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Reaction Concept in Electromagnetic Theory

V. H. RUMSEY
Ohio State University, Columbus, Ohio

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A physical observable called the reaction is defined to simplify the formulation of boundary value problems in electromagnetic theory. To illustrate its value it is used to obtain formulas for scattering coefficients, transmission coefficients, and aperture impedances. An approximate solution to problems of this type is obtained by replacing the correct source (of the scattered field for example) with an approximate source which is adjusted so that its reaction with certain "test" sources is correct. This insures that the approximate source "looks" the same as the correct source according to the physical tests which are inherent in the problem. The formulas so obtained have a stationary character (for the cases considered) and thus the results could also be obtained from a variational approach. However the physical approach has two important advantages. It is general whereas the variational technique has to be worked out for each problem. It is conceptually simple and leads directly to results which might not be uncovered by the variational approach because of the complexity of the mathematical formulation. The problem of scattering by a dielectric body is used to illustrate this latter point.

INTRODUCTION

THE classical analysis of electromagnetic waves is based on the theory of fields which satisfy Maxwell's equations. It is interesting to examine this concept from the point of view of an experimenter whose objective is to use the theory to correlate his measurements. Suppose that we attempt to measure the field radiated by some source of electromagnetic energy by observing the signal received at the terminals of an antenna placed at the point of observation. By moving the antenna around we can obtain a considerable amount of information about the given field, but it is very difficult to relate this information to the classical field parameters, e.g., the electric field. Indeed from a literal point of view the postulate of electric field might be questioned on the grounds that any experiment designed to measure the electric field at a point must necessarily consist of measuring the effect of the field over a small but finite region, and therefore the postulate is incompatible with the process of performing an observation. This suggests that it is desirable to introduce into the theory a fundamental observable which represents measurements which can actually be performed.

DEFINITIONS AND PROPERTIES OF REACTION

We introduce a quantity, called the "reaction," which is defined as follows. Let the source of a monochromatic electromagnetic field consist of the volume distributions of electric and magnetic current $d\mathbf{J}$ and $d\mathbf{K}$ (i.e., the electric dipole moment contained in a volume V is equal to the vector $\int\int\int_V d\mathbf{J}$). (The word source is used in the sense that everywhere in the given region there is no field when the source is absent. Thus currents which may be induced in various parts of the region are not counted as sources because they vanish when the true source is turned off.) Let $\mathbf{E}(a)$ and $\mathbf{H}(a)$ represent the electric and magnetic fields generated by the source distributions $d\mathbf{J}(a)$ and $d\mathbf{K}(a)$, which we call the source a for simplicity. Similarly for some other source b, which generates a field at the same frequency and in the same environment. Define a complex number [the usual $\exp(\pm i\omega t)$ time convention is employed], denoted by $\langle a,b\rangle$ as follows:

$$\langle a,b\rangle=\int\int\int_V[\mathbf{E}(b)\cdot d\mathbf{J}(a)-\mathbf{H}(b)\cdot d\mathbf{K}(a)], \quad (1)$$

where the volume V contains the source a.

The reciprocity theorem[1] states that

$$\langle a,b\rangle=\langle b,a\rangle, \quad (2)$$

provided that all media are isotropic and that a and b can be contained in a finite volume.

The scalar $\langle a,b\rangle$ is a measure of the reaction (or coupling) between the sources a and b. We think of "reaction" as a physical observable like mass, length, charge, etc.: Eq. (1) is to be understood as a formula for the measure of reaction. For example, in electrostatic theory, let the source a consist of the volume distribution of charge $dq(a)$: similarly for the source b. Then we could define the static reaction (a,b) by the relation

$$(a,b)=\int\int\int\mathbf{E}(b)dq(a), \quad (3)$$

which is analogous to Eq. (1). In this case the physical observable which (a,b) represents is the resultant force exerted by a on b. In the oscillating case let the source a consist of a unit current generator connected to the terminals of some antenna. Then it follows from Eq. (1) that $\langle a,b\rangle$ is equal to the open circuit voltage generated at the antenna terminals by the source b. Again for static fields the electric field is equal to (u,a) the reaction between the given source a and an infinitesimal unit charge u placed at the point of observation.

[1] S. A. Schelkunoff, *Electromagnetic Waves* (D. Van Nostrand Publishing Company, New York, 1943), p. 477. The proof given applies here with a minor extension.

For oscillating fields the component of electric field in a particular direction is equal to $\langle u,a\rangle$, the reaction between the given source a and an infinitesimal electric dipole u, of unit moment, placed parallel to this direction at the point of observation.

The complex number $\langle a,b\rangle$ consists of the sum of products of the form $V(a)I(b)$ where $V(a)$ represents the voltage generated by a across the terminals of a current generator of strength $I(b)$. If

$$V(a)=|V(a)|\exp(i\alpha),\quad I(b)=|I(b)|\exp(i\beta),$$

then the corresponding explicit time functions which these complex numbers represent are

$$V(a,t)=|V(a)|\cos(\omega t+\alpha),\quad I(b,t)=|I(b)|\cos(\omega t+\beta),$$

and the complex number,

$$V(a)I(b)=|V(a)I(b)|\exp[i(\alpha+\beta)],$$

represents the explicit time function,

$$|V(a)I(b)|\cos(2\omega t+\alpha+\beta) = |V(a)I(b)|[2\cos(\omega t+\alpha)\cos(\omega t+\beta)-\cos(\alpha-\beta)].$$

The first term in the square brackets is the instantaneous rate at which b works against a, and the second term is the average rate per cycle at which b works against a. Thus, $\langle a,b\rangle$ represents the instantaneous minus the average rate at which b expends energy on a. Note that if $\langle a,b\rangle=0$, then no energy is transferred from a to b at any time.

