The Problem of Multiple Scattering

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Multiple scattering can be regarded as a succession of elementary events. The distribution function for the particles which have gone through $n+1$ elementary events is the convolution of two functions. The first of these expresses the scattering law; the second one is the distribution function for particles which have gone through n events. It is well known that such convolutions can be calculated very easily by means of Fourier transforms if the elementary event is the traversal of a free path in an arbitrary direction. In this case, the Fourier transform of the convolution is the product of the Fourier transforms of the convolvents. In the case of more general scattering laws, integrals over products of the distribution function, and of represensations of the group which leaves the scattering law invariant, play the same role which the Fourier transforms play in the aforementioned case. From the present point of view, the exponential in the Fourier transform is a representation of the displacement group. It is shown that one can solve several problems of multiple scattering on the basis of the above observation. These problems include the scattering of a point particle without change of energy but an arbitrary angular distribution, and several more involved problems.

INTRODUCTION

GROSJEAN[1] has given, by direct calculation, a solution of the problem of multiple scattering in an infinite medium. The reason for taking up the same problem again is that the method of calculation to be presented is somewhat more transparent and also applicable to a wider range of problems. It is based on the symmetry of the problem considered and uses the theory of group representations.

By multiple scattering we mean a succession of "elementary events" which change the state of the system and which are statistically independent of each other. In the theory of multiple scattering, the elementary event is, in general, a collision and the subsequent traversal of a free path. It changes the state of the system from the one in which it is before a collision to the state in which it is before the next one. The statistical independence of elementary events will be guaranteed if the description of the state of the system is complete, i.e., if it extends to all the parameters. In many cases, this is not necessary. Thus, if one is interested only in the velocity distribution after a certain number of collisions, and if the medium in which the particle moves is homogeneous and extends over all space, one can suppress the position coordinates in the description of the state and consider, as elementary events, the changes in the velocity vector caused by the subsequent collisions. It is clear that the statistical independence of subsequent elementary events is preserved under the conditions specified even though the description of the state is incomplete in this case. The same holds in the case of spherically symmetric scattering with respect to the direction of the velocity if one is interested only in the density distribution of the particles, irrespective of the direction of their velocities. In the most important case of multiple scattering, in which a complete specification of the state of the system is necessary, the elementary act can contain, just as well, a free path with a *subsequent* collision. In this case, the elementary event changes the state from the one *after* a collision to the state *after* the next collision.

The condition under which we can give an explicit solution of the problem is as follows. (1) The probability that an elementary event changes the state in a certain way is invariant under a group G. (2) Every state of the system is obtainable from a single fixed state by the operations of the group G. In the case of multiple scattering in an infinite homogeneous and isotropic medium, the group G contains all rotations and displacements in space, i.e., is the Euclidean group. The conditions of homogeneity and isotropy of the scattering medium express the fact that the scattering law, i.e., the law of the elementary event, is invariant under G. If the moving particle has no structure and does not change its energy as a result of the collisions, its state is completely characterized by its position and the direction of its velocity. Hence, every state of the particle can be obtained from a standard one by a displacement and a rotation, i.e., by an element of G. As standard state one can choose, for instance, the one in which the particle is at the origin of the coordinate system and its velocity is parallel to the Z axis. In the second case mentioned in the preceding paragraph, in which one is interested only in the velocity distribution, the scattering law will have the symmetry of the rotation group and G will be this group. If the collision does not

[1] C. C. Grosjean, dissertation, Columbia University, 1951 (unpublished). This thesis also has rather extensive references to earlier literature. Among earlier papers, those of W. Bothe [Z. Physik **54**, 161 (1929)] and of S. Goudsmit and J. L. Saunderson [Phys. Rev. **54**, 773 (1939) and **58**, 36 (1940)] anticipate most nearly Grosjean's results. The last article solves, in particular, the same problem which is treated by Eqs. (16) of the present note. Grosjean's article goes further than these by being able to give a rigorous expression for the probability of a given displacement, not only for the probability of a given deflection. This corresponds to Eqs. (17) of the present note. For later developments of Grosjean's method, see also C. C. Grosjean, Koninkl. Vlaam. Acad. Wetenschap. Letter en Shone Kunsten België Jaarboek **13**, No. 36 (1951) and Physica **19**, 29 (1953).

change the energy of the moving particle, its state is given by the direction of the velocity vector and any such direction can be obtained from a standard one by rotation, i.e., by an element of G. Hence, our method will be applicable to both these cases and a similar discussion shows that it is applicable also in the last case mentioned above,—that of spherically symmetric scattering if the particle again does not change its energy in the course of the collisions. Further and more general examples will be given later.

EQUATIONS OF MULTIPLE SCATTERING

A.

Even though the method of solution to be adopted will be essentially the same in all cases to be considered, it seems worth while to distinguish two cases at this point. In the first case, there is only one group element which carries the standard state e of the particle into a given state s. In this case, the group element in question can be denoted by s itself, the unit element e corresponding to the standard state. This is the situation if the particle is restricted to move in a plane, no matter whether its "state" has to be characterized only by the direction of its velocity, or by this direction and the position of the particle. Let us denote in this case the probability distribution (per unit volume element of the invariant group space) after n elementary events by $f_n(s)$. The volume element is simply $d\varphi$, with φ the angle between direction of the velocity and the standard direction, if one excludes the consideration of the position of the particle. It is $d\varphi dxdy$, with x and y the rectangular coordinates of the particle, if one is interested also in the probability of the position of the particle. If the energy also changes in the course of collisions, it becomes a fourth parameter of s and the volume element will depend on the energy dependence of the law of scattering.

