Nonuniform Nuclear Charge Distributions and Measurements of Nuclear Electrical Radius*

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The dependence of three long wavelength electronic effects and of the nuclear Coulomb energy on the size and shape of the electrical charge distribution in heavy nuclei is examined. We note that the x-ray fine structure effect and the isotope shift effect measure the volume integral of $r^{2\sigma}$ weighted by the nuclear charge density. [$\sigma=\{1-(\alpha Z)^2\}^{\frac{1}{2}}$]. An earlier result of Feshbach—that medium-energy electron scattering measures the similarly weighted mean value of r^2—is derived more simply. For these electronic effects, therefore, exact calculations for one shape of charge distribution may easily be scaled to other shapes. The nuclear Coulomb energy has no such simple dependence, but measures very roughly the weighted mean value of $r^{0.8}$—nearly the same quantity as measured by the $2P\rightarrow 1S$ mu-mesonic transition in heavy nuclei.

The nuclear Coulomb energy sheds no light on the shape of the nuclear charge distribution because of its inherent insensitivity to change of shape. The x-ray fine structure, and the atomic isotope shift, cannot give detailed information about the nuclear charge distribution because of uncertainty in the magnitude of radiative corrections, and of nuclear compressibility, respectively. Medium-energy electron scattering results, coupled with mu-mesonic x-ray results, will in principle be able to delineate shapes. Present evidence from this source suggests a charge distribution nearly uniform or somewhat peaked toward the edge of the nucleus.

I. INTRODUCTION

IN the preceding paper[1] a set of conceivable charge distributions for the Pb nucleus was derived, limited by the requirement that for each the mu-mesonic $2P_{3/2}\rightarrow 1S$ transition energy has the observed value of 6.0 Mev.[2] In the present paper this set of charge distributions is used to calculate the contribution of the finite extent of the nuclear charge to the following phenomena:[3]

1. Electronic x-ray fine structure.[4]
2. Atomic isotope shift.[5–7]
3. "Medium" energy (5–40 Mev) electron scattering.[6]
4. Nuclear Coulomb energy.[6,8]

These effects have in common that their dependence on the nuclear charge distribution is rather easily calculable [in contrast to the mu-mesonic bound state energies[1,2,8] and the "high" energy ($\gtrsim$50 Mev) electron scattering[9–11]]. They also have in common, however, that their certainty of theoretical interpretation or their precision of measurement is less than the certainty or precision of the mu-mesonic transition energies.

We therefore adopt the mu-mesonic transition energy ($2P_{3/2}\rightarrow 1S$) as a fixed constant and find the dependence of the other phenomena on the shape of the nuclear charge distribution. The relative results are not at all sensitive to the particular choice of 6.0 Mev for the mu-mesonic transition energy. We discuss qualitatively some effects other than the finite extension of the nucleus which may contribute significantly to these phenomena, but we add no new insight into the detailed quantitative interpretation of these phenomena. It is the limited aim of this paper to demonstrate the dependence of these effects on the shape of the nuclear charge distribution, and to show to what extent a nonuniform charge distribution might explain the apparent discrepancies in values of nuclear radii found from different sources.

It is found that values of nuclear radii deduced from these sources should not differ among themselves by more than about 10 percent. Discrepancies considerably greater than this now exist, and exceed the uncertainties in the experimental data.[12] Only the electron scattering

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[1] D. L. Hill and K. W. Ford, preceding paper, Phys. Rev. **94**, 1617 (1954).

[2] V. L. Fitch and J. Rainwater, Phys. Rev. **92**, 789 (1953).

[3] References 4–8 are to recent theoretical work. References to experimental work and to earlier theoretical work may be found in these papers.

[4] A. L. Schawlow and C. H. Townes, Science **115**, 284 (1952).

[5] W. Humbach, Z. Physik **133**, 589 (1952).

[6] F. Bitter and H. Feshbach, Phys. Rev. **92**, 837 (1953).

[7] Wilets, Hill, and Ford, Phys. Rev. **91**, 1488 (1953).

[8] L. N. Cooper and E. M. Henley, Phys. Rev. **92**, 801 (1953).

[9] Hofstadter, Fechter, and McIntyre, Phys. Rev. **92**, 978 (1953).

[10] L. I. Schiff, Phys. Rev. **92**, 988 (1953).

[11] Yennie, Wilson, and Ravenhall, Phys. Rev. **92**, 1325 (1953).

[12] This conclusion is somewhat at variance with that of Bitter and Feshbach in their recent report (reference 6) on nuclear radii. Since our work overlaps theirs to some extent, we mention here several points of difference. We find that the isotope shift for heavy nuclei depends on a weighted integral of $r^{2\sigma}=r^{(\sim 1.6)}$ rather than r^2. Their Fig. 1 consequently overestimates the effect of an altered radius on the isotope shift. Further we conclude that nuclear compressibility is important for isotope shifts and introduces a correction of about 25±15 percent in the derived radius. In spite of the contrary evidence from the x-ray fine structure, we agree with the general conclusion that the weight of evidence favors a nuclear electrical radius substantially smaller than the previously accepted $1.4\text{–}1.5\times 10^{-13}A^{1/3}$ cm.

can be used at present to get an idea of the shape of the nuclear charge distribution.

