Theory of the (d,p) Reaction*

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The Born approximation treatment of the (d,p) reaction is modified so as to eliminate integrations over the interior of the target nucleus where the distortion of the incident wave is most severe. This treatment yields the S. T. Butler result when certain additional physical assumptions are made. An approximate expression for the (d,p) cross section is given in which (a) the Coulomb interaction is taken into account, (b) the effects due to the interaction of the deuterons and the protons with the target nucleus are expressed in terms of the boundary conditions for the wave functions of these particles at the nuclear surface, and (c) the effects arising from the fact that the mass of the target nucleus is not infinite are not entirely neglected. The methods presented can be used to get the corresponding result for the (d,n) reaction.

THERE are two approximate treatments of the (d,p) reaction that currently have been of interest. One is the boundary condition matching treatment introduced by Butler[1] and the other is the Born approximation treatment.[2] When the Coulomb interaction and the interaction between the proton and the residual nucleus are neglected, the boundary condition matching treatment gives a simple analytic expression for the (d,p) cross section which agrees surprisingly well with experiment. This result has proved to be a useful tool for determining the spins and parities of nuclear states. However, when the effects mentioned above are not neglected, the boundary condition matching cross section involves integrals which are very difficult to evaluate.

On the other hand, the Born approximation (d,p) cross section is not much more complicated when the Coulomb interaction and the interaction between liberated particle and the target nucleus are taken into account. It is also a simple matter to include the effects due to the scattering of the incident deuteron beam in the Born approximation treatment. However, the Born approximation involves replacing the true wave function describing the interaction of a beam of deuterons with the target nucleus Ψ by the wave function for the unperturbed incident beam Ψ^0. This is a very dubious approximation since there must be considerable distortion of the wave function inside and near the target nucleus.

We demonstrate below that the interior of the target nucleus can be eliminated from the Born approximation calculation. This cuts out the region where the distortion of the incident wave is most severe so that replacing Ψ by Ψ^0 in the modified treatment may not be a bad approximation.

Elimination of the interior of the target nucleus from the treatment introduces a set of parameters which may be approximately identified with the scattering amplitudes for free particles interacting with the residual nucleus. If one assumes that the liberated particles do not interact with the residual nucleus and sets these amplitudes equal to zero, neglects the Coulomb interaction, and assumes the n-p interaction to have zero range, then our (d,p) cross section reduces to the boundary condition matching expression for the (d,p) cross section.

We will discuss the (d,p) process only, but the extension of the discussion to the (d,n) process is straightforward.

II. GENERAL EXPRESSION FOR THE (d,p) CROSS SECTION

To simplify the following discussion we will neglect the Coulomb interaction, assume all the particles under discussion have zero spin, and assume that the mass of the target nucleus is infinite. These approximations will be removed in Sec. IV of this paper.

Our object is to calculate the cross section for a deuteron to interact with a target nucleus to yield a final state in which the neutron is captured by the target nucleus with (negative) energy $E_f=\hbar^2K_N^2/2M$ and the proton emerges with asymptotic momentum $\hbar\mathbf{K}_P$. K_N and K_P are related by the conservation of energy

$$\frac{\hbar^2K_N^2}{2M}+\frac{\hbar^2K_P^2}{2M}=E, \tag{1}$$

where E is the kinetic energy of the incident deuteron minus its binding energy. Consequently, all protons liberated with asymptotic momentum $\hbar\mathbf{K}_P$ must be associated with neutrons captured with energy $\hbar^2K_N^2/2M$. Thus our problem is simply to calculate the flux of protons liberated with asymptotic momentum $\hbar\mathbf{K}_P$.

Let $\Psi(\xi,\mathbf{r}_N,\mathbf{r}_P)$ be the wave function describing the interaction of a beam of deuterons with the target nucleus. $\mathbf{r}_N$ and $\mathbf{r}_P$ are the coordinates of the neutron and proton, respectively, measured from the center of

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[1] S. T. Butler, Proc. Roy. Soc. (London) **A208**, 559 (1951); F. L. Friedman and W. Tobocman, Phys. Rev. **92**, 93 (1953).

[2] A. B. Bhatia *et al.*, Phil. Mag. **43**, 485 (1952); R. Huby, Nature **166**, 552 (1950); Proc. Roy. Soc. (London) **A215**, 385 (1952); P. B. Daitsch and J. B. French, Phys. Rev. **87**, 900 (1952); N. Austern, Phys. Rev. **89**, 318 (1953); E. Gerjuoy, Phys. Rev. **91**, 645 (1953).

mass of the target nucleus. The coordinates of the A nucleons making up the target nucleus are lumped together in the symbol ξ. Let

$$\psi_f(\xi,\mathbf{r}_N)=\sum_{lm}\psi_f^{lm}(\xi,r_N)Y_l^m(\theta_N)$$

be the wave function for the residual nucleus. Then the wave function for the protons liberated as a result of neutrons being captured into the state ψ_f is

$$\mathfrak{f}_f(\mathbf{r}_P)=\int d\mathbf{r}_N d\xi \psi_f^*\Psi \quad (2)$$

for large r_P. The current per unit solid angle of these protons for large r_P is

$$F_f(\mathbf{K}_P)=\frac{\hbar}{2Mi}r_P^2\left(\mathfrak{f}_f^*\frac{\partial}{\partial r_P}\mathfrak{f}_f-\mathfrak{f}_f\frac{\partial}{\partial r_P}\mathfrak{f}_f^*\right), \quad (3)$$

so that the (d,p) cross section is

$$\sigma(\mathbf{K}_p)=\sum_f\frac{F_f(\mathbf{K}_P)}{\mathcal{J}}. \quad (4)$$

$\mathcal{J}$ is the current density of the incident deuteron beam and the sum over f includes all states of the residual nucleus having energy E_f. We choose our normalization so that the asymptotic form of the incident deuteron beam is the plane wave,

$$\Psi^0=\chi(|\mathbf{r}_P-\mathbf{r}_N|)\Phi_I(\xi)\exp[i\mathbf{K}_D\cdot\tfrac{1}{2}(\mathbf{r}_P+\mathbf{r}_N)]. \quad (5)$$

Therefore

$$\mathcal{J}=\hbar K_D/M_D \quad (6)$$

and

$$\sigma(\mathbf{K}_P)=\frac{M_D}{\hbar K_D}\sum_f F_f(\mathbf{K}_P). \quad (7)$$

The wave function Ψ is a solution of the Schrödinger equation,

$$(H_I+T_N+T_P+V_{IN}+V_{IP}+V_{NP}-E)\Psi=0, \quad (8)$$

where H_I is the Hamiltonian operator for the target nucleus, T is the kinetic energy operator, and V_{ij} is the operator for the energy of interaction of particles i and j. Expand Ψ in terms of the complete set of eigenfunctions of the operator $H_I+T_N+V_{IN}$.

$$\Psi(\xi,\mathbf{r}_N,\mathbf{r}_P)=\sum_i \mathfrak{f}_i(\mathbf{r}_P)\psi_i(\xi,\mathbf{r}_N), \quad (9)$$

where the sum is understood to include an integral over the continuum of unbound states. Combining Eqs. (8) and (9) gives the Schrödinger equation for $\mathfrak{f}_f$:

$$\begin{aligned}\left(T_P-\frac{\hbar^2K_P^2}{2M}\right)\mathfrak{f}_f(\mathbf{r}_P)&=-\int d\mathbf{r}_N d\xi\psi_f^*[V_{NP}+V_{IP}]\Psi\\&=-\sum_i\langle f|V_{NP}+V_{IP}|i\rangle\mathfrak{f}_i(\mathbf{r}_P)\\&=-V_{FP}\mathfrak{f}_f(\mathbf{r}_P).\end{aligned} \quad (10)$$

This is just the form of the $(A+2)$-nucleon Schrödinger equation we would use to discuss the scattering of free protons by the residual nucleus.