The probability that an elementary event change the state s into a unit volume element at t will be denoted by $P(s,t)$. The invariance of this probability under the operations of the group means that for every group element u

$$P(s,t)=P(us,ut). \tag{1}$$

This equation already uses our assumption that there is a correspondance between states and the elements of the group which leaves the law of the elementary event invariant. Writing, in particular, $u=s^{-1}$ in (1), one obtains

$$P(s,t)=P(e,s^{-1}t)=P(s^{-1}t). \tag{1a}$$

This equation expresses the fact that the scattering law remains the same no matter whether one uses e or s as the basic state. The probability distribution $f_{n+1}(t)$ is given therefore by

$$f_{n+1}(t)=\int f_n(s)P(s,t)ds=\int f_n(s)P(s^{-1}t)ds, \tag{2}$$

where ds signifies the invariant group integration. (Right and left invariant group integrals are identical in all the groups to be considered.) It is very natural that the probability distribution after $n+1$ collisions be given by the convolution of the probability distribution after n collisions and the scattering law.

Because of the somewhat abstract nature of the derivation of (2), it may be worth while to illustrate it on the aforementioned examples. If, in two-dimensional scattering, the position of the particle is suppressed, the group G is simply the group of rotations in two dimensions and can be characterized by an angle φ. The state (φ_0) of the particle is obtained from the standard state, with the velocity directed parallel to the X axis, by a rotation with φ_0; i.e., it is the state in which the velocity includes an angle φ_0 with the X axis. If t and s in (2) are characterized by the angles φ and φ_0 this equation becomes

$$f_{n+1}(\varphi)=\int f_n(\varphi_0)P(\varphi-\varphi_0)d\varphi_0, \tag{3}$$

since the angle which characterizes the group element $s^{-1}t$ becomes $\varphi-\varphi_0$. According to its definition,

$$P(\varphi')=\sigma(\varphi')/\sigma, \tag{3a}$$

where $\sigma=\int\sigma(\varphi')d\varphi'$ is the total cross section. Hence, (3) gives the usual way to calculate the angular distributions successively.

If one wishes to obtain the spatial distribution of the particle as well as its velocity, the group G becomes the group of motions in two dimensions. It can be characterized by a displacement by x, y and a rotation by φ (the latter preceding the former). The group element,

$$s=\begin{Vmatrix}\cos\varphi_0 & -\sin\varphi_0 & x_0\\ \sin\varphi_0 & \cos\varphi_0 & y_0\\ 0 & 0 & 1\end{Vmatrix}=\begin{Vmatrix}1 & 0 & x_0\\ 0 & 1 & y_0\\ 0 & 0 & 1\end{Vmatrix}\cdot\begin{Vmatrix}\cos\varphi_0 & -\sin\varphi_0 & 0\\ \sin\varphi_0 & \cos\varphi_0 & 0\\ 0 & 0 & 1\end{Vmatrix},$$

characterizes the state in which the particle is at x_0, y_0 and the direction of the velocity includes an angle φ_0 with the X axis. A direct method to obtain P in terms of the cross section will be given in the section Calculation of $P(s,t)$. In the present case we wish to verify only *a posteriori* that (2) gives the well-known equation (3e) if

$$P(x',y',\varphi')=\sigma(\varphi')r^{-1}e^{-\sigma r}\delta(\psi-\varphi'), \tag{3b}$$

where r and ψ are abbreviations for the polar coordinates of $x'=r\cos\psi$ and $y'=r\sin\psi$. The δ function in (3b) expresses the fact that the displacement is always in the direction of the velocity. For the verification of (2), let us denote the parameters of the group element s by x_0, y_0, φ_0, those of t by x, y, φ; the parameters of $s^{-1}t$

then become

$$x'=(x-x_0)\cos\varphi_0+(y-y_0)\sin\varphi_0,$$
$$y'=-(x-x_0)\sin\varphi_0+(y-y_0)\cos\varphi_0, \quad (3c)$$
$$\varphi'=\varphi-\varphi_0.$$

The polar coordinates of x', y' are r and $\psi=\alpha-\varphi_0$, where r and α are polar coordinates for $x-x_0=r\cos\alpha$, $y-y_0=r\sin\alpha$. Hence

$$P(x',y',\varphi')=\sigma(\varphi-\varphi_0)r^{-1}e^{-\sigma r}\delta(\alpha-\varphi),$$

and with the invariant volume element for ds given above, (2) becomes

$$f_{n+1}(x,y,\varphi)=\int\int\int dx_0dy_0d\varphi_0 f_n(x_0,y_0,\varphi_0)$$
$$\times\sigma(\varphi-\varphi_0)r^{-1}e^{-\sigma r}\delta(\alpha-\varphi). \quad (3d)$$

This becomes the well-known equation,

$$f_{n+1}(x,y,\varphi)=\int\int\int dr d\alpha d\varphi_0 f_n(x_0,y_0,\varphi_0)$$
$$\times\sigma(\varphi-\varphi_0)e^{-\sigma r}\delta(\alpha-\varphi)$$
$$=\int\int dr r\varphi_0 f_n(x-r\cos\varphi,\ y-r\sin\varphi,\ \varphi_0)$$
$$\times\sigma(\varphi-\varphi_0)e^{-\sigma r}, \quad (3e)$$

if one replaces the integration variables x_0, y_0 by r and α and carries out the integration over α. It may be arguable that the well-known (3e) is simpler than (2) but we shall see that (2) suggests more directly a method of solution than does (3e).

B.

The preceding calculation applies in the case in which G has only one element which transforms the standard state e into a given state s. This is not true if, for instance, a particle without structure moves in three- rather than two-dimensional space. The reason therefore is that there are rotations—those about the direction of the velocity in the standard state—which leave the standard state unchanged. The rest of the present section contains a discussion of the group theoretic description of the state and of the scattering law. While the former discussion is essential, the discussion following (5a) is not necessary if one is interested only in using the present method. The result of this discussion is (9) which becomes evident at any rate if one calculates P explicitly.