Observe that

$$\langle a,(b+c)\rangle=\langle a,b\rangle+\langle a,c\rangle, \tag{4}$$

where a, b, and c represent any three sources radiating at the same frequency in the same region. Note also that if A represents any scalar and Aa represents the source a increased in strength by the factor A, then

$$\langle Aa,b\rangle=A\langle a,b\rangle. \tag{5}$$

Before proceeding any further with the formal development let us now discuss how the concept of reaction can be used, in order to see what further developments are needed.

APPLICATIONS OF THE REACTION CONCEPT

Consider the problem of scattering by a perfectly conducting body, of surface S, which is irradiated by the source g (see Fig. 1). Let $\mathbf{J}(c)$ represent the surface distribution of electric current which is induced on the scatterer by g. The scattered field, $\mathbf{E}(c)$, is defined as the field that would be generated by $\mathbf{J}(c)$ (acting as a source) if the scatterer were absent (the total field is then the superposition of the incident and scattered fields). We postulate that if $\mathbf{J}(c)$ were known the scattered field $\mathbf{E}(c)$ could be calculated. It is known that the tangential component of $\mathbf{E}(c)$ on S is equal to minus the tangential component of the incident electric field, because the scatterer is a perfect conductor. Thus, although $\mathbf{J}(c)$ is unknown it is possible to calculate the reaction between the source $\mathbf{J}(c)$ and some known current distribution, $\mathbf{J}(a)$, on the surface S, for

FIG. 1. A hypothetical surface S in the presence of a source g.

$$\langle a,c\rangle=\iint_S \mathbf{J}(a)\cdot\mathbf{E}(c)dS \tag{6}$$

and the tangential component of $\mathbf{E}(c)$ on S is known.

Suppose that we wish to calculate the "echo," i.e., the signal at g due to the scattered field. If we think of g as a unit current generator connected to the terminals of some antenna, the open circuit voltage at these terminals generated by the scattered field is equal to $\langle g,c\rangle$. Thus the problem is to calculate $\langle g,c\rangle$. Let $\mathbf{J}(a)$ represent an assumed distribution of electric current on S which we propose to adjust so that it approximates $\mathbf{J}(c)$. We would like to adjust $\mathbf{J}(a)$ so that

$$\langle g,a\rangle=\langle g,c\rangle, \tag{7}$$

for then the echo obtained by substituting $\mathbf{J}(a)$ for $\mathbf{J}(c)$ would be correct. Obviously this is too much to expect: indeed we find that Eq. (7) cannot be enforced because we cannot calculate $\langle g,c\rangle$. We can interpret Eq. (7) as the condition that the approximate source a (i.e., $\mathbf{J}(a)$) should "look" the same as the correct source c (i.e., $\mathbf{J}(c)$) to the source g, in the sense that a and c produce the same signal at g. Thus we can think of g as a "test" source which is used to test for any difference between a and c. This suggests that we regard Eq. (7) as a special case of the more general restriction

$$\langle x,a\rangle=\langle x,c\rangle, \tag{8}$$

which expresses the condition that a and c should "look" the same to an arbitrary test source x. The problem is now a matter of trying to enforce Eq. (8) for every "available" test source, i.e., every x for which $\langle x,a\rangle$ and $\langle x,c\rangle$ can be calculated. The only sources in the problem are g, c and a. We have seen that g is not "available" because $\langle g,c\rangle$ cannot be calculated and we find that c is not available because $\langle c,c\rangle$ cannot be calculated. Thus a is the only "available" test source. We therefore adjust the approximation a to satisfy the condition

$$\langle a,a\rangle=\langle a,c\rangle \tag{9}$$

and use the value of a so obtained in place of the correct source c.

To carry out the calculation of the echo we assume some current distribution $\mathbf{J}(a)$ whose level can be adjusted, i.e., let

$$\mathbf{J}(a)=U\mathbf{J}(u),\quad\text{or}\quad a=Uu, \tag{10}$$

where $\mathbf{J}(u)$ is fixed and U is an adjustable constant. Substituting for a in Eq. (9) gives

$$U=\langle u,c\rangle/\langle u,u\rangle. \tag{11}$$

Now the echo $\sim-\langle c,g\rangle$, where

$$\begin{aligned}-\langle c,g\rangle&=\int\int_S \mathbf{J}(c)\cdot\mathbf{E}(c)dS\\&=\langle c,c\rangle\simeq\langle a,a\rangle=U^2\langle u,u\rangle=\langle u,c\rangle^2/\langle u,u\rangle\\&=\left[\int\int_S \mathbf{J}(u)\cdot\mathbf{E}(c)dS\right]^2\Big/\\&\qquad\int\int_S \mathbf{J}(u)\cdot\mathbf{E}(u)dS.\end{aligned} \tag{12}$$

In Eq. (12) $\mathbf{J}(u)$ represents the assumed current distribution, $\mathbf{E}(u)$ represents the electric field generated by $\mathbf{J}(u)$ (in the absence of the scatterer), and $\mathbf{E}(c)$ represents the tangential component at S of the incident electric field.