Using the Green's function for the operator $T_P-(\hbar^2K_P^2/2M)$, we can transform Eq. (10) into an integral equation:

$$\begin{aligned}\mathfrak{f}_f(\mathbf{r})&=-\frac{M}{2\pi\hbar^2}\int d\mathbf{r}_P\frac{e^{iK_P|\mathbf{r}_P-\mathbf{r}|}}{|\mathbf{r}_P-\mathbf{r}|}V_{FP}\mathfrak{f}_f(\mathbf{r}_P)\\&=-\frac{M}{2\pi\hbar^2}\int d\mathbf{r}_P d\mathbf{r}_N d\xi\frac{e^{iK_P|\mathbf{r}_P-\mathbf{r}|}}{|\mathbf{r}_P-\mathbf{r}|}\psi_f^*[V_{NP}+V_{IP}]\Psi.\end{aligned} \quad (11)$$

Incorporated into Eq. (11) is the asymptotic boundary condition that all the liberated protons are outgoing at infinity. Now substituting Eq. (11) into Eq. (7) gives

$$\sigma(\mathbf{K}_P)=\frac{MM_DK_P}{(2\pi)^2\hbar^4K_D}\sum_f\left|\int d\mathbf{r}_P d\mathbf{r}_N d\xi e^{-i\mathbf{K}_P\cdot\mathbf{r}_P}\times\psi_f^*[V_{NP}+V_{IP}]\Psi\right|^2. \quad (12)$$

Replacing Ψ in Eq. (12) by Ψ^0, the wave function for the incident beam of deuterons, gives the Born approximation result for the (d,p) cross section. As long as the cross section for interaction with the target nucleus is small, Ψ^0 should be a fair approximation to Ψ outside the range of V_{IP} and V_{IN}. However, inside the target nucleus the wave function must be considerably distorted so that the Born approximation breaks down. Even if one believed the Born approximation, one would not be able to evaluate the stripping cross section because the form of ψ_f inside the target nucleus is not known.

We propose to modify the Born approximation treatment by eliminating (to a good approximation) the troublesome range of integration in favor of a set of parameters involving the logarithmic derivative of $\mathfrak{f}_f$ at the nuclear surface.

First we analyze $\mathfrak{f}_f$ into a sum of spherical harmonics:

$$\mathfrak{f}_f(\mathbf{r}_P)=\sum_{L,M}\mathfrak{f}_{fLM}(r_P)Y_L^M(\theta_P). \quad (13)$$

Introducing this expansion into Eq. (11) gives

$$\mathfrak{f}_{fLM}(r)=\frac{2K_PM}{i\hbar^2}\int d\mathbf{r}_P j_L(K_Pr_<)h_L^{(1)}(K_Pr_>)\times Y_L^{-M}(\theta_P)V_{FP}\mathfrak{f}_f(\mathbf{r}_P), \quad (14)$$

where

$$r_<=\begin{cases}r_P & r_P<r\\ r & r<r_P\end{cases} \quad\text{and}\quad r_>=\begin{cases}r_P & r_P>r\\ r & r>r_P\end{cases}.$$

In this expression j_L represents the spherical Bessel function and $h_L^{(1)}$ represents the spherical Hankel function of the first kind. Now for $r>R$, Eq. (14) may be written

$$\mathfrak{f}_{fLM}(r)=\frac{2K_PM}{i\hbar^2}\int_R^\infty d\mathbf{r}_P j_L(K_Pr_<)h_L^{(1)}(K_Pr_>)\times Y_L^{-M}(\theta_P)V_{FP}\mathfrak{f}_f(\mathbf{r}_P)-I_{fLM}h_L^{(1)}(K_Pr), \quad (15)$$

where I_{fLM} does not depend on r. I_{fLM} can be expressed in terms of the first term on the right of the above equation and the logarithmic derivative of $\mathfrak{f}_{fLM}$ at $r=R$. This can be accomplished by means of the Schrödinger equation and Green's theorem. However, it is simpler to take the logarithmic derivative of both sides of Eq. (15), set $r=R$, and solve for I_{fLM}. The result is

$$I_{fLM}=\beta_{fLM}^*\frac{2K_PM}{i\hbar^2}\int_R^\infty d\mathbf{r}_P h_L^{(1)}(x_Pr_P)Y_L^{-M}(\theta_P)\times V_{FP}\mathfrak{f}_f(r_P), \quad (16)$$

where

$$\beta_{fLM}^*=\left[\frac{\frac{\partial}{\partial r}j_L(K_Pr)-j_L(K_Pr)\frac{\partial}{\partial r}\log\mathfrak{f}_{fLM}(r)}{\frac{\partial}{\partial r}h_L^{(1)}(K_Pr)-h_L^{(1)}(K_Pr)\frac{\partial}{\partial r}\log\mathfrak{f}_{fLM}(r)}\right]_{r=R}. \quad (17)$$

Using Eq. (15) instead of Eq. (11) in Eq. (7) gives us an alternate expression for the (d,p) cross section:

$$\sigma(\mathbf{K}_P)=\frac{MM_DK_P}{(2\pi)^2\hbar^4K_D}\sum_f|A_f|^2,$$
$$A_f=\int d\mathbf{r}_Nd\xi\int_R^\infty d\mathbf{r}_P\mathcal{E}^*(\mathbf{K}_P,\mathbf{r}_P)\times\psi_f^*(\xi,\mathbf{r}_N)[V_{NP}+V_{IP}]\Psi=\int_R^\infty d\mathbf{r}_P\mathcal{E}^*(\mathbf{K}_P,\mathbf{r}_P)V_{FP}\mathfrak{f}_f(\mathbf{r}_P), \quad (18)$$

where

$$\mathcal{E}^*(\mathbf{K}_P,\mathbf{r}_P)=e^{-i\mathbf{K}_P\cdot\mathbf{r}_P}-4\pi\sum_{L,M}i^{-L}Y_L^M(\theta_{KP})\times Y_L^{-M}(\theta_P)\beta_{fLM}^*h_L^{(1)}(K_Pr_P)=4\pi\sum_{L,M}i^{-L}Y_L^M(\theta_{KP})Y_L^{-M}(\theta_P)\times[j_L(K_Pr_P)-\beta_{fLM}^*h_L^{(1)}(K_Pr_P)]. \quad (19)$$

Associated with the target nucleus and the residual nucleus are characteristic radii R_I and R_F such that $V_{IP}=0$ for $r_P>R_I$ and $V_{FP}=0$ for $r_P>R_F$. We wish to choose R no smaller than R_I so that V_{IP} drops out of the stripping amplitude. We would like to choose R as large as possible since the difference between Ψ and Ψ^0 tends to decrease as we move away from the target nucleus, and we intend to replace Ψ by something like Ψ^0 eventually. However, we cannot choose $R>R_F$, since then our expression for the scattering amplitude A_f becomes simply $\mathfrak{f}_f(R)$ which is of no use to us. We will find that the choice $R=R_I$ is most convenient.