Let us denote the number of parameters which characterize the states by n_1, the number of parameters of the group G by n. There will be then an $n-n_1$ parametric subgroup E, the elements of which carry the standard state $\mathcal{E}$ into itself:

$$e_1\mathcal{E}=\mathcal{E} \text{ if } e_1 \text{ contained in } E. \quad (4)$$

Two group elements, s and s', will carry $\mathcal{E}$ into the same state $\mathcal{S}$ if

$$\mathcal{S}=s\mathcal{E}=s'\mathcal{E}; \quad \mathcal{E}=s^{-1}s'\mathcal{E}. \quad (4a)$$

$s^{-1}s'$ is an element of the subgroup E, i.e., if s and s' are in the same left coset of E. One can say that in the more general case now considered the left cosets of E correspond to the different states of the particle rather than the group elements themselves.

We shall see that expressions of the form (2) are very easily evaluated and we wish to define, therefore, also in the present case a probability function $f(s)$, depending on the group elements rather than on left cosets. This $f(s)$, if integrated over the elements of the cosets which correspond to a set of states, will give the probability of the states of the set. In order to express this analytically, it is useful to introduce two types of parameters, ζ and ϵ, for the group G. There are n_1 parameters ζ and they have the same value for all elements of a left coset and serve to distinguish these cosets, i.e., to characterize the states of the particle. There are $n-n_1$ parameters ϵ and they distinguish the various elements of the left cosets. One can choose, for instance, the parameters ζ and ϵ in such a way that the element with parameters ζ and ϵ become

$$s(\zeta,\epsilon)=s(\zeta,0)e(\epsilon), \quad (4b)$$

where $e(\epsilon)$ are the elements of the subgroup E and $s(\zeta,0)$ is a continuous function of $\mathcal{S}(\zeta)$. If we denote the probability of the states $\mathcal{S}(\zeta)$ per unit range of ζ by $F(\zeta)$, we shall demand of $f(s)$

$$F(\zeta)=\int f(s)g(\zeta,\epsilon)d\epsilon, \quad (5)$$

where the integration over ϵ is to be extended over the subgroup E, $g(\zeta,\epsilon)$ is the weight factor which makes $g(\zeta,\epsilon)d\zeta d\epsilon$ the invariant integral, and s is the group element with the parameters ζ, ϵ. This equation expresses the postulate that the integral of f over a coset give the probability of the state to which the coset corresponds.

Clearly, (5) does not determine $f(s)$ completely and we can further postulate that it have the same value for each element of a coset, i.e., be a constant along the path of integration of (5)

$$f(s)=f(se_1) \text{ if } e_1 \text{ contained in } E. \quad (5a)$$

All our probability functions shall have the property (5a).

In a similar way, we shall try to replace the transition probability $P(\mathcal{S}_1,\mathcal{S}_2)=P(\zeta_1,\zeta_2)$ by a function $P(s,t)$ of the group elements s,t. This has to be done in such a way that if

$$F_{n+1}(\zeta_2)=\int F_n(\zeta_1)P(\zeta_1,\zeta_2)d\zeta_1, \quad (6)$$

and if F_n and f_n correspond to each other in the sense of (5), (5a), then

$$f_{n+1}(s_2)=\int f_n(s_1)P(s_1,s_2)ds_1 \tag{6a}$$

also correspond to F_{n+1} in the same sense, no matter what the functional form of F_n is. Since f_{n+1}, as a probability function, will have to satisfy (5a), we can conclude at once that

$$P(s_1,s_2)=P(s_1,s_2e_2) \text{ if } e_2 \text{ contained in } E. \tag{7a}$$

Expressing now both F in (6) by the corresponding f, we find

$$\int f_{n+1}(s_2)g(\zeta_2,\epsilon_2)d\epsilon_2=\int\int f_n(s_1)g(\zeta_1,\epsilon_1)P(\zeta_1,\zeta_2)d\epsilon_1 d\zeta_1$$

$$=\int f_n(s_1)P(\zeta_1,\zeta_2)ds_1,$$

where s_2 and s_1 are the group elements with the parameters ζ_2,ϵ_2 and ζ_1,ϵ_1, respectively, and the second line follows from the definition of $g(\zeta,\epsilon)d\zeta d\epsilon$ as the invariant group integral. The condition that the f_{n+1} obtained from (6) satisfy (6a) is, therefore,

$$\int ds_1\int f_n(s_1)P(s_1,s_2)g(\zeta_2,\epsilon_2)d\epsilon_2=\int ds_1 f_n(s_1)P(\zeta_1,\zeta_2).$$

Since f_n is not an arbitrary function of s_1 but satisfies (5a), the last equation does not fully determine $P(s_1,s_2)$ and we are free to let it depend on ϵ_1 in an arbitrary fashion. The simplest choice is

$$\int P(s_1,s_2)g(\zeta_2,\epsilon_2)d\epsilon_2=P(\zeta_1,\zeta_2), \tag{7}$$

according to which we have, in addition to (7a),

$$P(se_1,s_2)=P(s_1,s_2) \text{ if } e_1 \text{ contained in } E. \tag{7b}$$

According to (7a) and (7b), $P(s_1,s_2)=P(s_1',s_2')$ if s_1 and s_1' are in the same left coset of E and the same holds of s_2, s_2'. This is indeed the most natural convention; it renders all quantities $f(s)$, $P(s,t)$ functions only of the ζ parameters of their (group element) variables. It follows that $P(s_1,s_2)$ can be taken out of the integral sign of (7) and this equation, together with (7a), (7b), completely determines $P(s_1,s_2)$ once $P(\zeta_1,\zeta_2)$ is given.