The approximation can be improved by starting from an assumed distribution $\mathbf{J}(a)$ which contains a number of adjustable constants. Thus, let

$$a=Ll+Mm+\cdots, \tag{13}$$

where L, M, $\cdots$ represent adjustable constants and, l, m, $\cdots$ represent fixed source distributions which are assumed. The problem is to find the linear combination of l, m, $\cdots$, denoted by a, which best approximates the correct source $\mathbf{J}(c)$, denoted by c. Here we can enforce the conditions:

$$\begin{aligned}\langle a,l\rangle&=\langle c,l\rangle,\\\langle a,m\rangle&=\langle c,m\rangle,\\\cdot\quad&\quad\cdot\\\cdot\quad&\quad\cdot\\\cdot\quad&\quad\cdot\end{aligned} \tag{14}$$

which ensure that a and c "look" the same to l, m, $\cdots$. If the assumed set of source distributions l, m, $\cdots$ constitute a complete orthogonal set, the equations (14) represent the condition that a and c are identical in every respect. Substituting for a from (13) in (14) gives the following equations for the constants:

$$\begin{aligned}L\langle l,l\rangle+M\langle m,l\rangle+\cdots&=\langle c,l\rangle,\\L\langle l,m\rangle+M\langle m,m\rangle+\cdots&=\langle c,m\rangle,\\\cdot\qquad\quad\cdot\qquad\quad&\quad\cdot\\\cdot\qquad\quad\cdot\qquad\quad&\quad\cdot\\\cdot\qquad\quad\cdot\qquad\quad&\quad\cdot\end{aligned} \tag{15}$$

In terms of a matrix notation, the constants L, M, $\cdots$ are given explicitly by

$$\begin{bmatrix}L\\M\\\cdot\\\cdot\\\cdot\end{bmatrix}=\begin{bmatrix}\langle l,l\rangle&\langle l,m\rangle&\cdots\\\langle m,l\rangle&\langle m,m\rangle&\cdots\\\cdot&\cdot&\\\cdot&\cdot&\\\cdot&\cdot&\end{bmatrix}^{-1}\begin{bmatrix}\langle l,c\rangle\\\langle m,c\rangle\\\cdot\\\cdot\\\cdot\end{bmatrix}. \tag{16}$$

By substituting for L, M, $\cdots$ in (13), the approximate value of the echo is given by (see Eq. (12)):

$$\langle a,a\rangle=\langle a,c\rangle=(\langle l,c\rangle\langle m,c\rangle\cdots)\times\begin{bmatrix}\langle l,l\rangle&\langle l,m\rangle&\cdots\\\langle m,l\rangle&\langle m,m\rangle&\cdots\\\cdot&\cdot&\\\cdot&\cdot&\\\cdot&\cdot&\end{bmatrix}^{-1}\begin{bmatrix}\langle l,c\rangle\\\langle m,c\rangle\\\cdot\\\cdot\\\cdot\end{bmatrix}. \tag{17}$$

These results can also be obtained by means of the variational technique. For example, let δa represent a slight change of the source distribution represented by a. If both a and $a+\delta a$ satisfy Eq. (9), then $2\langle a,\delta a\rangle=\langle a,\delta a\rangle+\langle\delta a,a\rangle=\langle\delta a,c\rangle$ to the first order. If δa represents a slight change about the correct distribution c, we can substitute c for a in this equation and thus obtain $\langle\delta a,c\rangle=0$. Hence, the expressions $\langle a,a\rangle$ and $\langle a,c\rangle$ are stationary for variations of a about c, if a satisfies Eq. (9), i.e., if $a=u\langle u,c\rangle/\langle u,u\rangle$ [see Eqs. (10) and (11)]. Thus the expression $x=\langle a,c\rangle=\langle a,a\rangle=\langle u,c\rangle^2/\langle u,u\rangle$ is stationary for variations of the assumed distribution $\mathbf{J}(u)$ about the correct distribution $\mathbf{J}(c)$, and Eqs. (14) and (15) can be obtained by setting $\partial x/\partial L=0$, $\partial x/\partial M=0$, etc. However, the fact that an expression is stationary for variations of an assumed distribution about the correct distribution does not justify the assumption that it will yield the "best" approximation when the assumed distribution is completely arbitrary. The reaction approach does show that the approximation is the best in a physical sense, i.e., in the sense that the approximate source "looks" the same as the correct source to any source in the problem which can be used for such an observation. More precisely, the reaction between the approximate source and every available test source is correct, it being understood that a test source is "available" if its reaction with the correct source can be calculated. Note that this comparison of the variational method with the reaction method is, so far, based on the specific problem of scattering by a perfect conductor. There are other problems (some of which we consider later) where a number of different approximations can be obtained from the variational method, and there is no way of deciding which approximation is best. In the reaction approach such problems yield an excessive number of test sources, i.e., the number of independent test sources exceeds the number of adjustments which can be made in the approximation, so that it is not possible to make the reaction with every test source correct. In this case it is necessary to decide what selection of the available test sources is most likely to yield the best approximation. From the physical point of view the answer is clearly that selection which most nearly represents the actual physical observation which we are trying to approximate. The physical approach has the added advantages of being general (whereas the stationary formulation has to be established for each specific problem), and of providing a simple understanding of the type of approximation which is being used. In short, the fundamental advantage of the reaction method is its conceptual simplicity which leads directly to results which might not be uncovered by the variational approach because of the complexity of the mathematical formulation. These points are illustrated by the examples which follow.