II. THEORY OF EFFECTS OF EXTENDED NUCLEAR CHARGE

1. Electronic X-Ray Fine Structure

Schawlow and Townes[4] fit the experimental electronic $2p$ fine structure splitting among the heavy elements by adding to the point-nucleus formula of Christy and Keller[13] a correction term for the displacement of the $2p_{1/2}$ level by the finite extent of the nucleus. The $2p_{3/2}$ level is displaced negligibly by comparison. The $2p_{1/2}$ displacement is upward (diminishing the doublet splitting) and arises mainly from the finite value of the small component of the $2p_{1/2}$ wave function at the origin.

Although first-order perturbation theory is inadequate to give the magnitude of the energy shift, the interesting result of a number of exact calculations is that the correct dependence on the size and shape of the nuclear charge distribution is the same as given by the perturbation formula. According to first-order perturbation theory, the energy shift of the $2p_{1/2}$ level may be written:

$$\Delta E_1 = 4\pi \int_0^\infty \rho_N(r) V_e^{(0)}(r) r^2 dr, \tag{1}$$

where $\rho_N(r)$ is the nuclear charge density (assumed spherically symmetric) and $V_e^{(0)}(r)$ is the potential due to the (unperturbed) $p_{1/2}$ electron, normalized to be zero at the origin. The unperturbed electronic charge density is given by

$$\rho_e^{(0)}(r) = -(e/r^2)(F_0^2+G_0^2), \tag{2}$$

where F_0 and G_0 are the small and large components of the point nucleus solution of the radial Dirac equation, with the normalization,

$$4\pi \int_0^\infty (F_0^2+G_0^2)dr = 1. \tag{3}$$

In the vicinity of the nucleus,

$$\rho_e^{(0)}(r) = -Aer^{2\sigma-2}, \tag{4}$$

where A is a normalization constant, and

$$\sigma = [1-(Z/137)^2]^{\frac{1}{2}};$$

and

$$V_e^{(0)}(r) = [4\pi Ae/2\sigma(2\sigma+1)]r^{2\sigma}. \tag{5}$$

Then the perturbation energy shift, Eq. (1), takes the form

$$\Delta E_1 = [4\pi Ze^2 A/2\sigma(2\sigma+1)]\langle r^{2\sigma}\rangle_{\text{Av}}. \tag{6}$$

Thus the property of the charge distribution determined by experiment is the average value of $r^{2\sigma}$, defined by

$$\langle r^{2\sigma}\rangle_{\text{Av}} = (Ze)^{-1} \int_0^\infty r^{2\sigma}\rho_N(r) 4\pi r^2 dr. \tag{7}$$

In spite of the small value of the energy shift, the perturbing potential is large and the modification of the wave function within the nucleus is large. A number of authors[5,14–17] have considered the required modification in the perturbation formulas (1) or (6). Formula (1) is clearly an upper limit to the shift because the electronic charge density $\rho_e^{(0)}(r)$, Eq. (2), is larger everywhere in the nucleus than the true charge density,

$$\rho_e(r) = -(e/r^2)(F^2+G^2). \tag{8}$$

Here F and G are the exact solutions of the Dirac equations in the correctly modified field [also normalized as in (3)]. If the argument is applied in reverse by regarding the modified field as unperturbed, and the point-nucleus field as the perturbed field, then it is similarly clear that replacing $\rho_e^{(0)}(r)$ by $\rho_e(r)$, and $V_e^{(0)}(r)$ by $V_e(r)$, $[\nabla^2 V_e = -4\pi\rho_e]$, in Eq. (1), leads to a lower limit on the energy shift. These obvious remarks are inserted in order to make reasonable the nearly exact formula of Broch.[15] Broch's formula is equivalent to the use in Eq. (1) of an effective electronic potential $V_e^B(r)$ satisfying

$$\nabla^2 V_e^B(r) = -4\pi\rho_e^B(r), \tag{9}$$

with

$$\rho_e^B(r) = -(e/r^2)(F_0F+G_0G). \tag{10}$$

Broch's result is in error only by a factor

$$4\pi \int_0^\infty (F_0F+G_0G)dr,$$

which is extremely close to 1 because of the very small fraction of the electronic charge which is within the nucleus.

The correct energy shift due to the finite extent of the nucleus at $Z=82$ is about 0.75 times the shift given by Eq. (6). The significant result of a number of exact calculations, for the present discussion, is that this correction factor is very nearly independent of the shape and size, R, of the nucleus. For potentials within the nucleus as widely different as zero (Broch[15]), $-Ze^2/R$, and $(-Ze^2/R)[(3/2)-\frac{1}{2}(r/R)^2]$ (Crawford and Schawlow[16]), the correction factor varies by only 4 percent. The exact results therefore are well represented by the form of Eq. (6):

$$\Delta E = \varphi(Z)\langle r^{2\sigma}\rangle_{\text{Av}}. \tag{11}$$

For any shape and size of nuclear charge distribution,

[13] R. F. Christy and J. M. Keller, Phys. Rev. **61**, 147 (1942).

[14] J. Rosenthal and G. Breit, Phys. Rev. **41**, 459 (1932).

[15] E. K. Broch, Arch. Math. Naturvidenskab. **48**, 25 (1945).

[16] M. F. Crawford and A. L. Schawlow, Phys. Rev. **76**, 1310 (1949).