The invariance of the law governing the elementary event can be expressed in the following way. Let us assume that the distribution will be given by $f_{n+1}(s_2)$ if it was $f_n(s_1)$ one event before. Then if the distribution is $h_n(s_1)=f_n(us_1)$, it will go over into $h_{n+1}(s_2)=f_{n+1}(us_2)$ after another event. This will hold for every group element u. One easily verifies that the h satisfy (5a) if the f do. Expressing the above condition analytically, one finds that

$$f_{n+1}(us_2)=h_{n+1}(s_2)=\int h_n(s_1)P(s_1,s_2)ds_1$$

$$=\int f_n(us_1)P(s_1,s_2)ds_1 \tag{8}$$

must be a consequence of

$$f_{n+1}(t)=\int f_n(s)P(s,t)ds. \tag{8a}$$

Since ds_1 indicates the invariant group integration, one can replace in (8) us_1 by s. Writing, furthermore, t for us_2, (8) goes over into

$$f_{n+1}(t)=\int f_n(s)P(u^{-1}s,u^{-1}t)ds, \tag{8b}$$

and this will be a consequence of (8a) if

$$\int f_n(s)P(s,t)ds=\int f_n(s)P(u^{-1}s,u^{-1}t)ds. \tag{8c}$$

Writing out the integration in terms of ζ,ϵ,

$$\int f_n(\zeta)P(s,t)g(\zeta,\epsilon)d\zeta d\epsilon$$

$$=\int f_n(\zeta)P(u^{-1}s,u^{-1}t)g(\zeta,\epsilon)d\zeta d\epsilon,$$

where s is the group element with the parameters ζ,ϵ. Since the last equation must hold for all f_n,

$$P(s,t)\int g(\zeta,\epsilon)d\epsilon=P(u^{-1}s,u^{-1}t)\int g(\zeta,\epsilon)d\epsilon. \tag{8d}$$

Both $P(s,t)$ and $P(u^{-1}s,u^{-1}t)$ could be placed before the integral sign because they are independent of ϵ. Setting then $u=s$, one again obtains (1a) for P and also obtains, from (6a), the same Eq. (2) which holds in the case in which there is only one group element which transforms the standard state into a given state. The only difference between the two cases is, therefore, the additional condition (5a) on the distribution functions, and the conditions (7a), (7b) on $P(s,t)$, which must be satisfied in order to make (2) valid also if there are group elements which leave the standard state unchanged and hence all the elements of a left coset transform the standard state into the same state. It is worth noting that (7a), (7b) give

$$P(e_1se_2)=P(s) \text{ for } e_1,\ e_2 \text{ contained in } E. \tag{9}$$

Hence, P depends in general on even fewer independent variables than f. The example of the following section will illustrate this point.

CALCULATION OF $P(s,t)$

In the case of the motion of a particle in two-dimensional space without energy change, we obtained the expressions (3a), (3b) for P by inspection; insertion of these expressions into (2) gave the relatively simple and well-known equations for multiple scattering for this case and thus verified (3a) and (3b) *a posteriori*. We shall not follow this procedure in the case of the scattering of a particle in three-dimensional space without energy change, principally because the explicit form of the equation of multiple scattering in three dimensions [i.e., the analog of (3e)] is very complicated. In fact, one of the simplifications which the method here presented introduces is the avoiding of these explicit equations and the possibility of evaluating (2) without writing down this equation in any other form. In order to use and interpret the solution, it is necessary, however, to express $P(s,t)$ in terms of the quantities commonly used (i.e., the differential cross section) and to establish the connection between the $f_n(s)$ and the commonly used angular distribution $F(\Omega)$. These connections are given, in principle, by (7), (7a), (7b), and by (5), (5a) but the corresponding relation will be explicitly evaluated now for the aforementioned case, i.e., scattering of a particle without change of energy under disregard of its position. It will be given also for the case in which the position of the particle is considered also.

The group G in the first case is the three-dimensional rotation group. As standard state, we choose the one in which the velocity is parallel to the Z axis. The subgroup E which leaves the standard state invariant consists of the rotations about Z. It is reasonable to choose the polar angles φ, ϑ of the velocity direction as the variables ζ describing the state. Equation (4b) then becomes

$$s(\varphi,\vartheta,\epsilon)=s(\varphi,\vartheta,0)Z(\epsilon), \tag{10}$$

where $Z(\epsilon)$ is a rotation by ϵ about the Z axis and we can choose for $s(\varphi,\vartheta,0)$ any rotation which turns the Z axis into the φ, ϑ direction. If we choose

$$s(\varphi,\vartheta,0)=Z(\varphi)X(\vartheta), \tag{10a}$$

the angles φ, ϑ, ϵ become the Eulerian parameters of the group elements. The distribution function f will depend, by (5a), only on φ and ϑ and we have, by (5) for the probability of a unit range in φ and ϑ:

$$F(\varphi,\vartheta)=\int_0^{2\pi} f(\varphi,\vartheta)\sin\vartheta d\epsilon=2\pi f(\varphi,\vartheta)\sin\vartheta, \tag{10b}$$

$\sin\vartheta d\varphi d\vartheta d\epsilon$ being the invariant volume element expressed in terms of Eulerian parameters.

According to (9), the function $P(s)$ depends only on the parameter ϑ—the simplicity of this condition is a consequence of (10a), i.e., of the use of Eulerian parameters to describe the group elements. A similar simplification can be accomplished, however, also in most other cases by choosing three sets of parameters ζ_1, ζ_2, ϵ defined by the equation

$$s(\zeta_1,\zeta_2,\epsilon)=e(\zeta_1)\bar{s}(\zeta_2)e(\epsilon), \tag{11}$$

where $e(\zeta_1)$ and $e(\epsilon)$ cover the subgroup E and the $\bar{s}(\zeta_2)$ are so chosen that all elements of the group are obtained by letting ζ_2 vary over a suitable domain.