TRANSMISSION CALCULATIONS

The problem of scattering by a perfect conductor can be used to illustrate the formulation of transmission problems. We are now interested in the signal received at an arbitrary point whereas in the echo problem the point of reception was at the given source g. As before the total field is represented as the sum of contributions from g and c (the induced electric current distribution on S) radiating as if the scatterer were absent. The contribution from g can be calculated easily since g is given. We therefore put up a source h at the point of observation (see Fig. 2) and the problem is to calculate the reaction $\langle c,h\rangle$ evaluated in the absence of the scatterer. Let the notation be as follows: c generates the same field as g inside of S (as before), d generates the same field as h inside of S, a is the approximation for c (as before), b is the approximation for d. The sources a, b, c, and d are all distributions of electric current on S, and all sources radiate as if the scatterer were absent. We therefore adjust the approximations a and b to satisfy the conditions:

$$\langle a,x\rangle=\langle c,x\rangle, \tag{18}$$

and

$$\langle b,y\rangle=\langle d,y\rangle, \tag{19}$$

where x and y represent any test sources inside of S. The available test sources are represented by $x=a$, $x=b$, $y=a$ and $y=b$. Since Eq. (18) cannot be satisfied for both values of x, we have to decide which value is likely to give the better approximation: similar remarks apply to Eq. (19). Now the quantity which we are trying to approximate is $\langle c,h\rangle$ where

$$\langle c,h\rangle=\int\int_S \mathbf{J}(c)\cdot\mathbf{E}(h)dS$$
$$=\int\int_S \mathbf{J}(c)\cdot\mathbf{E}(d)dS=\langle c,d\rangle. \tag{20}$$

To approximate $\langle c,d\rangle$ we replace c by its approximation a, or d by b, or both. Thus there are three possible approximations which are represented by $\langle a,d\rangle$, $\langle c,b\rangle$ and $\langle a,b\rangle$. We see that these are all the same if we choose $x=b$ and $y=a$ which is therefore the best choice.

FIG. 2. Two sources g and h in the presence of a conductor.

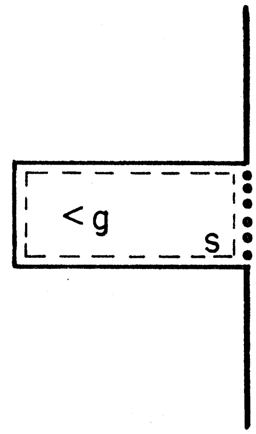


FIG. 3. A source g radiating through a waveguide.

An explicit formula for $\langle c,d\rangle$ is obtained by setting

$$a=Uu \quad\text{and}\quad b=Vv, \tag{21}$$

where U and V are adjustable constants. Proceeding as before [see Eq. (10)], we obtain

$$\langle c,d\rangle\simeq\langle c,v\rangle\langle d,u\rangle/\langle u,v\rangle, \tag{22}$$

where u and v represent assumed electric current distributions. In terms of integrals over S, Eq. (22) becomes

$$\langle c,d\rangle\simeq\left[\int\int_S \mathbf{E}(g)\cdot\mathbf{J}(v)dS\right]\left[\int\int_S \mathbf{E}(h)\cdot\mathbf{J}(u)dS\right]\Big/\int\int_S \mathbf{E}(u)\cdot\mathbf{J}(v)dS, \tag{23}$$

where $\mathbf{J}(u)$ and $\mathbf{J}(v)$ are the current distributions which we assume to approximate the current distributions induced on the scatterer by g and h, respectively.

IMPEDANCE CALCULATIONS

Suppose that the given source g radiates into space through a length of uniform wave guide as shown in Fig. 3. This represents a type of problem which has been treated successfully by means of the variational approach.[2]

We postulate that it is impractical to calculate the field which is radiated through the aperture (the aperture is shown as the line of dots in Fig. 3), but that it is practical to calculate the field (in the waveguide) that would be obtained if the aperture were covered with a conducting plate.

Let $\mathbf{J}(c)$ represent the electric current distribution that would be induced by g on a conducting plate covering the aperture. Let $\mathbf{E}(g,1)$ represent the field generated by g with the plate in position, $\mathbf{E}(g,2)$ represent the field generated by g with the plate removed, and $\mathbf{E}(c,2)$ the field that would be generated

[2] *Waveguide Handbook*, edited by N. Marcuvitz (McGraw-Hill Book Company, Inc., New York, 1951).