[17] P. Brix and H. Kopfermann, Festschr. Akad. Wiss. Göttingen, Math.-Phys. Kl. **17** (1951); Phys. Rev. **85**, 1050 (1952).

the energy shift depends on the charge distribution only through the quantity $\langle r^{2\sigma}\rangle_{\text{Av}}$ defined by (7). The relation (11) appears to be equally valid for $p_{1/2}$ and $s_{1/2}$ states and is confirmed by recent exact calculations of Humbach[5] and of Schawlow and Townes.[4] Humbach examined distributions ranging from charge concentrated at center of nucleus to charge in a shell at edge of nucleus. His results are in good agreement with the relation (11). For example, at $Z=80$ he finds the shift for a shell of charge to be greater than the shift for uniform charge of the same radius by a factor 1.51. The ratio of $\langle r^{2\sigma}\rangle_{\text{Av}}$ for these two distributions is 1.54. For the same two distributions, Schawlow and Townes give the ratio of the $p_{1/2}$ shifts over the range $Z=70$ to 90 to be $11.2/7.32=1.53$. For these distributions, the ratios of $\langle r^{2\sigma}\rangle_{\text{Av}}$ are $(1/3)(3+2\sigma)=1.57$ at $Z=70$, 1.54 at $Z=80$, and 1.50 at $Z=90$. For comparison, the ratio of $\langle r^2\rangle_{\text{Av}}$ for the distributions is 1.67.

The useful relation (11) can be understood in part by noting that, for any given shape of charge distribution, the true charge density, Eq. (8), at the origin varies approximately as $R^{2\sigma-2}$. Thus the energy shift for the upper and lower limiting perturbation formulas and for the exact formula varies with nuclear size as $R^{2\sigma}$ for any shape of charge distribution. No simple argument has been found, however, to show that indeed the exact shift varies as $\langle r^{2\sigma}\rangle_{\text{Av}}$ from one shape to another.

2. Atomic Isotope Shift

The theoretical work referred to in the preceding section was directed mainly toward the explanation of the electronic isotope shift, but was equally applicable to the directly measured $2p_{1/2}$ shift. That part of the isotope shift (presumed to be the major part) arising from the nuclear volume effect is just the difference in the energy shifts (usually of an s electron) between isotopes of the same element. In order to see the dependence of the shift on the nuclear charge distribution, it is adequate to work from Eq. (11). The isotope shift is then

$$\delta\Delta E=\varphi(Z)\delta\langle r^{2\sigma}\rangle_{\text{Av}}. \tag{12}$$

If the shape of the charge distribution is the same for both isotopes, that is if

$$\rho_N(r)=\rho_0 f(r/r_0), \tag{13}$$

and if the function $f(x)$ is the same for both isotopes, then (12) becomes

$$\delta\Delta E=\Delta E[2\sigma\delta r_0/r_0]. \tag{14}$$

If, moreover, the nuclear size increases regularly as $A^{1/3}$, then

$$\delta r_0/r_0=\delta A/3A, \tag{15}$$

and

$$\delta\Delta E=(2\sigma/3A)\varphi(Z)\langle r^{2\sigma}\rangle_{\text{Av}}, \tag{16}$$

having the same dependence on the nuclear charge distribution as the direct shift ΔE. Equation (16) is incorrect for at least two reasons.[7] Nuclear deformation invalidates the assumption of a constant functional form $f(x)$ for the charge distribution and has a large effect. Nuclear compressibility invalidates the assumption of an $A^{1/3}$ law of radius and may also have a rather large effect. The first can be adequately corrected for and the second, which can be only roughly guessed, should not markedly affect the proportionality of $\delta\Delta E$ to $\langle r^{2\sigma}\rangle_{\text{Av}}$. Thus after correction for nuclear deformation, the isotope shift, like the direct shift, is proportional to $\langle r^{2\sigma}\rangle_{\text{Av}}$.

3. Medium-Energy Electron Scattering

Feshbach[18] has studied the dependence of "medium" energy electron scattering on the form of the nuclear charge distribution. "Medium" energy is defined by the conditions (a) electron energy≫rest energy and (b) electron wavelength≫nuclear radius, and therefore comprises for heavy nuclei about the range 5 to 40 Mev. Feshbach's results are that in this region only the s-wave phase shift is important in the deviation from Coulomb scattering, and that this phase shift depends only on the mean value of r^2 over the charge distribution, i.e., on

$$\langle r^2\rangle_{\text{Av}}=(Ze)^{-1}\int_0^\infty r^2\rho_N(r)4\pi r^2dr. \tag{17}$$

We give here a simpler derivation of the latter result, which makes more evident its limits of validity. The radial Dirac equations in a central field are:

$$\begin{aligned}(dF/d\xi)-(k/\xi)F+(\epsilon-1-U)G=0,\\ (dG/d\xi)+(k/\xi)G-(\epsilon+1-U)F=0,\end{aligned} \tag{18}$$

in which ϵ and U are the total energy and the potential energy in units of mc^2, ξ is the radius in units of $\hbar/mc$, and k, the angular quantum number, is $\pm(j+1/2)$ for $j=l\mp 1/2$ (the negative of Feshbach's k). Equations (18) may be reduced to a single integral equation for the ratio F/G: either[19]

$$(F/G)=-\xi^{2k}\int_0^\xi[(\epsilon-1-U)+(\epsilon+1-U)(F/G)^2]\xi^{-2k}d\xi,\quad k<0, \tag{19}$$

or

$$(G/F)=\xi^{-2k}\int_0^\xi[(\epsilon-1-U)(G/F)^2+(\epsilon+1-U)]\xi^{2k}d\xi,\quad k>0. \tag{20}$$

Since the contribution of the kth partial wave to the scattering is determined[20] only by the ratio F/G at

[18] H. Feshbach, Phys. Rev. **84**, 1206 (1951).

[19] Here the upper limit of integration is chosen to be less than the first node of G (for $k<0$), or less than the first node of F (for $k>0$). See the asymptotic expressions in Appendix II of reference 1.