Since $P(s,t)$ depends only on $s^{-1}t$, it is sufficient to determine $P(e,t)=P(t)$ where e is the unit element. We then have, from (7),

$$\int_0^{2\pi} P(\vartheta)\sin\vartheta d\epsilon=\sigma(\vartheta)\sin\vartheta/\sigma. \tag{10c}$$

Since we have set $s_1=e$ the right side represents the probability that an elementary event change the velocity, originally parallel to Z, into a velocity at ϑ, φ within unit range of $d\vartheta$ and $d\varphi$. Since the differential cross section $\sigma(\vartheta)$ is usually defined per unit solid angle, the probability of the transition to unit $d\vartheta d\varphi$ range becomes $\sigma(\vartheta)\sin\vartheta$. It is an obvious consequence of the spherical symmetry of the problem that this probability is independent of φ. This does not give an additional condition, however, because the symmetry of $P(s)$ is fully described already in (9).

Summarizing, we have

$$f(s)=f(\varphi,\vartheta,\epsilon)=(2\pi\sin\vartheta)^{-1}F(\varphi,\vartheta), \tag{12}$$

$$P(s)=P(\varphi,\vartheta,\epsilon)=(2\pi)^{-1}\sigma(\vartheta)/\sigma. \tag{12a}$$

It may be useful to repeat that $F(\varphi,\vartheta)$ is not the probability per unit solid angle but per unit $d\varphi d\vartheta$ interval. No $\sin^{-1}\vartheta$ appears in the expression for $f(s)$ if F is given in terms of the distribution function per unit solid angle.

The expressions for the case in which one wishes to consider not only the direction of the velocity but also the position of the particle can be derived with equal ease. The group G for this case is the group of motions in three dimensions (Euclidean group) defined in the same fashion as the group of motions in two dimensions was. The distribution function $F(x,y,z,\varphi,\vartheta)$ depends on the position of the particle as well as the direction of its velocity. The group elements depend on six parameters: the Eulerian angles $\varphi,\vartheta,\epsilon$ of the rotation, and the three components x,y,z of the subsequent displacement. The subgroup E contains only the rotations about Z; it is the same group as in the preceding example. The connection between the group theoretic f and the usual F is, in complete analogy to (12)

$$f(s)=f(x,y,z,\varphi,\vartheta,\epsilon)=(2\pi\sin\vartheta)^{-1}F(x,y,z,\varphi,\vartheta). \tag{13}$$

The form of $P(s)$ is simpler if we consider as fundamental process the traversal of a free path and a subsequent collision (i.e., if we adopt the second point of view of the Introduction). The probability of the tran-

sition from the standard state into unit interval at the state $s=(x,y,z,\varphi,\vartheta)$ then becomes $\delta(x)\delta(y)e^{-\sigma z}\sigma(\vartheta)\sin\vartheta$. Hence, application of (7) to the case $s_1=e$, $s_2=s$ gives

$$P(s)\int \sin\vartheta d\epsilon=\delta(x)\delta(y)e^{-\sigma z}\sigma(\vartheta)\sin\vartheta$$

or

$$P(s)=P(x,y,z,\varphi,\vartheta,\epsilon)=(2\pi)^{-1}\sigma(\vartheta)e^{-\sigma z}\delta(x)\delta(y). \quad (13a)$$

One convinces oneself easily that this expression satisfies (9).

The above expressions could have been foreseen without the detailed derivation given above. In fact, the numerical factors in (12) to (13a) need not be known for actual calculations since the successively calculated distribution functions can easily be normalized *a posteriori*.

EVALUATION OF THE EXPRESSION (2)

The reason that (2) can be evaluated particularly simply if s and t are elements of a group is that the group integral of the product of the convolute and the matrix of a representation is the product of two matrices which are obtained from the convolvents in a similar fashion. In fact, we obtain from $D(s)D(u)=D(su)$:

$$\int f_{n+1}(t)D(t)dt=\int\int f_n(s)P(s^{-1}t)D(t)dsdt$$

$$=\int\int f_n(s)P(u)D(su)dsdu$$

$$=\int f_n(s)D(s)ds\cdot\int P(u)D(u)du. \quad (14)$$

The second line is obtained by substituting $t=su$ and noting the invariance of the group integral with respect to such a substitution.[2] Since $D(s)$ is, in general, a matrix, all expressions in (14) are matrices, with the number of dimensions of D. Written out in more detail, (13) reads

$$\int f_{n+1}(s)D(s)_{\kappa\lambda}dt$$

$$=\sum_\mu\int f_n(s)D(s)_{\kappa\mu}ds\int P(s)D(s)_{\mu\lambda}ds. \quad (14a)$$

This equation holds no matter whether the representation D is reducible or irreducible. If D is the regular representation, one is led back to (2); the simplest results are obtained if D is irreducible. If (14) holds for a function f_{n+1} and all irreducible representations, this f_{n+1} satisfies (2).

[2] Equation (14) must have been known already to G. Frobenius.

It follows from the above that

$$\int f_n(s)D(s)ds=\Phi^{(n)}=\Phi^{(0)}\Pi^n;$$

$$\Pi=\int P(s)D(s)ds. \quad (14b)$$

Again, the Φ and Π are matrices with as many dimensions as D. The evaluation of (14b) is often made very much easier by the fact that f_0 and P satisfy (5a) and (9), i.e., actually do not depend on all the group variables. In particular, if one assumes the representations in the form in which the matrices corresponding to elements of E are in the reduced form, only those $\int f_0(s)D(s)_{\lambda\nu}ds$ will be different from zero in which ν corresponds to a unit representation of E and both μ and ν must correspond to such representations if $\int P(s)D(s)_{\mu\nu}ds$ is to be finite.