by $\mathbf{J}(c)$ (acting as a source) with the plate removed. Then,

$$\mathbf{E}(g,1)=\mathbf{E}(g,2)+\mathbf{E}(c,2), \tag{24}$$

for points in the wave-guide region (inside of the dashed curve S shown in Fig. 3). If g consists of a unit current generator connected to a pair of terminals, then it follows from (24) that

$$Z_1=V_1=V_2+\langle c,g\rangle=Z_2+\langle c,g\rangle, \tag{25}$$

where V_1 and V_2 are the voltages at these terminals with and without the plate respectively, Z_1 and Z_2 are the corresponding impedances, and $\langle c,g\rangle$ is evaluated with the plate removed. The problem is to calculate an approximation for Z_2, assuming that Z_1 is known. For example, suppose that the wave-guide structure consists of a biconical horn as illustrated in Fig. 4. The source g consists of a unit current generator connected to the input. Then

$$Z_1=-i\mu^{\frac{1}{2}}\epsilon^{-\frac{1}{2}}\pi^{-1}\log(\cot\alpha/2)\tan\omega l\mu^{\frac{1}{2}}\epsilon^{-\frac{1}{2}},$$
$$\mathbf{J}(c)=[2\pi l\cos\omega l\mu^{\frac{1}{2}}\epsilon^{\frac{1}{2}}\sin\theta]^{-1}\boldsymbol{\theta},$$

where l and θ are given in Fig. 4, ω represents the frequency, μ and ϵ represent the permeability and inductive capacity of the medium inside of S, and $\boldsymbol{\theta}$ represents a unit vector in the spherical coordinate system shown in Fig. 4.

The specific example which we have chosen to illustrate impedance calculations is representative of the general problem in which g is inside of as illustrated in Fig. 5. It is essentially the same as the scattering problem in which g is outside of S. Here, as in the scattering problem, $-\mathbf{J}(c)$ generates the same field as g on the source-free side of S. Here, $\langle g,c\rangle$ cannot be calculated because tangential $\mathbf{E}(g)$ at S is unknown, whereas $\langle g,c\rangle$ cannot be calculated in the scattering problem because $\mathbf{J}(c)$ is unknown. In this problem we therefore assume some approximation for $\mathbf{E}(c)$, the tangential component of electric field in the aperture (or over S in the more general terminology). We postulate that it is possible to calculate $\mathbf{E}(a)$, the field which fits this assumed distribution of tangential $\mathbf{E}(c)$ on both sides of S. By definition the tangential component of $\mathbf{E}(a)$ at S is continuous but the tangential component of $\mathbf{H}(a)$ obviously is not continuous. In short, the field $\mathbf{E}(a)$ is generated by a certain distribution of electric currents $\mathbf{J}(a)$ on S which can be calculated from the assumed distribution of tangential electric field. If this assumed distribution were correct, then $\mathbf{J}(a)$ would turn out to be identical to $\mathbf{J}(c)$. We can see now that the impedance problem is formally identical to the scattering problem, with the modification that the approximate source a is now required to look the same as the correct source c to a test source x which is outside of S. Thus Eq. (9) applies again:

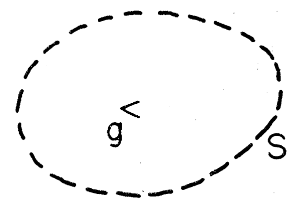


FIG. 5. A source g inside of a closed surface S.

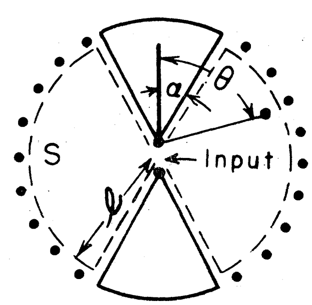


FIG. 4. A biconical horn.

$$\langle a,a\rangle=\langle a,c\rangle. \tag{9}$$

Instead of Eq. (10), we now put

$$\mathbf{E}(a)=U\mathbf{E}(u), \tag{10a}$$

where $\mathbf{E}(u)$ represents the assumed distribution of tangential $\mathbf{E}(c)$ over S and U is an adjustable constant. Equation (10a) is equivalent to Eq. (10) because it implies that

$$a=Uu, \tag{10}$$

as before. The approximation for $\langle c,g\rangle$ given by (12) also applies except that we express $\langle u,c\rangle$ in the form $\iint_S \mathbf{E}(u)\cdot\mathbf{J}(c)dS$. Thus, the approximation for the impedance obtained from Eq. (25) (replacing c by a) is

$$Z_2=Z_1-\langle u,c\rangle^2/\langle u,u\rangle. \tag{26}$$

The problem of transmission through the aperture can be treated by placing a test source h at the point of observation outside of S and proceeding as in Eqs. (18)–(23).

SCATTERING BY A DIELECTRIC

The problem of scattering by a dielectric brings out some interesting features which have not been encountered up to this point. Let the given source g radiate in the presence of a dielectric scatterer of surface S, as in Fig. 1. The scattered field is again defined as the difference between the fields generated by g with and without the scatterer. The problem is to find a source, a, which generates a field outside of S which is approximately equal to the scattered field (and such that it is possible to calculate the field generated by a).