The elimination of V_{IP} by choosing $R=R_I$ leads to the opportunity for an important simplification of the scattering amplitude A_f. With V_{IP} absent, the integrand is proportional to V_{NP}. The short range of V_{NP} and the fact that $r_P>R_I$ cuts down the contribution to the integral due to the range $0<r_N<R_I$. In fact, assuming V_{NP} to have zero range, as is often done, completely removes the interior of the target nucleus from the ranges of the integrations over r_N and r_P. Thus we have to a good approximation eliminated the interior of the target nucleus by introducing the parameters β_{fLM}^*.

III. INTERPRETATION OF THE PARAMETERS β_{fLM}^*

We have seen that $\mathfrak{f}_f(\mathbf{r}_P)$ satisfies the Schrödinger equation for a proton interacting with the residual nucleus. However we cannot immediately conclude that $[(\partial/\partial r)\log\mathfrak{f}_{fLM}(r)]_{RI}$ is the logarithmic derivative appropriate to the scattering of free protons out of an incident beam by the residual nucleus. First of all R_I is smaller than the radius of the residual nucleus and secondly the boundary conditions for the two problems are not obviously the same.

Actually the difference between R_I and R_F cannot be very great. Taking $R_I=NA^{\frac{1}{3}}$ and $R_F=N(A+1)^{\frac{1}{3}}$ gives

$$\frac{R_F-R_I}{R_F}\approx\frac{1}{3(A+1)}, \quad (20)$$

so that

for carbon, $R_F-R_I\approx0.026R_F$;
for aluminum, $R_F-R_I\approx0.012R_F$;
for iron, $R_F-R_I\approx0.006R_F$.

Since the logarithmic derivative of $\mathfrak{f}_{fLM}$ must be continuous, we expect that its value must be essentially the same at R_I as at R_F.

In order for the boundary conditions for $\mathfrak{f}_{fLM}(r_P)$ at $r_P=R_I$ to be that same as those for free incident protons, all the protons represented by $\mathfrak{f}_f$ must originate at values of r_P greater than R_I.

Because of the large extension of the deuteron and its small binding energy, the deuteron bond must be broken as soon as one member interacts with the target nucleus. In this process the other member of the deuteron is left free just outside the surface of the residual nucleus. Thus we can picture two sequences for the (d,p) process. (a) The neutron interacts with the target nucleus to form the residual nucleus leaving the proton outside. (b) The proton interacts with the target nucleus leaving the neutron free outside. The neutron is subsequently captured by the proton + target nucleus system and following this the proton is ejected to leave the residual nucleus. Sequence (a) is called stripping. Clearly sequence (b) is much less probable than sequence (a). Sequence (a) is a one-step process while sequence (b) is a three-step process. The one step of (a) is equally as probable as the first step of (b). Taking the Coulomb interaction into account will tend to favor the one step of (a) over the first step of (b). But then the probability for each of the two succeeding steps of sequence (b) is much less than one. We conclude that sequence (b) is of little importance for the (d,p) reaction. Thus almost all the protons are liberated outside $r_P=R_I$. This is our reason for choosing $r_P=R_I$.

Assuming that all the protons are liberated outside the target nucleus allows us to identify $[(\partial/\partial r)\times\log\mathfrak{f}_{fLM}(r)]_{R_I}$ with the logarithmic derivative appropriate for free protons incident on the residual nucleus. From what we have just said this assumption is equivalent to neglecting the contribution of compound nucleus formation to the (d,p) reaction. The extent to which $[(\partial/\partial r)\log\mathfrak{f}_{fLM}(r)]_{R_I}$ can be identified with the logarithmic derivative for incident free protons is a measure of the importance of stripping for the (d,p) reaction.

We have shown that although R_I is smaller than R_F, the difference between them is very small so that the logarithmic derivative of $\mathfrak{f}_{fLM}$ at R_I is very nearly the same as the logarithmic derivative at R_F. We have also shown that practically all the protons produced in the (d,p) process are liberated outside $r_P=R_I$. Consequently the logarithmic derivative of $\mathfrak{f}_{fLM}$ at $r_P=R_I$ is essentially the same as the logarithmic derivative appropriate to the interaction of free protons incident on the residual nucleus.

The value of the logarithmic derivative of $\mathfrak{f}_{fLM}$ is given by the particular theory of nuclear reactions we choose to employ. For example, we may assume that the surface of the residual nucleus presents an inpenetrable barrier to the liberated protons. This was done by J. Horowitz and A. M. L. Messiah [J. phys. radium **14**, 695 (1953)]. Alternatively, we may use the continuum theory of nuclear reactions[3] to estimate the logarithmic derivatives. According to this theory, we should write

$$\left[\frac{\partial}{\partial r}\log\mathfrak{f}_{fLM}(r)\right]_{R_I}=-\frac{[1+iKR_I]}{R_I},\tag{21}$$

where

$$K=(K_0^2+K_P^2)^{\frac{1}{2}},\quad K_0\approx 1\times 10^{-13}\ \mathrm{cm}^{-1}.$$

Equation (17) now becomes

$$\beta_{fLM}{}^*=\frac{j_L(K_PR_I)}{h_L^{(1)}(K_PR_I)}\times\left[\frac{R_I(\partial/\partial r)\log j_L(K_Pr)+1+iKR_I}{R_I(\partial/\partial r)\log h_L^{(1)}(K_Pr)+1+iKR_I}\right]_{R_I}.\tag{23}$$

If one supposes that the liberated protons have no interaction with the residual nucleus, one sets $\beta_{fLM}{}^*=0$. It will be shown below that this assumption leads to the boundary condition matching result for the (d,p) cross section.