In the first case discussed in the last section, it follows either from (9) or more directly from (12a), that assuming the customary form of the irreducible representations[3] only the 0,0 element of $\Pi(l)$ is different from zero. One has

$$\Pi(l)_{\nu\mu}=\int P(s)D^{(l)}(s)_{\nu\mu}ds$$

$$=\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi}\frac{\sigma(\vartheta)}{2\pi\sigma}e^{i\nu\varphi}d^{(l)}{}_{\nu\mu}(\vartheta)e^{i\mu\epsilon}d\varphi\sin\vartheta d\vartheta d\epsilon$$

$$=(2\pi/\sigma)\delta_{\nu 0}\delta_{\mu 0}\int_0^{\pi}\sigma(\vartheta)d_{00}{}^{(l)}(\vartheta)\sin\vartheta d\vartheta. \quad (15)$$

Hence, in all powers of the "matrix" $\Pi(l)$ only the 0–0 element is different from zero and this is the corresponding power of (15). Similarly, it follows from (5a), or more directly from (12), that only the 0 column of

$$\Phi^{(n)}(l)=\int f_n(s)D^{(l)}(s)ds \quad (15a)$$

contains elements which are different from 0. This is, of course, consistent with (14b) and (15).

Let us consider, for instance, the angular distribution obtained by means of Born's first approximation, for Rutherford scattering on a shielded nucleus. For a potential proportional to $r^{-1}e^{-\alpha r}$ the angular distribution becomes, with $\beta=\alpha^2/2k^2$, where k is the wave number of the particle,

$$\frac{\sigma(\vartheta)}{\sigma}=\frac{\beta(2+\beta)}{4\pi(1+\beta-\cos\vartheta)^2}. \quad (16)$$

[3] See, e.g., E. P. Wigner, *Gruppentheorie und ihre Anwendung* (Friedr. Vieweg, Braunschweig, 1931), Chap. XV.

In order to evaluate the integral (15), it is most convenient to write $\cos\vartheta = x$ and to use d_{00} in the form

$$d_{00}{}^{(l)}(\vartheta) = P^l(\vartheta) = \frac{1}{2^l l!}\frac{d^l(x^2-1)^l}{dx^l}. \tag{16a}$$

This gives for $\Pi(l)_{00}$ (we shall omit the indices 0 for convenience)

$$\Pi(l) = 2\pi\int_{-1}^{1}\frac{\beta(2+\beta)}{4\pi(1+\beta-x)^2}\frac{1}{2^l l!}\frac{d^l(x^2-1)^l}{dx^l}dx$$

$$= \frac{\beta(2+\beta)}{2^{l+1}l!}\int_{-1}^{1}\frac{(l+1)!(1-x^2)^l}{(1+\beta-x)^{l+2}}dx. \tag{16b}$$

The second line is obtained by l-fold partial integration. For $l=0$, this gives (as always) $\Pi(0)=1$ for $l=1$,

$$\Pi(1) = 1+\beta-\tfrac{1}{2}\beta(2+\beta)\ln(2+\beta)/\beta. \tag{16c}$$

This is the forward bias after one collision; after n collisions the forward bias in the nth power of this. (16c) has been obtained already in the earlier articles of reference 1.

Let us consider, as a second example, the motion of a particle without energy change in two-dimensional space. The unitary irreducible representations of the corresponding group—the Euclidean group of the plane—are infinite-dimensional and can be characterized by a continuous variable k:

$$D^{(k)}(r,\psi,\varphi)_{m'm} = e^{im'\psi}J_{m'-m}(kr)e^{im(\varphi-\psi)}. \tag{17}$$

The notation is the same as in (3b): the group element (r,ψ,φ) is given by the matrix of the equation preceding (3b) if one sets therein $x=r\cos\psi$, $y=r\sin\psi$, $\varphi_0=\varphi$. The very simple derivation of (16) is given in the Appendix.[4] J_n is the Bessel function of order n. We define again

$$\Pi(k)_{m'm} = \int P(s)D^{(k)}(s)_{m'm}ds$$

$$= \int\sigma(\varphi)r^{-1}e^{-\sigma r}\delta(\psi-\varphi)e^{im'\psi}J_{m'-m}(kr)$$
$$\times e^{im(\varphi-\psi)}rdrd\psi d\varphi; \tag{17a}$$

the $P(s)$ was taken from (3b). The integration over ψ can be carried out at once, and one obtains

$$\Pi(k)_{m'm} = \int_0^{2\pi}\sigma(\varphi)e^{im'\varphi}d\varphi\int_0^{\infty}e^{-\sigma r}J_{m'-m}(kr)dr$$

$$= \frac{\sigma_{m'}}{(k^2+\sigma^2)^{\frac{1}{2}}}\xi^{m'-m}, \tag{17b}$$

where

$$\sigma_m = \int_0^{2\pi}\sigma(\varphi)e^{im\varphi}d\varphi; \quad \xi = \frac{\sigma}{k} - \left(1+\frac{\sigma^2}{k^2}\right)^{\frac{1}{2}}. \tag{17c}$$

[4] For representations of similar groups, see also E. Inonu and E. P. Wigner, Nuovo cimento **9**, 705 (1952).

One sees that $\Pi(k)$ is again of rank 1. One easily proves by induction that

$$[\Pi(k)]^{n+1} = \frac{[2\pi\sigma(0)]^n}{[\sigma^2+k^2]^{\frac{1}{2}n}}\Pi(k), \tag{17d}$$

where $\sigma(0)$ is the differential cross section in the forward direction:

$$2\pi\sigma(0) = \sum\sigma_m. \tag{17e}$$

It would have been possible, of course, to transform (17) in such a way that, similar to (15), only one element of the $\Pi(k)$ matrix would have been different from zero. However, the rather arbitrarily chosen form of (17) has caused no difficulty.