Let $\mathbf{E}(g,1)$ represent the field generated by g in the presence of the scatterer, and

$$\mathbf{J}(c)=\mathbf{n}\times\mathbf{H}(g,1),\quad \mathbf{K}(c)=\mathbf{E}(g,1)\times\mathbf{n}, \tag{27}$$

where $\mathbf{n}$ represents a unit vector normal to S pointing inwards. The surface distributions of electric and magnetic current $\mathbf{J}(c)$ and $\mathbf{K}(c)$, are the equivalent currents of Schelkunoff.[3] Let the combination of $\mathbf{J}(c)$ and $\mathbf{K}(c)$

[3] S. A. Schelkunoff, Phys. Rev. 56, 308 (1939).

be represented by the symbol c. Then the combination of g and $-c$ generates a field which is zero everywhere inside of S. Therefore we can remove the scatterer without affecting the field generated by $(g-c)$. It follows that the scattered field is equal to the field which c generates outside of S, in the absence of the scatterer. Thus the problem is to find an approximation for c. Observe that c generates the same field as g inside of S, regardless of the medium inside of S. Thus we require that the approximation for c, represented by a, should look the same as g to a test source x inside of S, and for the purpose of simplifying calculations we can perform this test in "free space" (see Fig. 6).

At this point it is desirable to extend the notation [introduced in Eq. (24)] in which a number represents the environment, as follows. Let $\langle p,1,q\rangle$, $\langle p,2,q\rangle$, and $\langle p,3,q\rangle$ represent the reaction between sources p and q in the presence of the scatterer, in free-space, and in an infinite homogeneous dielectric medium, respectively. Thus the test illustrated by Fig. 6 is represented by the equation

$$\langle g,2,x\rangle=\langle a,2,x\rangle. \tag{28}$$

Observe that c generates zero field outside of S, in the presence of the scatterer. The region outside of S can therefore be filled with the same material as the inside of S without affecting the result. We therefore require a to satisfy the requirement,

$$\langle a,3,y\rangle=0, \tag{29}$$

where y represents any test source outside of S (see Fig. 7).

The coarsest approximation is obtained by representing a in the form

$$a=Ll+Mm, \tag{30}$$

where l and m represent the assumed distributions of electric and magnetic current on S respectively, and L and M are adjustable constants. Then the possible choices of x and y are represented by $x=l$, $x=m$, $x=a$, $y=l$, $y=m$, and $y=a$ (only four of which are independent). Observe that if we set $x=l$ in the expression $\langle a,2,x\rangle$, we must interpret this as the limit obtained as l approaches a from the inside of S as illustrated in Fig. 8. Similarly, the substitute for y must be taken just outside of S. An expression of the form $\langle a,2,a\rangle$ is evaluated by imagining two sources a slightly displaced from each other in the manner of Fig. 8. Considering the symmetry of Eqs. (28) and (29), the obvious

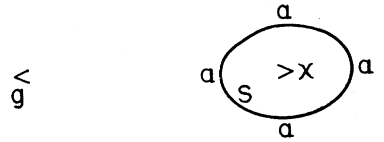


FIG. 6. A test source x set up to detect the difference between sources a and g.

choice appears to be $x=a$ and $y=a$. A further point in support of this choice is the fact that the echo $=\langle g,2,c\rangle=\langle c,2,c\rangle$ is approximated by $\langle c,2,a\rangle$ or $\langle a,2,a\rangle$ and the choice $x=a$ makes these two results the same. The choice $y=a$ is also supported by the fact that if a were correct (i.e., if $y=c$), its radiation would be confined to the region occupied by the source under test (which is also a), thus producing, in a sense, the maximum irradiation of the source under test.

Substituting $x=a$ in Eq. (28) and $y=a$ in Eq. (29) gives the following equations for the constants L and M:

$$L\langle g,2,l\rangle+M\langle g,2,m\rangle=L^2\langle l,2,l\rangle+LM\langle (l),2,m\rangle +LM\langle l,2,(m)\rangle+M^2\langle m,2,m\rangle, \tag{31}$$

$$0=L^2\langle l,3,l\rangle+LM\langle (l),3,m\rangle +LM\langle l,3,(m)\rangle+M^2\langle m,3,m\rangle, \tag{32}$$

where the notation (l) in $\langle (l),2,m\rangle$ indicates that l is just inside of m (see Fig. 8), e.g.,

$$\langle (l),2,m\rangle=\int\int_S \mathbf{J}(l)\cdot\mathbf{E}(m,2,\text{internal})dS, \tag{33}$$

where

$$\mathbf{n}\times[\mathbf{E}(m,2,\text{internal})-\mathbf{E}(m,2,\text{external})]=\mathbf{K}(m) \quad \text{[see Eq. (27)]}. \tag{34}$$

The approximation for the echo is then given by substituting a for c in $\langle g,2,c\rangle$.

It can be shown that the formula for the echo obtained in this way is stationary for variations of the assumed distribution of electric current $\mathbf{J}(l)$ and magnetic current $\mathbf{K}(m)$ about the correct distribution. The interesting point is that a number of stationary formulas for the echo can be derived, all of which are based on assumed distributions $\mathbf{J}(l)$ and $\mathbf{K}(m)$ and involve the same "free space" calculations, but differ in the way that the calculations are combined. Here is a case where the physical approach through the reaction concept leads to a result which probably would not have been uncovered by the variational technique, although, once the result has been established, it is possible to see how it could have been obtained by means of a variational approach.