IV. EFFECTS DUE TO SPIN, THE COULOMB INTERACTION, AND THE FINITE MASS OF THE TARGET NUCLEUS

The derivation of the previous sections can be carried out without neglecting the effects listed above. We will not give the details of the calculation, but we will merely state the results. For this purpose we write Eq. (18) for the (d,p) cross section in the following form:

$$\sigma(\mathbf{K}_P)=\frac{M^2M_DK_P}{2\pi^2\hbar^6RK_D}\sum_f\Big|\sum_{l,m}\sqrt{\gamma_l}\,B_f{}^{lm}\Big|^2,\tag{23}$$

where

$$B_f{}^{lm}=\int d\mathbf{r}_Nd\xi\int_{R_I}^{\infty}d\mathbf{r}_P\frac{\psi_f{}^{lm*}(\xi,r_N)}{\phi_f{}^{lm*}(R_I)}Y_l{}^{-m}(\theta_N)\times\mathcal{E}^*(\mathbf{K}_P,\mathbf{r}_P)[V_{NP}+V_{IP}]\Psi,\tag{24}$$

$$\gamma_l=\frac{\hbar^2R_I}{2M}|\phi_f{}^{lm}(R_I)|^2,\tag{25}$$

$$\phi_f{}^{lm}(R_I)=\int d\xi\Phi_I{}^*(\xi)\psi_f{}^{lm}(\xi,R_I),$$

$\Phi_I(\xi)=$ the wave function for the target nucleus.

To take the Coulomb interaction into account, one merely replaces the proton wave functions by their Coulomb analogs. Thus the function $\mathcal{E}^*(\mathbf{K}_P,\mathbf{r}_P)$ is replaced by

$$\mathcal{E}^{c*}(\mathbf{K}_P,\mathbf{r}_P)=4\pi\sum i^{-L}Y_L{}^M(\theta_{KP})Y_L{}^{-M}(\theta_P)\times e^{-i\sigma_L(\eta_P)}\left\{\frac{F_L(\eta_P,K_Pr_P)-\beta_{LM}{}^{c*}H_L(\eta_P,K_Pr_P)}{K_Pr_P}\right\},\tag{26}$$

where

$$\beta_{LM}{}^{c*}=\left[\frac{\dfrac{\partial}{\partial r}\dfrac{F_L(\eta_P,K_Pr)}{K_Pr}-\dfrac{F_L(\eta_P,K_Pr)}{K_Pr}\dfrac{\partial}{\partial r}\log\mathfrak{f}_{fLM}(r)}{\dfrac{\partial}{\partial r}\dfrac{H_L(\eta_P,K_Pr)}{K_Pr}-\dfrac{H_L(\eta_P,K_Pr)}{K_Pr}\dfrac{\partial}{\partial r}\log\mathfrak{f}_{fLM}(r)}\right]_{r=R_I},$$

[3] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952), p. 340.

$$\eta_P = Ze^2M/\hbar^2K_P,$$

$$\sigma_L(\eta_P) = \arg\{\Gamma(L+1+i\eta_P)\},$$

$$H_L(\eta,\rho) = F_L(\eta,\rho) - iG_L(\eta,\rho).$$

F_L and G_L are defined as the regular and irregular solutions of the differential equation

$$\left\{\frac{d^2}{d\rho^2}+\left[1-\frac{2\eta}{\rho}-\frac{L(L+1)}{\rho^2}\right]\right\}F_L(\eta,\rho)=0.$$

They are normalized so that they have the asymptotic forms

$$F_L(\eta,\rho)\to\sin[\rho-\eta\ln 2\rho-\tfrac{1}{2}L\pi+\sigma_L(\eta)],$$

$$G_L(\eta,\rho)\to\cos[\rho-\eta\ln 2\rho-\tfrac{1}{2}L\pi+\sigma_L(\eta)],$$

in the limit as $\rho\to\infty$.

The effect of taking the finiteness of the mass of the target nucleus into account is to

(a) introduce the factor

$$(1+M_D/M_I)^{-1}(1+M/M_I)^{-1}[1+M/(M+M_I)]^{-1}$$

into the expression for the cross section,

(b) replace $\mathbf{r}_P$ by $\boldsymbol{\varrho}_P=\mathbf{r}_P-\mathbf{r}_NM/(M+M_I)$ in the argument of $\mathcal{E}^{c*}$ and as the variable of integration in B_f^{lm}, and

(c) replace $V_{NP}+V_{IP}$ by $V_{NP}+\Delta V=V_{NP}+V_{IP}+Ze^2(1/r_P-1/\rho_P)$ in B_f^{lm}.

For $M_I=\infty$, ΔV makes no contribution to B_f^{lm}. For finite M_I the contribution of ΔV to B_f^{lm} cannot be very large since while ρ_P ranges from R_I to ∞ the fact that ψ_f^{lm} describes a bound state of the residual nucleus means that r_N is restricted to a relatively small region around the origin. It follows that $\mathbf{r}_P-\boldsymbol{\varrho}_P\cong\mathbf{r}_N/(A+1)$ is small and consequently ΔV must be negligible over practically the entire range of integration.

Let I be the spin of the target nucleus, and let J be the spin of the residual nucleus. Taking the conservation of angular momentum and parity into account gives in place of Eq. (23)

$$\sigma(\mathbf{K}_P)=\frac{M^2M_DK_P}{2\pi^2\hbar^6R_IK_D}\sum_{l,m}{}'\frac{(2J+1)\gamma_l}{2(2I+1)(2l+1)}|B_f^{lm}|^2. \quad (27)$$

The primed summation symbol is used to indicate that the sum over l is restricted to the range $||I-J|-\frac{1}{2}|\leq l\leq I+J+\frac{1}{2}$, l being restricted to even integers or odd integers depending on the change of parity. This simple result is the consequence of the assumption that ΔV is not spin dependent and the assumption that it is a good approximation to replace Ψ by a wave function having the same spin dependence as the incident deuteron beam.

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We conclude that the expression which should be compared with experiment is

$$\sigma(\mathbf{K}_P)=\frac{[1+M/(M+M_I)]^{-1}}{(1+M/M_I)(1+M_D/M_I)}\frac{M_DM^2}{(2\pi)^2\hbar^6R_I}\times\frac{(2J+1)}{(2I+1)}\frac{K_P}{K_D}\sum_{l,m}{}'\frac{\gamma_l}{2l+1}|B_f^{lm}|^2, \quad (28)$$

where

$$B_f^{lm}=\int d\mathbf{r}_Nd\xi\int_{R_I}^{\infty}d\boldsymbol{\varrho}_P\frac{\psi_f^{lm*}(\xi,r_N)}{\phi_f^{lm*}(R_I)}Y_l^{-m}(\theta_N)\times\mathcal{E}^{c*}(\mathbf{K}_P,\boldsymbol{\varrho}_P)[V_{NP}+\Delta V]\Psi, \quad (29)$$

J = spin of the residual nucleus, I = spin of the target nucleus, and γ_l = the reduced width of the level into which the neutron is captured. The sum over l is restricted as in Eq. (27). If the level into which the neutron is captured is characterized by a definite value of l as well as definite values of J and parity, then only those terms in Eq. (28) corresponding to that value of l are to be retained.