[20] George Parzen, Phys. Rev. **80**, 355 (1950).

some radius $\xi = a$ beyond which the potential is pure Coulomb, it follows that distributions which are equivalent for scattering must have equal values of the integrals

$$\int_0^a (\epsilon-1-U)\left[1+\frac{\epsilon+1-U}{\epsilon-1-U}\left(\frac{F}{G}\right)^2\right]\xi^{-2k}d\xi, \quad k<0; \quad (21)$$

or

$$\int_0^a (\epsilon+1-U)\left[1+\frac{\epsilon-1-U}{\epsilon+1-U}\left(\frac{G}{F}\right)^2\right]\xi^{2k}d\xi, \quad k>0. \quad (22)$$

These exact expressions go over to the equivalent integrals of Feshbach for the $s_{1/2}(k=-1)$ and $p_{1/2}(k=1)$ phase shifts if the second terms in the square brackets are negligible compared to 1. [Two integrations by parts then convert (21) and (22) to (17).] For electron *kinetic* energy within the nucleus great compared to mc^2, $(\epsilon\pm 1-U)/(\epsilon\mp 1-U)\cong 1$; and for electron wavelength great compared to nuclear size, F/G (or $G/F)\ll 1$ for $k<0$ (or $k>0$) at all positions in and near the nucleus. For $\epsilon - U = 120$ within the nucleus (e.g., 40-Mev electrons on Pb), the second term in square brackets contributes only a few percent to the integrals (21) and (22). So long as only the $|k|=1$ partial waves are important, therefore, it is an excellent approximation to say that the electron scattering measures $\langle r^2\rangle_{\text{Av}}$. Higher partial waves of course measure higher moments of the charge distribution. The evident reason that bound state $s_{1/2}$ and $p_{1/2}$ energy shifts measure $\langle r^{2\sigma}\rangle_{\text{Av}}$, while $s_{1/2}$ and $p_{1/2}$ electron scattering measure $\langle r^2\rangle_{\text{Av}}$, is that the former depends on F^2+G^2, while the latter depends only on the ratio F/G.

It is to be noted that at small angles the electron scattering in the Born approximation also depends only on $\langle r^2\rangle_{\text{Av}}$. The Born approximation is proportional to the square of a form factor:[10]

$$F' = (Ze)^{-1}\int_0^\infty \rho_N(r)[\sin qr/qr]4\pi r^2 dr, \quad (23)$$

where q is the momentum change of the electron divided by $\hbar$. At small angles (small q), this form factor is given approximately by:

$$F' = 1-(1/6)q^2\langle r^2\rangle_{\text{Av}}, \quad (24)$$

with $\langle r^2\rangle_{\text{Av}}$ defined by (17).

4. Nuclear Coulomb Energy

In the absence of correlations among proton positions in the nucleus, the nuclear Coulomb energy is

$$E_c = [(Z-1)/2Z]\int_0^\infty \rho_N(r)V_N(r)d(\text{vol}), \quad (25)$$

where $V_N(r)$ is the electrostatic potential which satisfies

$$\nabla^2 V_N = -4\pi\rho_N, \quad (26)$$

and approaches $-Ze/r$ outside the nucleus at large r. The Coulomb energy depends on the charge distribution in a more complicated way than do the other three effects discussed above. Since the integral on the right of (22) contains the product $\rho_N(r)\rho_N(r')$, the Coulomb energy cannot be expressed in terms of a single integral over the charge distribution. For any given shape of charge distribution, the Coulomb energy varies inversely as the radial extent of the distribution. For different distributions, intermediate in shape between uniform and exponential and having the same Coulomb energy, $\langle r^{0.8}\rangle_{\text{Av}}$ is roughly constant.

III. DEPENDENCE ON SHAPE OF CHARGE DISTRIBUTION

A convenient way to discuss the effect of changing the shape of the nuclear charge distribution is in terms of an equivalent radius of a uniform distribution. That is, if different experimental effects are interpreted in terms of a uniform distribution, a radius will be found for each, which we call the equivalent radius, R_{eq}. In case the distribution is in fact uniform, the R_{eq}'s should be the same for all effects and equal to the nuclear radius. Otherwise they will differ.

1. Charge distributions which are equivalent for isotope shift and x-ray fine structure have equal values of $\langle r^{2\sigma}\rangle_{\text{Av}}$ [Eq. (7)]. In order to treat the same set of functional forms for the charge distribution as considered in the preceding paper[1] (I), we turn to the dimensionless formulation introduced there. We introduce the new functional of the charge distribution f,

$$K_f(\sigma;x) = \int_0^x f(\xi)\xi^{2\sigma+2}d\xi. \quad (27)$$

Then

$$\langle r^{2\sigma}\rangle_{\text{Av}} = r_0^{2\sigma}K_f(\sigma;\infty)/I_f(\infty), \quad (28)$$

where $I_f(\infty)$ is defined by I, (6). For a uniform distribution,

$$\langle r^{2\sigma}\rangle_{\text{Av}} = R^{2\sigma}[3/(2\sigma+3)]. \quad (29)$$

Equating (28) and (29), one gets for the equivalent radius relevant to this first set of phenomena,

$$R_{eq}^{(1)} = \left[\frac{2\sigma+3}{3}\cdot\frac{K_f(\sigma;\infty)}{I_f(\infty)}\right]^{1/2\sigma}\cdot r_0. \quad (30)$$

Expressions for $K_f(\sigma;\infty)$ are given in an appendix to this paper, and expressions for $I_f(\infty)$ in an appendix to I.