Another approach is to set down the equations for continuity of tangential $\mathbf{E}$ and $\mathbf{H}$ which are:

$$\begin{aligned}\mathbf{n}\times\mathbf{E}(g,2)-\mathbf{n}\times\mathbf{E}(c,2,\text{external}) &=\mathbf{n}\times\mathbf{E}(c,3,\text{internal}),\\ \mathbf{n}\times\mathbf{H}(g,2)-\mathbf{n}\times\mathbf{H}(c,2,\text{external}) &=\mathbf{n}\times\mathbf{H}(c,3,\text{internal}).\end{aligned} \tag{35}$$

We multiply these equations by tangential $\mathbf{H}$ and $\mathbf{E}$, respectively, and integrate over S. This is represented by the equation

$$\langle g,2,z\rangle-\langle (a),2,z\rangle=\langle a,3,(z)\rangle, \tag{36}$$

where z represents a test source distributed over S and the correct source c has been replaced by the approximate source a. Here it is necessary to use every available test source, i.e., $z=l$ and $z=m$ in order to evaluate L and M. We then obtain two equations which are essentially the same as those used by Crowley[4] for a certain class of impedance calculations. The formula for the echo obtained from Eq. (36) is stationary but it is different from that given by Eqs. (28) and (29). An important point of difference is that the approximation given by Eq. (36) does not give the same value for the two possible approximations for the echo, $\langle g,2,a\rangle$ and $\langle a,2,a\rangle$.

There is a simple connection between Eqs. (28), (29), and (36). It is obtained from the relation

$$\langle (a),2,b\rangle-\langle a,2,(b)\rangle=\langle (a),3,b\rangle-\langle a,3,(b)\rangle, \qquad (37)$$

which we shall establish [see Eq. (47)]. In Eq. (37), a and b represent any two sources, and 2 and 3 represent any two environments. Thus, Eq. (36) can be written in the form

$$\langle g,2,z\rangle=\langle a,2,(z)\rangle+\langle (a),3,z\rangle. \qquad (38)$$

Bearing in mind that in Eqs. (28) and (29) x is inside of S and y is outside of S, we see that Eq. (36) is satisfied if Eq. (28) is satisfied for $x=z$ and Eq. (29)

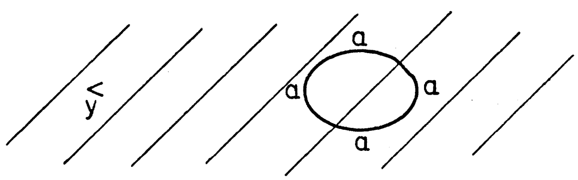
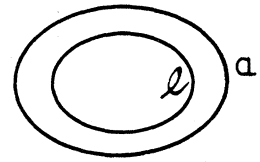


FIG. 7. A test source y set up to detect the external field from a.

for $y=z$. In this sense, Eqs. (28) and (29) embody the information contained in Eq. (36). We obtain different formulas for the echo because we choose to enforce Eq. (36) for two different values of z, $z=l$ and $z=m$, whereas we enforce Eqs. (28) and (29) for $x=a$ and $y=a$. To put it another way, if $x=y=z=a$ then the same value of a is obtained from any two of the three equations (28), (29), and (36). It is interesting to note that the value of $\langle g,2,a\rangle$ (the echo) obtained from (36) is stationary provided Eq. (36) is enforced for the single value of z, $z=a$. On the other hand, if (28) and (29) are enforced for $x=a$ and $y=a$, then all three quantities $\langle g,2,a\rangle$, $\langle a,2,a\rangle$ and $\langle a,3,a\rangle$ are stationary.

To give a specific comparison between these two approaches, the echo from an infinite plane dielectric slab for a plane wave at normal incidence, has been computed from Eqs. (28) and (29) with $x=a$ and $y=a$, and from Eq. (36) with $z=l$ and $z=m$. We note at the

FIG. 8. The source l just inside of the source a.



outset that the results of this comparison must be regarded as suggestive rather than conclusive for not only is the example specific, as opposed to general, but the assumed distribution is also specific. Note also that the assumed distribution, which is usually a function of position, in this case consists of four discrete values corresponding to the front and the back faces of the slab. Thus, if our approximation contains four independent adjustable constants, any set of tests which determines them uniquely is bound to yield the correct solution.

The two methods were compared on the basis of the type of approximation represented by Eq. (30), which contains two adjustable constants. A crude approximation is obtained by assuming equal and opposite electric and magnetic currents on the front and back faces, e.g., the distributions l and m can be represented by the pair of values $(+1,\,-1)$. For thin slabs and low dielectric constants both methods give an echo which is four times the correct value, and for low dielectric constants, the maximum echo, as a function of slab thickness, is about twice the correct value. The situation here is that the differences between the two methods are overshadowed by the crudeness of the assumed distribution, which emphasizes the overriding importance of starting from an assumed distribution which is nearly correct. A better approximation is obtained by assuming that l and m are represented by the distribution $(+1,\,-\exp it)$ where t represents the electrical thickness, i.e., $t=\omega\mu^{\frac{1}{2}}\epsilon^{\frac{1}{2}}d$, where $d=$ thickness of slab and μ and ϵ are the constants of the dielectric. This is the type of assumption used in physical optics and neglects reflection from the back face. For an incident field of one volt per meter the following formulas are obtained. The correct solution is

$$|\langle g,2,c\rangle|=\frac{2K|1-\exp 2it|}{\eta_0|1-K^2\exp 2it|}. \qquad (39)$$

The approximation given by Eqs. (28) and (29) is

$$|\langle g,2,a\rangle|=\frac{2K|1-\exp(it/1+K)|^2}{\eta_0|1-2\exp(2it/1+K)+\exp 2it|}. \qquad (40)$$

The approximation given by Eq. (36) is

$$|\langle g,2,a\rangle| =\frac{2K|1-\exp it/1+K|^2|1-\exp 2it|}{\eta_0|1-2\exp(2it/1+K)+\exp(4it/1+K)-K^2\exp(4it)+2K^2\exp[2it(2+K)/1+K]-K^2\exp(4it/1+K)|}. \qquad (41)$$

[4] T. H. Crowley, "Variational Impedance Calculations," Antenna Laboratory Report 478–5, Ohio State University (unpublished).