To evaluate B_f^{lm} we introduce the following approximations. ΔV is neglected so that the effects due to the finite mass of the target nucleus are only partially taken into account. The (unknown) terms in Ψ representing the dissociation and distortion of deuterons are dropped. That is to say, Ψ is approximated by the expression $\Psi_{\text{c.m.}}(\frac{1}{2}(\mathbf{r}_N+\mathbf{r}_P))\cdot\chi(\mathbf{r}_P-\mathbf{r}_N)\Phi_I(\xi)$, where χ is the normalized wave function for the internal motion of an unperturbed deuteron, Φ_I is the wave function for the target nucleus, and $\Psi_{\text{c.m.}}$ describes the motion of the centers of mass of the incident and scattered deuterons. For $\Psi_{\text{c.m.}}$ we write

$$\Psi_{\text{c.m.}}(\mathcal{R})=4\pi\sum_{L,M}i^LY_L^{-M}(\theta_{KD})Y_L^M(\theta_{\mathcal{R}})e^{-i\sigma_L}\times\left\{\frac{F_L(\eta_D,K_D\mathcal{R})-\delta_LH_L(\eta_D,K_D\mathcal{R})}{K_D\mathcal{R}}\right\}, \quad (30)$$

where $\mathcal{R}=(\mathbf{r}_P+\mathbf{r}_N)/2$. Lastly, V_{NP} is approximated by the zero range potential. This last step is the source of two important simplifications. First of all, the interior of the target nucleus is completely eliminated from the range of integration. Thus, since r_N is now restricted to a range in B_f^{lm} where it is always greater than R_I, ψ_f^{lm} can be replaced by $\Phi_I(\xi)h_l^{(1)}(K_Nr_N)$.

Secondly, our six-dimensional integral is reduced to a three-dimensional integral. In this three-dimensional integral the integration over angles can be performed leaving an expression for B_f^{lm} which is a sum of one-dimensional integrals.

The result of these approximations is

$$B_f^{lm}=-\frac{\hbar^2(8\pi\alpha)^{\frac{1}{2}}}{M|K_N|^3}\int_{R_I|K_N|(1+M/M_I)}^{\infty}d\mathbf{x}Y_l^{-m}(\theta_x)$$
$$\times\frac{h_l^{(1)}(ix)}{h_l^{(1)}(i|K_N|R_I)}\mathcal{E}^{c*}\left(\mathbf{K}_P,\frac{\mathbf{x}}{1+M/M_I}\right)\Psi_{\text{c.m.}}(\mathbf{K}_D,\mathbf{x})$$
$$=-\frac{4\pi\hbar^2(1+M/M_I)[2\alpha(2l+1)]^{\frac{1}{2}}}{M|K_N|K_DK_P}\sum_{L,\lambda}P_L^{|m|}(\cos\theta_{KP})$$
$$\times i^{\lambda-L}\exp(-i[\sigma_L(\eta_P)+\sigma_\lambda(\eta_D)])\Gamma_{L\lambda}^{lm}f_{L\lambda}^{l},\qquad(31)$$

where

$$\Gamma_{L\lambda}^{lm}=\frac{(2\lambda+1)(2L+1)}{(2l+1)}\left(\frac{(L-|m|)!}{(L+|m|)!}\right)^{\frac{1}{2}}\times(L\lambda00|L\lambda l0)(L\lambda m0|L\lambda lm),$$

$$f_{L\lambda}^{l}=\int_X^{\infty}dx\frac{h_l^{(1)}(ix)}{h_l^{(1)}\left(\dfrac{iX}{1+M/M_I}\right)}[F_\lambda(\eta_D,ax)-\delta_\lambda H_\lambda(\eta_D,ax)][F_L(\eta_P,bx)-\beta_{Lm}^{c*}H_L(\eta_P,bx)],$$

$$a=\frac{K_D}{|K_N|},\quad b=\frac{K_P}{|K_N|(1+M/M_I)},$$

$$X=R_I|K_N|(1+M/M_I),$$

$$\eta_P=\frac{Ze^2M(M_I+M)}{\hbar^2K_P(M_I+2M)},$$

$$\eta_D=\frac{Ze^2M_DM_I}{\hbar^2K_D(M_I+M_D)},$$

$\sigma_L(\eta)=\arg\Gamma(1+L+i\eta)$,
$H_L(\eta,\rho)=F_L(\eta,\rho)-iG_L(\eta,\rho)$,
$F_L(\eta,\rho)=$ the regular radial Coulomb function,
$G_L(\eta,\rho)=$ the irregular radial Coulomb function,
$h_l^{(1)}(x)=$ the spherical Hankel function,
$\hbar^2\alpha^2/M=$ binding energy of the deuteron,
$M=$ the nucleon mass,
$M_D=$ the deuteron mass,
$M_I=$ the target nucleus mass,
$R_I=$ the radius of the target nucleus,
$\hbar\mathbf{K}_D=$ asymptotic momentum of the incident deuterons relative to the center of mass,
$\hbar\mathbf{K}_P=$ asymptotic momentum of the liberated protons relative to the center of mass,
$\hbar^2K_N^2(M+M_I)/2MM_I=$ energy of the captured neutron,
$(L\lambda m0|L\lambda lm)=$ the Clebsch-Gordan coefficient.

K_D, K_P, K_N, and α are related by the conservation of energy by

$$\frac{\hbar^2K_N^2(M+M_I)}{2MM_I}+\frac{\hbar^2K_P^2}{2M}\frac{(M_I+2M)}{(M_I+M)}=\frac{\hbar^2K_D^2}{2M_D}\frac{(M_I+M_D)}{M_I}-\frac{\hbar^2\alpha^2}{M}.\qquad(32)$$

β_{Lm}^{c*} is defined after Eq. (26).

Setting $Z=0$, $\beta_{Lm}^{c*}=0$, and $\delta_\lambda=0$ causes Eq. (31) to have the form

$$B_f^{lm}=-\frac{\hbar^2(8\pi\alpha)^{\frac{1}{2}}}{Mh_l^{(1)}(K_NR_I)}\int_{R_I(1+M/M_I)}^{\infty}d\mathbf{r}Y_l^{-m}(\theta_r)\times h_l^{(1)}(K_Nr)\exp\left[i\left(\mathbf{K}_D-\frac{\mathbf{K}_P}{1+M/M_I}\right)\cdot\mathbf{r}\right].\qquad(34)$$

Carrying out the integration over angles gives

$$B_f^{lm}=-\frac{4\pi\hbar^2i^l\delta_{m0}[2\alpha(2l+1)]^{\frac{1}{2}}}{Mh_l^{(1)}(K_NR_I)}\int_{R_I(1+M/M_I)}^{\infty}drr^2\times h_l^{(1)}(K_Nr)j_l(\kappa r),\qquad(35)$$

where $\boldsymbol{\kappa}=\mathbf{K}_D-\mathbf{K}_P[1+(M/M_I)]^{-1}$ and δ_{m0} is the Kronecker delta symbol. The radial integral can be performed to give

$$B_f^{lm}=\frac{4\pi\hbar i^l\delta_{m0}[2\alpha(2l+1)]^{\frac{1}{2}}}{M(K_N^2-\kappa^2)}\left\{r^2\left[\frac{\partial}{\partial r}j_l(\kappa r)-j_l(\kappa r)\frac{\partial}{\partial r}\log h_l^{(1)}(K_Nr)\right]\right\}_{r=R_I(1+M/M_I)}.\qquad(36)$$

By means of the conservation of energy the denominator is changed:

$$B_f^{lm}=-\frac{2\pi\hbar i^l\delta_{m0}[2\alpha(2l+1)]^{\frac{1}{2}}}{M(\alpha^2+k_d^2)}\left\{r^2\left[\frac{\partial}{\partial r}j_l(\kappa r)-j_l(\kappa r)\frac{\partial}{\partial r}\log h_l^{(1)}(K_Nr)\right]\right\}_{r=R_I(1+M/M_I)},\qquad(37)$$

where $\mathbf{k}_d=\mathbf{K}_P-\frac{1}{2}\mathbf{K}_D$. This expression is the boundary condition matching scattering amplitude when V_{NP} is assumed to have zero range.