2. Medium energy electron scattering (5–40 Mev) measures approximately $\langle r^2\rangle_{\text{Av}}$. The equivalent radii for charge distributions having equal values of $\langle r^2\rangle_{\text{Av}}$ are given by:

$$R_{eq}^{(2)} = \left[\frac{5}{3}\cdot\frac{K_f(1;\infty)}{I_f(\infty)}\right]^{\frac{1}{2}} r_0, \quad (31)$$

which follows immediately from (30).

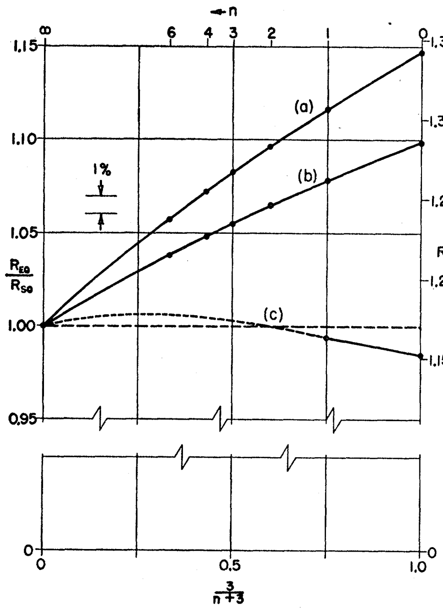

FIG. 1. Equivalent electrical radii of Pb for several nuclear charge-sensitive effects plotted *vs* an arbitrarily chosen shape parameter for family I of charge distributions. The right scale is $R_0=R_{eq}/A^{1/3}$ in units of 10^{-13} cm. (a) Radii determined from medium energy electron scattering. (b) Radii determined from x-ray fine structure effect and isotope shift. (c) Radii determined from nuclear Coulomb energy. Horizontal dashed line gives radii determined from mu-mesonic $2P\rightarrow 1S$ transition, to which the other curves are normalized.

3. For charge distributions possessing the same Coulomb energy, the integral

$$4\pi\int_0^\infty \rho_N(r)V_N(r)r^2dr \qquad (32)$$

is constant. We introduce the corresponding dimensionless integral

$$L_f(\infty)=\int_0^\infty f(x)J_f(x)x^2dx. \qquad (33)$$

$J_f(x)$ is the dimensionless potential defined by I, (7). Then the Coulomb energy is

$$E_c=\tfrac{1}{2}Z(Z-1)e^2r_0^{-1}[L_f(\infty)/I_f(\infty)]. \qquad (34)$$

For a uniform distribution,

$$E_c=(3/5)Z(Z-1)e^2R^{-1}. \qquad (35)$$

Therefore the equivalent radius for Coulomb energy is defined by

$$R_{eq}^{(3)}=(6/5)[I_f(\infty)/L_f(\infty)]r_0. \qquad (36)$$

Expressions for $L_f(\infty)$ are given in the appendix.

Figures 1 and 2 show these three equivalent radii plotted as a function of a suitable shape parameter for charge distribution families I and II, subject to the condition that the mu-mesonic $2P_{3/2}\rightarrow 1S$ transition energy is a constant, 6.0 Mev. Z is taken to be 82. The left edge of each graph corresponds to the uniform distribution, the right, to the exponential. To make clear the meaning of these curves, we suppose for illustration that the Pb nucleus has an exponential charge distribution. If then the phenomena here considered are all interpreted exactly on the assumption of a uniform distribution, and the calculated radii put in the form $R=R_0A^{1/3}$, one should conclude from the electronic x-ray fine structure and isotope shift[21] that the nucleus has $R_0=1.28\times 10^{-13}$ cm; from the mu-mesonic $2P\rightarrow 1S$ transition that $R_0=1.17\times 10^{-13}$ cm; from the Coulomb energy that $R_0=1.15\times 10^{-13}$ cm; and from the medium energy electron scattering that $R_0=1.34\times 10^{-13}$ cm. For small changes of the normalizing value 1.17×10^{-13} cm from the mu-meson experiment, the relative position of the curves is not appreciably altered.

The apparent qualitative features of Figs. 1 and 2 are: (a) The Coulomb energy is very insensitive to change of shape of the charge distribution. (b) The equivalent radii deduced from electronic x-ray data, isotope shift, and electron scattering are never smaller than that deduced from mu-mesonic transitions (exception: a distribution both peaked toward the edge of the nucleus and with a very rapid fall-off beyond the peak). (c) Radii deduced in these different ways should not differ by more than about 10 percent for any reasonable shape of charge distribution. The numbers plotted in Figs. 1 and 2 are given in Table I.

IV. COMPARISON WITH EXPERIMENT AND DISCUSSION

1. X-Ray Fine Structure

R_{eq} deduced from x-ray data by Schawlow and Townes[4] is $1.77\times 10^{-13}A^{1/3}$ cm, with an assigned prob-

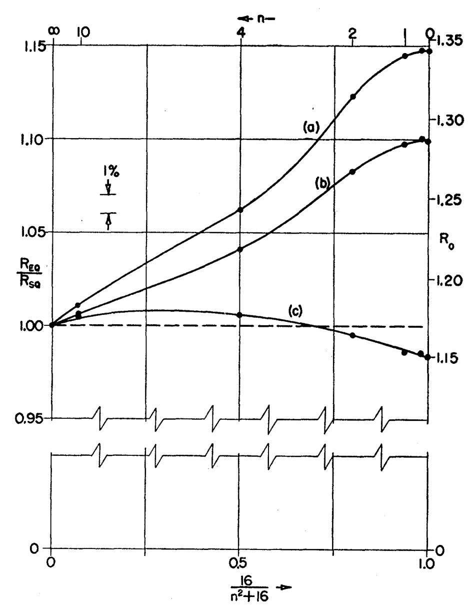

FIG. 2. Equivalent electrical radii for Pb for family II charge distributions. Labelling and significance of curves same as in Fig. 1.

