)!} \times \int dT_M (T_M)^{\frac{1}{2}} (E - 2M - 2T_M)^{3n-1}. \qquad (26)$$

Here T_n is the kinetic energy of one of the two heavy particles of mass M in the center-of-momentum system. The coefficient of dT_M therefore represents the spectrum

$$S_{m+2}(T_M) = \frac{\mathcal{S}M^{3/2}\Omega^{m+1}(T_M)^{\frac{1}{2}}}{\mathcal{G}2^{7/2}\pi^{2m+5}(3n-1)!}(E - 2M - 2T_M)^{3n-1}. \qquad (27)$$

IV. APPLICATIONS

The foregoing results may be used to compute the probabilities of various processes. The probability for formation of particles of masses $M_1, \cdots, M_n$ is

$$P_n(M_1 \cdots M_n) = \frac{S_n(M_1 \cdots M_n)}{\sum_n' S_n(M_1 \cdots M_n)}. \qquad (28)$$

[5] See reference 3, p. 576, Eq. (13). Fermi omits explicit reference to the factor $\mathcal{S}/\mathcal{G}$.

TABLE I. Probabilities for emission of m_1, m_2, m_3 particles of mass $M_1=176$, $M_2=967$, $M_3=1848$ (in units of the electron mass) at the indicated energy E (laboratory system). Case of distinguishable particles.

m_1	m_2	m_3	2.5 Bev	3.9 Bev	4.71 Bev
0	0	2	0.5846	0.007388	0.0008
1			0.4127	0.2302	0.0647
2			0.0027	0.3679	0.2957
3			0	0.09187	0.2547
4				0.00503	0.0601
5				0.00007	0.0046
0	1	2		0.05910	0.0111
1				0.1391	0.0936
2				0.03006	0.0954
3				0.0010	0.0207
4				0.000005	0.0012
0	2	2		0.0562	0.0288
1				0.00942	0.0365
2				0.00013	0.0071
3				0	0.0003
0	3	2		0.00248	0.0148
1	3	2		0.000005	0.0024
0	4	2			0.0008
0	0	4			0.0063
1	0	4			0
0	1	4			0.0001
1	1	4			0

TABLE II. Probabilities for emission of m_1, m_2, m_3 particles of mass $M_1=176$, $M_2=967$, $M_3=1848$ (in units of the electron mass) at the indicated energy E (laboratory system). Case of indistinguishable particles.

m_1	m_2	m_3	2.5 Bev	3.9 Bev	4.71 Bev
0	0	2	0.5854	0.0108	0.0018
1			0.4132	0.3367	0.1412
2			0.0014	0.2690	0.3228
3			0	0.0224	0.0927
4				0.0003	0.0055
5				0	0
0	1	2		0.08642	0.0242
1				0.2035	0.2042
2				0.02198	0.1042
3				0.00024	0.0075
4				0	0.0001
0	2	2		0.0411	0.0315
1				0.0069	0.0398
2				0.0001	0.0039
3				0	0
0	3	2		0.0006	0.0054
1	3	2		0	0.0009
0	4	2			0.0001
0	0	4			0.0138
1	0	4			0.0001
0	1	4			0.0003
1	1	4			0

$\sum_n'$ is to be derived from the statistical weights S_n by omitting those terms, from the sum over all states, which are incompatible with heavy-particle conservation.

Since the total cross section is assumed to be fixed, that part of it leading to a given final state is

$$\sigma_n(M_1\cdots M_n)=P_n(M_1\cdots M_n)\sigma_{\text{total}}. \quad (29)$$

Tables I and II give the probabilities for emission of various numbers of neutral pions and a possible heavy meson of mass $967mc$. For purposes of comparison they have been computed assuming particles of a given mass are: (a) distinguishable, (b) indistinguishable. Spin and isotopic spin conservation have been neglected.[6] For the case of distinguishable particles they differ somewhat from those given in Fermi's article, since the pion rest mass has been included in the sum occurring in Eq. (25). This amounts to writing $E=|\mathbf{p}|+\mu$ for extremely relativistic particles and has the effect of insuring that not more pions are emitted than are compatible with the available energy. The curves giving the energy spectra of inelastically scattered neutrons have been similarly handled. They have been transformed to the laboratory system by using the fact that this model, which neglects angular momentum conservation, yields an isotropic angular distribution in the center-of-mass system.

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[6] This corresponds to setting $\mathcal{S}=1$ in the expression for S_n.