We see that Eq. (31) gives a spherical harmonic expansion of B_f^{lm} of the form

$$B_f^{lm}=\sum_L P_L^{|m|}(\cos\theta)\sum_\lambda A_{L\lambda}^{lm}f_{L\lambda}^{l}.$$

The fact that $A_{L\lambda}^{lm}$ is proportional to $(L\lambda00|L\lambda l0)$ limits the sum over λ to the values $\lambda=l+L$, $l+L-2$, $l+L-4$, $\cdots$, $|L-l|$. So for each value of L we must evaluate no more than $l+1$ radial integrals. For cases where the energy of the incident deuterons is about 15

Mev, the sum over L can be broken off at $L\approx 15$ without introducing great error. For smaller incident energies the range of L that is important is correspondingly smaller.

Since $h_l^{(1)}(ix)$ contains the factor e^{-x} we will usually introduce an error of about 1 percent if we replace the upper limit of infinity for the radial integrations by $X[1+(4.6/X)]$. X generally has values in the neighborhood of 4, so we see that the important range of integration for the radial integrals is from about $r=R$ to about $r=2R$.

The radial integrals appear to be suited to numerical evaluation. However, since the radial Coulomb functions are not adequately tabulated for this problem, the evaluation of these integrals will have to be done with an automatic computer or by means of some approximate method.

VI. AN ALTERNATE DERIVATION

We present here an alternate treatment of the (d,p) process. While this treatment leads to a result very similar to that given above, the approximations made are somewhat different and the final result is related very directly to the boundary condition matching result.

Again we neglect the Coulomb interaction, assume that all particles have zero spin, and assume that the mass of the target nucleus is infinite for the sake of simplicity. Let $\mathfrak{f}(\mathbf{K}_P,\mathbf{r}_P)$ be the wave function describing the motion of the liberated protons with asymptotic momentum $\hbar\mathbf{K}_P$. For large r_P the outgoing part of $\mathfrak{f}$ has the same form as the outgoing part of $e^{i\mathbf{K}_P\cdot\mathbf{r}_P}$. The (d,p) cross section is proportional to the rate at which deuterons are broken up to liberate protons with wave function $\mathfrak{f}(\mathbf{K}_P,\mathbf{r}_P)$

Assume $\mathfrak{f}$ is a member of a complete set of orthonormal functions. Then the wave function for the $A+1$ nucleons associated with the proton in the state $\mathfrak{f}(\mathbf{K}_P,\mathbf{r}_P)$ is

$$\psi(\mathbf{K}_P,\xi,\mathbf{r}_N)=\int d\mathbf{r}_P\mathfrak{f}^*(\mathbf{K}_P,\mathbf{r}_P)\Psi(\xi,\mathbf{r}_N,\mathbf{r}_P). \tag{38}$$

The rate at which protons are liberated into state $\mathfrak{f}(\mathbf{K}_P,\mathbf{r}_P)$ is just equal to the rate at which the nucleons described by $\psi(\mathbf{K}_P,\xi,\mathbf{r}_N)$ are forming residual nuclei. This is just the net flux of neutrons into the target nucleus predicted by ψ:

$$J(\mathbf{K}_P)=-\frac{\hbar R^2}{2Mi}\int d\xi d\Omega_N \times\left[\psi^*(\mathbf{K}_P,\xi,r_N)\frac{\partial}{\partial r_N}\psi(\mathbf{K}_P,\xi,\mathbf{r}_N) -\psi(\mathbf{K}_P,\xi,\mathbf{r}_N)\frac{\partial}{\partial r_N}\psi^*(\mathbf{K}_P,\xi,\mathbf{r}_N)\right]_{r_N=R}. \tag{39}$$

In terms of J the (d,p) cross section is

$$\sigma(\mathbf{K}_P)=\int_{K_{P-}}^{K_{P+}}dK\sigma_1(\mathbf{K}),\quad \sigma_1(\mathbf{K})=\frac{K^2}{\mathcal{J}}J(\mathbf{K})=\frac{M_D}{\hbar K_D}K^2J(\mathbf{K}). \tag{40}$$

The integration is over a resonance due to a level in the residual nucleus.

Equation (40) would provide an exact expression for the (d,p) cross section except for the fact that we must assume $\mathfrak{f}$ is a member of a complete set of functions. This means that the interaction between the proton and the residual nucleus is assumed to be elastic so that it can be represented by a real potential function $\bar{V}_{FP}(r_P)$. This assumption rules out the possibility for compound nucleus formation so that the (d,p) process is pictured in this treatment as going entirely by stripping.

To evaluate Eq. (40) for the cross section we proceed as before. Namely, we write the integral equation for $\psi(\mathbf{K}_P,\xi,\mathbf{r}_N)$, eliminate the interior of the target nucleus, and substitute the result into Eq. (40).

First we define $\mathfrak{f}(\mathbf{K}_P,\mathbf{r}_P)$ to be the solution of

$$\left(T_P+\bar{V}_{FP}-\frac{\hbar^2K_P^2}{2M}\right)\mathfrak{f}(\mathbf{K}_P,\mathbf{r}_P)=0, \tag{41}$$

which is the time reflection of the wave function for a beam of protons incident with momentum $-\hbar\mathbf{K}_P$ on the scattering center $\bar{V}_{FP}$. Let $\Phi_i(\xi)$ be the orthonormal regular solutions of

$$(H_I-E_i)\Phi_i(\xi)=0.$$

Then we expand the wave function $\Psi(\xi,\mathbf{r}_N,\mathbf{r}_P)$ as follows:

$$\Psi(\xi,\mathbf{r}_N,\mathbf{r}_P)=\int d\mathbf{K}_P\psi(\xi,\mathbf{r}_N,\mathbf{K}_P)\mathfrak{f}(\mathbf{K}_P,\mathbf{r}_P) =\sum_{l,m,i}\int d\mathbf{K}_P\phi_i^{lm}(\mathbf{K}_P,r_N) \times Y_l^m(\theta_N)\Phi_i(\xi)\mathfrak{f}(\mathbf{K}_P,\mathbf{r}_P). \tag{42}$$