[21] It is assumed for the moment that corrections for radiative effects and nuclear compressibility are known and included.

able error of 15 percent. This figure is to be contrasted with the much lower prediction from Figs. 1 and 2 of $(1.17 \text{ to } 1.29)\times 10^{-13}A^{1/3}$. In discussing this and succeeding discrepancies, we retain the idea that the R_0 of 1.17×10^{-13} cm determined by the mu-mesonic experiment is accurate. (We note, however, that the mean R_0 for elements near Pb might be slightly larger than this value because of the closed shell structure and negligible deformation of the Pb^{208} nucleus.) That part of the $2p$ fine structure which is taken to be due to the finite extent of the nucleus is distinguished from the remainder by its greater Z dependence. Hence any large contributing error is to be sought in an effect with similar Z dependence, that is, an effect which depends on the $p_{1/2}$ electron density at the nucleus. One such effect is nuclear polarization. This effect, however, should act in the direction opposite to the finite volume effect and thereby increase the nuclear radius discrepancy. The effect has in any case been estimated to be small.[22] The Lamb shift probably provides the major uncertainty in $p_{1/2}$ energy shift. A part of the radiative correction may be attributed to an anomalous electron magnetic moment, which has been already included by Schawlow and Townes. The remainder of the radiative correction has its source in virtual events occurring within an electron Compton wavelength of the nucleus. This part of the Lamb shift has a Z dependence stronger than the leading term in the fine structure splitting, but weaker than the nuclear volume effect. In the absence of any detailed analysis of the Lamb shift in heavy elements, it is tempting to suspect that this effect may account for the large radius deduced by Schawlow and Townes.[23]

2. Isotope Shift

If one corrects for nuclear deformation,[7] but ignores nuclear compressibility, one may deduce from the comprehensive data of Brix and Kopfermann[17] a "spherically symmetric, incompressible" radius of heavy nuclei given by $0.92\times 10^{-13}A^{1/3}$ cm, with a probable error of about 10 percent. This value is smaller than that given in reference 6 because of the assumption used there that isotope shift effect $\sim R^2$. The effect of a finite compressibility is to increase this number. If the nuclear radius varies as $A^{1/3}$ along the line of stable elements, the addition of neutrons only to a nucleus will increase the radius fractionally by[7]

$$\gamma[\delta A/3A], \tag{37}$$

where γ is a number less than 1.0. The equivalent nuclear radius will then be greater than that deduced on the incompressible assumption by a factor $(\gamma)^{-1/2\sigma}$.

TABLE I. Equivalent radii for nonuniform charge distributions (near $Z=82$). R=radius of uniform distribution. r_0=radial constant of distribution yielding same $2P_{3/2}\to 1S$ mu-mesonic transition energy as uniform distribution. $R_{eq}^{(1)}$=electrical radius determined by experiment which measures $\langle r^{2\sigma}\rangle_{Av}$. $R_{eq}^{(2)}$=electrical radius determined by experiment which measures $\langle r^2\rangle_{Av}$. $R_{eq}^{(3)}$=electrical radius determined by nuclear Coulomb energy.

Shape parameter	r_0/R	$R_{eq}^{(1)}/R$	$R_{eq}^{(2)}/R$	$R_{eq}^{(3)}/R$
		Family I		
$n=0$	0.2564	1.099	1.147	0.985
$n=1$	0.2039	1.079	1.117	0.994
$n=2$	0.1693	1.065	1.097	
$n=3$	0.1447	1.055	1.083	
$n=4$	0.1264	1.048	1.072	
$n=6$	0.1008	1.038	1.057	
$n=\infty$	1.0000	1.000	1.000	1.000
		Family II		
$n=0$	same as Family I			
$n=0.5$	0.2567	1.101	1.148	0.986
$n=1.0$	0.2551	1.098	1.145	0.986
$n=2.0$	0.4839	1.083	1.124	0.996
$n=4.0$	0.7554	1.041	1.062	1.006
$n=10.0$	0.9459	1.006	1.011	1.005
$n=\infty$	same as Family I			
		Family IV		
exp.	same as $n=0$			
mod. exp.	same as $n=1$, Family I			
Gaussian	0.6651	1.032	1.052	1.000
mod. Gauss.	0.5533	1.021	1.035	1.003
uniform	same as $n=\infty$			

Values of γ are given in Table III of reference 7. At $Z=80$, Feenberg's estimate[24] of nuclear compressibility ($E_0''=100A\ mc^2$) leads to $\gamma=0.683$, and therefore to an isotope shift equivalent radius of $R_0=1.16\times 10^{-13}$ cm. The near agreement with the predicted R_0 (1.17 to 1.29×10^{-13}) is fortuitous. There is an uncertainty of 10 percent from the correction for nuclear deformation, and a change in the compressibility by 50 percent alters the radius by about 15 percent. The isotope shift does not provide at present an accurate way to measure nuclear radii, owing primarily to the uncertainty in nuclear compressibility. To the extent that the heavy element Lamb shift is a function of Z only and independent of the precise distribution of protons in the nucleus, the isotope shift will be unaffected by radiative corrections to the energy levels.