In order to obtain the expansion matrices $\Phi^{(n)}(k)$ of f_n from those of $[\Pi(k)]^n$, we have to multiply the matrix expanding f_0 with $[\Pi(k)]^n$. If the particle is, originally, in the standard state, its expansion matrices are all unit matrices and $\Phi^{(n)}(k)=[\Pi(k)]^n$. Hence, these matrices play a role similar to that of the source solution of diffusion equations. In order to return, from the Φ, to the distribution function f, one has to invert equations of the form (15a). This is a trivial matter if the group is closed and can be done easily also for the representations (17) by means of well-known expansion formulas for Bessel and trigonometric functions. The problem of inverting Eqs. (15a) for not closed groups has been attacked recently in a rather general form by Harish-Chandra and by Segal.[5] In many cases, such as dealt with in Eqs. (16), the expansion matrices have more immediate physical significance than the distribution function f itself.

CALCULATION OF TOTAL DENSITY FUNCTIONS

In many cases one is interested in the total distribution of all particles, irrespective of the number of elementary events they have passed through. Usually, this will be finite only if the elementary process can lead to absorption.

If the probability of absorption $1-\gamma$ is independent of the state of the particle before the elementary event, the total distribution function will be

$$f = \sum\gamma^n f_n. \tag{18}$$

The corresponding momentum matrix becomes

$$\Phi(l) = \int f(s)D^{(l)}(s)ds = \sum\gamma^n\Phi^{(n)}(l)$$

$$= \Phi^{(0)}(l)\sum\gamma^n\Pi(l)^n = \Phi_0(l)[1-\gamma\Pi(l)]^{-1}. \tag{18a}$$

The right side of (18a) can be written down at once in all cases considered in the preceding sections.

[5] I. E. Segal, Ann. Math. **52**, 272 (1950) and Harish-Chandra, Proc. Natl. Acad. Sci. U.S. **37**, 813 (1951).

APPLICABILITY OF THE METHOD

It has been realized already by Grosjean that one can successively evaluate the distribution functions for multiple scattering, if one wishes to take the energy changes into account, most easily if the probability of a certain fractional change in energy is only a function of the scattering angle. This is the case for elastic scattering on atoms at rest. It seems natural to ask, therefore, whether the calculations presented above, if applied to a larger than the Euclidean group, could be given a physical interpretation. We shall not try to solve this problem in general but point only to the simplest generalization of the Euclidean group, that formed by the matrices

$$s=\begin{Vmatrix} cr & \mathbf{a} \\ 0 & 1 \end{Vmatrix}. \tag{19}$$

In (19), r is a real orthogonal matrix, $\mathbf{a}$ is a real vector, $c>0$. For $c=1$, the s form the Euclidean group; dropping this restriction increases the number of group parameters by 1. Clearly, c will be connected in some way with the velocity of the particle.

The condition of invariance means that the transition probability from s_1 to s_2 be equal, for all u, to the transition probability from us_1 to us_2. It is enough to demand this for a set of u from which all group elements can be obtained by multiplication. We choose the following u:

$$u_1=\begin{Vmatrix} 1 & \mathbf{a} \\ 0 & 1 \end{Vmatrix}, \quad u_2=\begin{Vmatrix} r & 0 \\ 0 & 1 \end{Vmatrix}, \quad u_3=\begin{Vmatrix} c1 & 0 \\ 0 & 1 \end{Vmatrix}. \tag{19a}$$

The above condition, applied to $u=u_1$ shows that the transition probability from the state $(c_1,r_1,\mathbf{a}_1)$ to $(c_2,r_2,\mathbf{a}_2)$ is equal to the transition probability from $(c_1, r_1, \mathbf{a}_1+\mathbf{a})$ to $(c_2, r_2, \mathbf{a}_2+\mathbf{a})$. This shows that $\mathbf{a}$ plays the role of the position vector of the particle, or is proportional to the position vector. The invariance condition is also satisfied with respect to u_2 if r continues to describe the direction of the velocity of the particle. Application to u_3 gives, finally, that the transition probabilities

$$(c_1,r_1,\mathbf{a}_1)\rightarrow(c_2,r_2,\mathbf{a}_2) \text{ and } (cc_1,r_1,c\mathbf{a}_1)\rightarrow(cc_2,r_2,c\mathbf{a}_2) \tag{19b}$$

are equal for all c. In other words, the probability of a path of length $|c(\mathbf{a}_1-\mathbf{a}_2)|$ at the value cc_1 of the first parameter is as great as the probability of the path $|\mathbf{a}_1-\mathbf{a}_2|$ at the value c_1 of the first parameter. Expressed still differently, the mean free path is proportional to the first parameter. For the first parameter itself, the change from c_1 to c_2 is as probable as from cc_1 to cc_2. The simplest interpretation of this is that

$$c=Cv^n; \tag{19c}$$

that is, c, and hence also the mean free path, is proportional to the n'th power of the velocity where $n\neq 0$ but can be arbitrary otherwise. Together with the preceding conditions, condition (19b) then stipulates that the angular distribution of the scattering be independent of the velocity and that the probability of a certain fractional change in energy depend only on the change in the direction of the velocity.

It seems likely that many other multiple scattering distributions can be calculated accurately, many others approximately, following the procedure outlined above. So far, no case has been encountered in which the calculation of the nth power of II, or of the reciprocal in (18a), would have been at all difficult. On the other hand, it does not appear to be possible to use the rotation (or Euclidean) group to calculate the multiple scattering of a particle with spin, by using the variable ϵ of (12) to describe the spin's motion. It seems that a more powerful method is needed to overcome the difficulties of this problem.