Introducing this expansion into Eq. (8) gives the Schrödinger equation for ϕ_i^{lm}:

$$[T_N^{(l)}-E_{Ni}]\phi_i^{lm}(\mathbf{K}_P,r_N) =-\int d\mathbf{r}_Pd\xi d\Omega_N\mathfrak{f}^*(\mathbf{K}_P,\mathbf{r}_P)\Phi_i^*(\xi)Y_l^{-m}(\theta_N) \times[V_{NP}+V_{IP}+V_{IN}-\bar{V}_{FP}]\Psi, \tag{43}$$

where

$$T_N^{(l)}=-\frac{\hbar^2}{2M}\left\{\frac{\partial^2}{\partial r_N^2}+\frac{2}{r_N}\frac{\partial}{\partial r_N}-\frac{l(l+1)}{r_N^2}\right\},$$

$$E_{Ni}=E-E_i-\frac{\hbar^2K_P^2}{2M}=\frac{\hbar^2K_{Ni}^2}{2M}.$$

By means of the Green's function for $T_N^{(l)}-E_{Ni}$ we get the integral equation for ϕ_i^{lm}:

$$\phi_i^{lm}(\mathbf{K}_P,r)=\frac{2K_{Ni}M}{i\hbar^2}\int d\mathbf{r}_N d\mathbf{r}_P d\xi j_l(K_{Ni}r_<)\times h_l^{(1)}(K_{Ni}r_>)Y_l^{-m}(\theta_N)\Phi_i^*(\xi)\times\mathfrak{f}^*(\mathbf{K}_P,\mathbf{r}_P)[V_{NP}+V_{IP}+V_{IN}-\bar{V}_{FP}]\Psi, \quad (44)$$

where

$$r_<=\begin{cases} r & r<r_N \\ r_N & r_N<r \end{cases}, \quad r_>=\begin{cases} r & r>r_N \\ r_N & r<r_N \end{cases}.$$

Equation (44) contains the asymptotic boundary condition that all the liberated neutrons at infinity are outgoing.

Let $V_{IN}=0$ for $r_N>R$. Proceeding as before we eliminate the contribution to ϕ_i^{lm} due to the range of integration $0\leq r_N\leq R$ by introducing the logarithmic derivative of ϕ_i^{lm} at $r_N=R$.

$$\phi_i^{lm}(\mathbf{K}_P,r)=\frac{2K_{Ni}M}{i\hbar^2}\int d\mathbf{r}_P d\xi\int_R^\infty d\mathbf{r}_N j_l(K_{Ni}r_<)\times h_l^{(1)}(K_{Ni}r_>)Y_l^{-m}(\theta_N)\Phi_i^*(\xi)\mathfrak{f}^*(\mathbf{K}_P,\mathbf{r}_P)\times[V_{NP}+V_{IP}-\bar{V}_{FP}]\Psi-\Pi_i^{lm}h_l^{(1)}(K_{Ni}r), \quad (45)$$

where $r>R$

$$\Pi_i^{lm}=\left[\frac{\frac{\partial}{\partial r}j_l(K_{Ni}r)-j_l(K_{Ni})\frac{\partial}{\partial r}\log\phi_i^{lm}(\mathbf{K}_P r)}{\frac{\partial}{\partial r}\log h_l^{(1)}(K_{Ni}r)-\frac{\partial}{\partial r}\log\phi_i^{lm}(\mathbf{K}_P r)}\right]_{R=r}\times\frac{2K_N M}{i\hbar^2}\bar{B}_i^{lm},$$

$$\bar{B}_i^{lm}=\int d\mathbf{r}_P d\xi\int_R^\infty d\mathbf{r}_N\frac{h_l^{(1)}(K_{Ni}r_N)}{h_l^{(1)}(K_{Ni}R)}Y_l^{-m}(\theta_N)\times\Phi_i^*(\xi)\mathfrak{f}^*(\mathbf{K}_P,\mathbf{r}_P)[V_{NP}+V_{IP}-\bar{V}_{FP}]\Psi.$$

For $r=R$, Eq. (45) becomes

$$\phi_i^{lm}(\mathbf{K}_P,R)=\frac{2M}{\hbar^2R^2}\bar{B}_i^{lm}\left[\frac{\partial}{\partial r}\log h_l^{(1)}(K_{Ni}r)-\frac{\partial}{\partial r}\log\phi_i^{lm}(\mathbf{K}_P,r)\right]_{r=R}^{-1}. \quad (46)$$

Substituting the expansion of $\psi(\xi,\mathbf{r}_N,\mathbf{K}_P)$ contained in Eq. (42) into Eq. (40) and using Eq. (46) gives

$$\sigma_1(\mathbf{K}_P)=-\frac{4K_P{}^2M_DM}{\hbar^4R^2K_D}\times\sum_{i,l,m}\frac{|\bar{B}_i^{lm}|^2 Im.\left[\frac{\partial}{\partial r}\log\phi_i^{lm}(\mathbf{K}_P,r)\right]_{r=R}}{\left|\frac{\partial}{\partial r}\log h_l^{(1)}(K_{Ni}r)-\frac{\partial}{\partial r}\log\phi_i^{lm}(\mathbf{K}_P,r)\right|^2_{R=r}}. \quad (47)$$

The (d,p) cross section due to a level of the residual nucleus results from integrating σ_1 over the corresponding resonance. To represent such a resonance we assume that for values of K_P near K_P', the

$$[\partial\log\phi_i^{lm}(\mathbf{K}_P,r)/\partial r]_R$$

for one value of i, say $i=f$, become very nearly equal to $[\partial\log h_l^{(1)}(K_{Ni}r)/\partial r]_R$. In this energy region we can write

$$\left[\frac{\partial}{\partial r}\log\phi_f^{lm}(\mathbf{K}_P,r)-\frac{\partial}{\partial r}\log h_l^{(1)}(K_{Nf}r)\right]_R=-\frac{1}{R\gamma_{lf}}\left[E_{Nf}-E_{Nf}'+\frac{i\Gamma}{2}\right]. \quad (48)$$

We substitute Eq. (48) into Eq. (47), integrate $\sigma_1(\mathbf{K}_P)dK_P$ over a narrow range, and let $\Gamma\to0$. The result is

$$\sigma(\mathbf{K}_P)=\frac{4\pi M^2K_PM_D}{\hbar^6RK_D}\sum_{l,m}\gamma_{lf}|\bar{B}_f^{lm}|^2. \quad (49)$$

Taking spin, the Coulomb interaction, and the finite mass of the target nucleus into account changes this to

$$\sigma(\mathbf{K}_P)=\frac{\left(1+\frac{M}{M+M_I}\right)^{-1}}{\left(1+\frac{M}{M_I}\right)\left(1+\frac{M_D}{M_I}\right)}\frac{M_DM^2}{\hbar^6R}\frac{(2J+1)}{(2I+1)}\frac{K_P}{K_D}\times{\sum_{l,m}}'\frac{\gamma_{lf}}{2l+1}|\bar{B}_f^{lm}|^2, \quad (50)$$

where

$$\bar{B}_f^{lm}=\frac{1}{h_l^{(1)}(K_NR)}\int d\boldsymbol{\varrho}_P d\xi\int_R^\infty d\mathbf{r}_N h_l^{(1)}(K_Nr_N)\times Y_l^{-m}(\theta_N)\Phi_i^*(\xi)\mathfrak{f}^{c*}(\mathbf{K}_P,\boldsymbol{\varrho}_P)\times\left[V_{NP}+V_{IP}+Ze^2\left(\frac{1}{r_P}-\frac{1}{\rho_P}\right)-\bar{V}_{FP}\right]\Psi, \quad (51)$$

$$\left[T_P+\bar{V}_{FP}+\frac{Ze^2}{\rho_P}-\frac{\hbar^2K_P{}^2}{2M}\right]\mathfrak{f}^c(\mathbf{K}_P,\boldsymbol{\varrho}_P)=0.$$

Equation (50) is identical with Eq. (28) except for a factor of $(2\pi)^3$ due to the difference in normalization between $\mathfrak{f}$ and $\mathcal{E}$ and except for the difference in the definitions of B and $\bar{B}$.