3. Electron Scattering

Bitter and Feshbach[6] analyze the experiments of Lyman *et al.*[25] and quote results of Raka *et al.*[26] on electrical radii deduced from medium energy electron scattering. For several heavy elements, values of R_0 from $(1.03 \text{ to } 1.2)\times 10^{-13}$ cm are obtained. These are lower than predicted by the mu-mesonic experiment. According to Figs. 1 and 2, the radius deduced from elec-

[22] Breit, Arfken, and Clendenin, Phys. Rev. **78**, 390 (1950).
[23] The possible significance of the radiative corrections to the $p_{1/2}$ level has also been mentioned by N. M. Kroll, Phys. Rev. **94**, 747(T) (1954) and C. H. Townes, Phys. Rev. **94**, 773(T) (1954).
[24] E. Feenberg, Revs. Modern Phys. **19**, 239 (1947).
[25] Lyman, Hanson, and Scott, Phys. Rev. **84**, 626 (1951).
[26] Hammer, Raka, and Pidd, Phys. Rev. **90**, 341 (1953).

TABLE II. Summary of radial measurements for Pb. Predicted electrical radii based on $2P_{3/2}\rightarrow 1S$ mu-mesonic transition energy of 6.0 Mev. Uncertainties in predicted and measured values discussed in text.

	$R_0 \equiv R_{eq}/A^{\frac{1}{3}}$ (units 10^{-13} cm)			
	Predicted		Measured	
Effect	Uniform	Exponential		Uniform
1. X-ray fine structure	1.17	1.29		1.77
2. Isotope shift	1.17	1.29	no compressibility	0.92
			standard compressibility	1.16
3. Medium energy Electron scattering	1.17	1.34		1.03–1.2
4. Coulomb energy	1.17	1.15		1.23

tron scattering should not be less than $1.17\times 10^{-13}A^{1/3}$ cm for distributions with maximum charge density at $r=0$, and could be as large as about $1.3\times 10^{-13}A^{1/3}$. The electron scattering results, therefore, although not yet of high accuracy, are suggestive of a charge distribution nearly uniform or even peaked at the edge of the nucleus.

4. Coulomb Energy

Charge distributions for Pb which are equivalent for the mu-mesonic transition are also approximately equivalent for Coulomb energy. In the realm of reasonable shapes, both measure approximately $\langle r^{0.8}\rangle_{\text{Av}}$. Proton correlation effects, important at low Z, are probably negligible for heavy elements.[8] But at high Z, there is no direct way to measure Coulomb energy. The balance of Coulomb repulsion *vs* effective surface tension, as deduced from fission thresholds[27] or collective rotational states of nuclei,[28] provides only a crude value for the Coulomb energy. The best measure of Coulomb energy in heavy elements is the semiempirical mass formula. A new determination of Coulomb energy from this source,[29] translated to equivalent radius by Bitter and Feshbach,[6] gives $R_0=1.23\times 10^{-13}$ cm, in rather good agreement with the prediction of $(1.16 \text{ to } 1.18)\times 10^{-13}$ cm—especially since the latter figure applies to Pb only, for which the mu-meson determined radius is somewhat smaller than the trend from other elements would imply.

The nuclear electrical radius data discussed here are summarized in Table II.

V. CONCLUSION

If one adopts the mu-mesonic $2P_{3/2}\rightarrow 1S$ transition energy as an accurate constraint on the nuclear charge distribution, then: (a) the electrical radius deduced from x-ray fine structure is considerably larger than can be explained by a change of shape of the nuclear charge distribution; (b) the electrical radius deduced from atomic isotope shift can be brought into agreement with any reasonable shape of charge distribution, but is uncertain by about 25 percent owing to uncertainty in the magnitude of nuclear compressibility; (c) the electrical radius deduced from medium-energy electron scattering is inconsistent with a strong central peaking of the charge density, and is even suggestive of a peaking at the edge of the nucleus; (d) the electrical radius deduced from Coulomb energy is only a few percent too large. The major uncertainty in the x-ray effect is the radiative corrections. The electronic $2p_{1/2}$ shift may provide a better way of measuring the heavy-element Lamb shift than of measuring the electrical size of the nucleus. Similarly, the isotope shift provides a better way of measuring nuclear compressibility than of measuring nuclear electrical radius. No major uncertainties in the electron scattering effect or in the nuclear Coulomb energy are known, and these two measurements of nuclear electrical radius are not in serious disagreement with the mu-meson determined electrical radius. The Coulomb energy is a check of consistency on the nuclear electrical size, but can cast no additional light on the shape of the charge distribution. A refinement in the scattering effect to a point where the error in the electrical radial determination is less than about 3 percent will make possible a definite conclusion about the shape of the charge distribution or will else reveal a discrepancy in the mu-meson and electron determined electrical radii.