APPENDIX

The irreducible representations of the group of motions in two dimensions can most easily be given in the Hilbert space of functions $f(\alpha)$ the variable α of which is restricted to the interval $0\leq\alpha<2\pi$. The operators $P_s=P_{r,\psi,\varphi}$ of the representation in this space are defined by

$$P_sf(\alpha)=P_{r,\psi,\varphi}f(\alpha)=e^{-ik(r\cos\psi\cos\alpha+r\sin\psi\sin\alpha)}f(\alpha-\varphi)$$
$$=e^{-ikr\cos(\psi-\alpha)}f(\alpha-\varphi). \tag{A1}$$

The group element s is a rotation by φ about the origin, followed by a displacement with a vector the polar coordinates of which are r and ψ. Borrowing concepts from quantum mechanics, one can say that f describes states in which the absolute value of the momentum is k but the direction of the momentum is variable and given by α. Rotation of this state by φ replaces α by $\alpha-\varphi$; displacement by the vector $r\cos\psi, r\sin\psi$ multiplies it with the exponential in (A1). The magnitude of k characterizes the representation; it can assume any positive value.

One can write (A1) also in the form

$$P_sf(\alpha)=\int\Delta^{(k)}(r,\psi,\varphi)_{\alpha\alpha'}f(\alpha')d\alpha', \tag{A2}$$

with a singular representation matrix:

$$\Delta^{(k)}(r,\psi,\varphi)_{\alpha\alpha'}=e^{-ikr\cos(\psi-\alpha)}\delta(\alpha-\varphi,\alpha'), \tag{A2a}$$

and the calculation of the text can be carried out also using this form of $\Delta^{(k)}$. The form of $D^{(k)}$ given in the text is obtained by using, instead of $f(\alpha)$, its Fourier expansion:

$$f(\alpha)=\sum_m f_me^{-im\alpha}. \tag{A3}$$

Substitution of this into (A1) gives

$$P_{r,\psi,\varphi}\sum_m f_m e^{-im\alpha}=\sum_m e^{-ikr\cos(\psi-\alpha)}e^{im(\varphi-\alpha)}f_m$$

$$=\sum_{mm'}D^{(k)}(r,\psi,\varphi)_{m'm}f_m e^{-im'\alpha}, \quad \text{(A4)}$$

where

$$D^{(k)}(r,\psi,\varphi)_{m'm}=(2\pi)^{-1}\int_0^{2\pi}e^{-ikr\cos(\psi-\alpha)}e^{im(\varphi-\alpha)}e^{im'\alpha}d\alpha$$

$$=(2\pi)^{-1}\int_0^{2\pi}e^{-ikr\cos\beta-i(m'-m)\beta}d\beta \times e^{i(m'-m)\psi+im\varphi}. \quad \text{(A5)}$$

Because of

$$J_n(z)=(2\pi)^{-1}i^{-n}\int_0^{2\pi}e^{-iz\cos\beta-in\beta}d\beta, \quad \text{(A5a)}$$

this differs from the expression given in the text only by the factor $i^{m'-m}$ which can be eliminated by a similarity transformation. It is worth noting that the transition from (17b) to (17c), i.e., the evaluation of the second integral of (17b), can be best carried out using the form (A5a) for J.

A similar calculation is possible also in the three-dimensional case but it is more laborious and will not be given here. It is, essentially, contained in the Appendix to Grosjean's last article.

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Calculated Values of the Parameters of Noble Gas Discharges

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Molar properties of gaseous discharges are determined by the transport cross section which is expressible in terms of the phase shifts defined in electron-atom scattering theory.

$\sigma_t(v)$ was calculated for helium, neon, and argon using values of phase shifts η_0 to η_6 derived by Westin from existing scattering data. Druyvesteyn distribution functions were calculated for E/p from 0.1 to 1.0. Average values were then calculated for collision frequency, drift velocity, diffusion coefficient, and average energy.

Also, collision frequency and elastic energy loss were calculated using Maxwell distributions for kT_e up to 10 ev. These calculations are particularly useful at very low energy, where extrapolation of phase shifts is more reliable than of the experimental data. Values of σ_t are compared with those of Barbiere (taken from direct scattering data); values of drift velocity are somewhat lower than Nielsen's data.

I. INTRODUCTION

THE theory of gas discharges reduces essentially to a study of the various types of collisions which take place among the electrons, atoms, and ions. In many practical cases where the average electron energy is low, the properties of the discharge are determined primarily by the differential cross section $\sigma(k,\vartheta)$ for elastic scattering of an electron by an atom. Here k is the electron wave number, and ϑ the angle through which the electron is deviated.

The ordinary total cross section[1] $\sigma(k)$ is defined by

$$\sigma(k)=\int_0^{\pi}\sigma(k,\vartheta)2\pi\sin\vartheta d\vartheta. \quad (1)$$

In gas discharges, however, since the energy lost by an electron in an elastic collision depends on ϑ, a more important quantity is the transport cross section $\sigma_t(k)$ defined by

$$\sigma_t(k)=\int_0^{\pi}\sigma(k,\vartheta)(1-\cos\vartheta)2\pi\sin\vartheta d\vartheta. \quad (2)$$

(This quantity is also referred to as the diffusion or momentum transfer cross section.) From collision theory, these two cross sections can be expressed[2] in terms of the phase shifts η_n introduced by the atom in the partial waves associated with successive units of angular momentum of the incident electron about the center of the atom:

$$\sigma(k)=\frac{4\pi}{k^2}\sum_{n=0}^{\infty}(2n+1)\sin^2\eta_n, \quad (3)$$

$$\sigma_t(k)=\frac{4\pi}{k^2}\sum_{n=0}^{\infty}(n+1)\sin^2(\eta_n-\eta_{n+1}). \quad (4)$$

Most of the molar properties of the electrons in a discharge, such as drift velocity, diffusion coefficient, etc., are given by some function of σ_t and k, averaged over a distribution function which in turn involves σ_t. Hence, a knowledge of the phase shifts η_n permits calculation of σ_t and this, in turn, enables one to derive the

[1] N. F. Mott and H. S. W. Massey, *Theory of Atomic Collisions* (Oxford University Press, London, 1949), Chap. 2.

[2] H. S. W. Massey and C. B. O. Mohr, Proc. Roy. Soc. (London) A144, 434 (1933).