Let us assume that V_{IP} can be approximated by a potential function $\bar{V}_{IP}(r_P)$. Then we would expect that $\bar{V}_{IP}$ is very nearly the same function of r_P as $\bar{V}_{FP}$ is of ρ_P. It follows that the contribution of

$$\Delta\bar{V}=V_{IP}-\bar{V}_{FP}+Ze^2\left(\frac{1}{r_P}-\frac{1}{\rho_P}\right)\approx\bar{V}_{IP}-\bar{V}_{FP}+Ze^2\left(\frac{1}{r_P}-\frac{1}{\rho_P}\right)$$

to $\bar{B}_f^{lm}$ is small and tends to vanish as $M_I\to\infty$. Upon dropping $\Delta\bar{V}$ we see that the range of integration $0\leqslant\rho_P\leqslant R(1+M/M_I)$ will make a negligible contribution to $\bar{B}_f^{lm}$ because of the short range of V_{NP} so let us drop this range of integration. Since ρ_P is now always greater than R in $\bar{V}_f^{lm}$, the form of $\mathfrak{f}^{c*}(\mathbf{K}_P,\boldsymbol{\varrho}_P)$ is

$$\mathfrak{f}^{c*}(\mathbf{K}_P,\boldsymbol{\varrho}_P)=\frac{4\pi}{(2\pi)^{\frac{3}{2}}}\sum_{L,M} i^{-L}Y_L^{-M}(\theta_{\rho P})Y_L^M(\theta_{KP})e^{-i\sigma_L(\eta_P)}\left[\frac{F_L(\eta_P,K_P\rho_P)-\bar{\beta}_L{}^{c*}H_L(\eta_P,K_P\rho_P)}{K_P\rho_P}\right], \tag{52}$$

$$\bar{\beta}_L{}^{c*}=\left[\frac{\dfrac{\partial}{\partial r}\dfrac{F_L(\eta_P,K_Pr)}{K_Pr}-\dfrac{F_L(\eta_P,K_Pr)}{K_Pr}\dfrac{\partial}{\partial r}\log\mathfrak{f}_{LM}{}^{c*}(\mathbf{K}_P,r)}{\dfrac{\partial}{\partial r}\dfrac{H_L(\eta_P,K_Pr)}{K_Pr}-\dfrac{H_L(\eta_P,K_Pr)}{K_Pr}\dfrac{\partial}{\partial r}\log\mathfrak{f}_{LM}{}^{c*}(\mathbf{K}_P,r)}\right]_R;$$

$\mathfrak{f}^{c*}$ is the wave function describing the scattering of protons of momentum $-\hbar\mathbf{K}_P$ by the potential $\bar{V}_{FP}(\rho_P)$. Hence the $\bar{\beta}_L{}^{c*}$ are the scattering amplitudes for free protons incident on $\bar{V}_{FP}$. Thus if $\bar{V}_{FP}$ is a good approximation to V_{FP}, then $\mathfrak{f}^{c*}$ is a good approximation to $\mathcal{E}^{c*}$ defined in Eq. (26).

Setting $\Delta\bar{V}=0$, $V_{NP}=$the zero range potential, and $\Psi=\Psi_{\text{c.m.}}\chi\Phi_I$ causes $\bar{B}_I^{lm}$ to be essentially equal to B_f^{lm} defined in Eq. (31) when $V_{FP}=\bar{V}_{FP}$. This derivation of Eq. (31) is not so satisfactory as our previous derivation since we are forced in the present derivation to replace V_{FP} by a real potential function of ρ_P and the contribution of $\Delta\bar{V}$ is not so obviously small as the contribution of ΔV. However in the present derivation the interpretation of $\bar{\beta}^{c*}$ is unambiguous and agrees with our previous interpretation of β^{c*}. γ_{lI} on the other hand can only approximately be identified with the reduced width of the level of the captured neutron in the residual nucleus.

The present derivation is also of interest because it is connected directly with the general boundary condition matching result. To demonstrate this we assume that $\bar{V}_{FP}=V_{IP}$ and $Z=0$. Then Eq. (51) becomes

$$\bar{B}_i^{lm}=\int d\boldsymbol{\varrho}_P d\xi\int_R^\infty d\mathbf{r}_N\frac{h_l^{(1)}(K_{Ni}r_N)}{h_l^{(1)}(K_{Ni}R)}Y_l^{-m}(\theta_N)\times\Phi_i^*(\xi)\mathfrak{f}^*(\mathbf{K}_P,\boldsymbol{\varrho}_P)V_{NP}\Psi \tag{53}$$

$$=-\frac{\hbar^2R^2}{2M}\left[\frac{\partial}{\partial r}A_i^{lm}-A_i^{lm}\frac{\partial}{\partial r}\log h_l^{(1)}(K_{Ni}r_N)\right]_{r=R},$$

where

$$A_i^{lm}=\frac{2MK_{Ni}}{i\hbar^2}\int d\boldsymbol{\varrho}_P d\mathbf{r}_N d\xi j_l(K_{Ni}r_<)h_l^{(1)}(K_{Ni}r_>)\times Y_l^{-m}(\theta_N)\Phi_i^*(\xi)\mathfrak{f}^*(\mathbf{K}_P,\boldsymbol{\varrho}_P)V_{NP}\Psi$$
$$=A_i^{lm}(\Psi).$$

Now set $\Psi=\Psi^0$, the wave function for the incident beam of deuterons and the target nucleus. But by Eq. (44) the transform of Ψ^0,

$$\phi_i^{0lm}(\mathbf{K}_P,r_N)=\int d\boldsymbol{\varrho}_P d\xi d\Omega_N Y_l^{-m}(\theta_N)\mathfrak{f}^*(\mathbf{K}_P,\rho_P)\Phi_i^*(\xi)\Psi^0,$$

is the solution of the integral equation

$$\phi_i^{0lm}=\delta_{iI}A_i^{lm}(\Psi^0). \tag{54}$$

Therefore

$$\bar{B}_i^{lm}=-\frac{\delta_{iI}\hbar^2R^2}{2M}\left[\frac{\partial}{\partial r_N}\phi_I^{0lm}-\phi_I^{0lm}\frac{\partial}{\partial r_N}\times\log h_l^{(1)}(K_Nr_N)\right]_R, \tag{55}$$

which is the general form of the boundary condition matching result.

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