APPENDIX: SOME INTEGRALS

We list here the integrals $K_f(\sigma;\infty)$ and $L_f(\infty)$ defined by Eqs. (27) and (33). The weighted mean value of $r^{2\sigma}$ over the nuclear charge distribution is proportional to K_f; the nuclear Coulomb energy is proportional to L_f. The families of charge distribution considered are defined in I.[1]

Family I

$$K_f(\sigma;\infty)=\Gamma(2\sigma+4+n)/[n!(2\sigma+3)],$$

$$L_f(\infty)=\binom{n+2}{2}-\binom{n+3}{3}^{-1}S,$$

$$S=(n+1)\left(\frac{1}{2}\right)^{2n+2}\binom{2n+7}{n}$$
$$\times[(n+4)F(-n-1,6;n+8;-1)$$
$$-(n+2)F(-n,6;n+8;-1)]+4\binom{n+5}{5}$$
$$-(n+2)\left(\frac{1}{2}\right)^{2n+2}\binom{2n+6}{n}$$
$$\times[(2n+7)F(-n-1,6;n+7;-1)$$
$$-(2n+4)F(-n,6;n+7;-1)].$$

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[27] D. L. Hill and J. A. Wheeler, Phys. Rev. **89**, 1102 (1953).
[28] A. Bohr and B. R. Mottelson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **27**, No. 16 (1953).
[29] A. E. S. Green and N. A. Engler, Phys. Rev. **91**, 40 (1953).

Family IIa $(n \leqslant 1)$

$$K_f(\sigma;\infty)=\frac{\Gamma(2\sigma+3)}{1-\frac{1}{2}e^{-n}}\left[\frac{1}{2}e^n-\sum_{k=0}^{\infty}\frac{n^{2\sigma+5+2k}}{\Gamma(2\sigma+6+2k)}\right],$$

$$L_f(\infty)=\{(2/15)n^5+(2/3)n^3+(5/4)n^2-2n+(21/8)+e^{-n}(n^2-2)-(5/16)e^{-2n}\}/\{(1-\tfrac{1}{2}e^{-n})[e^{-n}+2n+(1/3)n^3]\}.$$

Family IIb $(n \geqslant 1)$

$$[K_f(\sigma;\infty)]_{\mathrm{IIb}}=(1/n^{2\sigma+3})[K_f(\sigma;\infty)]_{\mathrm{IIa}},$$

$$[L_f(\infty)]_{\mathrm{IIb}}=(1/n^2)[L_f(\infty)]_{\mathrm{IIa}}.$$

Family IV

Exponential

$$K_f(\sigma;\infty)=\Gamma(2\sigma+3),\quad L_f(\infty)=\tfrac{5}{8}.$$

Modified Exponential

$$K_f(\sigma;\infty)=(2\sigma+4)\Gamma(2\sigma+3),\quad L_f(\infty)=63/32.$$

Gaussian

$$K_f(\sigma;\infty)=\tfrac{1}{2}\Gamma(\sigma+(3/2)),\quad L_f(\infty)=2^{-3/2}.$$

Modified Gaussian

$$K_f(\sigma;\infty)=\tfrac{1}{2}(\sigma+(5/2))\Gamma(\sigma+(3/2)),$$
$$L_f(\infty)=(83/80)2^{-1/2}.$$

Uniform

$$K_f(\sigma;\infty)=(1/2\sigma+3),\quad L_f(\infty)=\tfrac{2}{5}.$$

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Search for Anomalous Positively Charged Particles from P^{32}*

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With a small β-ray spectrometer of unique design, the electrically charged emanations of radioactive P^{32} have been analyzed in an attempt to verify evidence found by others of the emission of positively charged particles in concentrations of the order of 10^{-3} to 10^{-4} per β decay. The ratio of the yield of positively charged particles to that of negatively charged particles in the momentum interval $H\rho=700$ to 2700 gauss cm was found to be less than 8×10^{-6}. This ratio is about 100 times smaller than earlier determinations in the same momentum interval obtained with cloud chambers and small spectrometers, but agrees in order of magnitude with a previous result obtained with an ordinary-sized spectrometer. The hypothesis that the anomalous "positive particles" in question are unstable and detectable only at short distances from the source would account for a low positive-particle yield measured with an ordinary spectrometer but cannot account for the disparity between the present results and those arrived at repeatedly with cloud chambers and other "short path length" detectors. It appears that the previously reported "positive particle" ratios in the range 10^{-3} to 10^{-4} arise from spurious background effects.

INTRODUCTION

When a β emitter is placed in a magnetic cloud chamber there appear among the tracks which ostensibly emanate from the source a certain small fraction which exhibit a sense of curvature characteristic of positively charged particles. Tracks of this nature have been observed in studies of such diverse substances as RaA, RaC, RaE, Th(C+C″), UX, and P^{32}.[1] In some cases[2,3] the positive particle yields (per β decay) from the same substance as determined with different source and chamber geometries have differed by several orders of magnitude, but to the best of our knowledge the lowest yield observed in any cloud chamber study is 10^{-4} "positives" per β decay. In every cloud chamber investigation the "positive" yield has exceeded by at least one order of magnitude that expected from positrons created in the source by electrons (electron pair-production), source bremsstrahlung or nuclear γ rays.

Some investigators have chosen to regard the "positives" as a completely spurious phenomenon attributable to one or more of the following effects: (1) electrons scattered from the walls back to the source, (2) electrons returning to the source after traversing a complete circle partially out of view, (3) electrons emerging from the source and being multiply scattered so as to assume a reversed curvature. Others have examined the spurious effects in more or less detail and have concluded that the "positive" tracks are really produced by positively charged particles—either positrons or some light particle distinct from the positron.

In recent years several cloud-chamber investiga-

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[1] As the present paper concerns only P^{32} and as extensive bibliographies relating to the other substances listed have been given elsewhere (see, e.g., reference 2) we omit references to the latter.

[2] L. Smith and G. Groetzinger, Phys. Rev. **70**, 96 (1946).

[3] G. Groetzinger and F. Ribe, Phys. Rev. **87**, 1003 (1952).