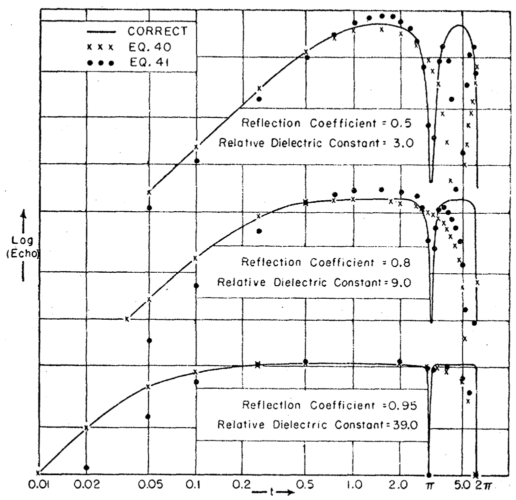


FIG. 9. The echo from a dielectric slab *versus* thickness.

In Eqs. (39), (40), and (41), K represents the reflection coefficient for $t=\infty$; i.e.,

$$K=(\epsilon_r-1)/(\epsilon_r+1), \tag{42}$$

where ϵ_r = relative dielectric constant of the dielectric.

These results are plotted in Fig. 9. It can be seen that the error in Eq. (40) is insignificant for $t\leqslant 1$ but becomes progressively worse for larger values of t until there is practically no correlation with the correct result. Equation (41) is in error for small values of t by the factor $(K+1)$, but, unlike Eq. (40), it is correct in the vicinity of $t=\pi$, although otherwise it shows practically no correlation with the correct result for large values of t. The behavior at $t=\pi$ is interesting because the assumed distribution is correct at this value of t, and therefore both (40) and (41) might be expected to reduce to the correct result at $t=\pi$. The failure of Eq. (40) at $t=\pi$ is due to the fact that the external field generated by l or m in a homogeneous dielectric is zero. Thus, while Eq. (29) is certainly satisfied it does not yield any information about the constants L and M, i.e., for $t=\pi$ the test represented by Eq. (29) turns out to be trivial for the particular assumed distribution under consideration. This suggests that instead we should put the test source y inside of S, but then we find that in order to arrive at an enforceable condition we are brought back to Eq. (36).

FURTHER PROPERTIES OF THE REACTION

The foregoing example of scattering by a dielectric shows the need for a more elaborate formulation to handle cases where Eq. (1) is inadequate. When a and b [see Eq. (1)] consist of surface distributions over the same surface S the field at a due to b is discontinuous and consequently the integral in Eq. (1) is not defined. In this case there are two possible values for the reaction which are obtained by imagining that a and b are slightly separated. If a is imagined to be just outside of b, we have

$$\langle a,(b)\rangle=\int\int_S[\mathbf{J}(a)\cdot\mathbf{E}(b,\text{external})-\mathbf{K}(a)\cdot\mathbf{H}(b,\text{external})]dS; \tag{43}$$

and for b just outside of a, we have

$$\langle (a),b\rangle=\int\int_S[\mathbf{J}(a)\cdot\mathbf{E}(b,\text{internal})-\mathbf{K}(a)\cdot\mathbf{H}(b,\text{internal})]dS. \tag{44}$$

It can be shown[5] that

$$\langle a,(b)\rangle=\langle (b),a\rangle, \tag{45}$$

under the same conditions as are required for the usual reciprocity theorem [see Eq. (2)]. It follows that

$$\langle (a),b\rangle-\langle a,(b)\rangle=\int\int_S[\mathbf{J}(a)\times\mathbf{K}(b)-\mathbf{J}(b)\times\mathbf{K}(a)]\cdot\mathbf{n}dS. \tag{46}$$

Thus

$$\langle (a),1,b\rangle-\langle a,1,(b)\rangle=\langle (a),2,b\rangle-\langle a,2,(b)\rangle, \tag{47}$$

where 1 and 2 represent any two environments.

The formula (1) is also inadequate if a or b cannot be contained in a finite volume. For example the reaction between infinite traveling wave line sources can be defined as follows. Let the source distribution be represented by $\mathbf{J}(x,y)\exp\gamma z$ and $\mathbf{K}(x,y)\exp\gamma z$, where the constant γ is such as to represent waves which are attenuated as they travel along the z axis, and the medium is independent of z but is otherwise heterogeneous. Then

$$\langle a,b\rangle=\int\int_\Sigma[\mathbf{J}(a,x,y)\sigma\mathbf{E}(b,x,y)+\mathbf{K}(a,x,y)\sigma\mathbf{H}(b,x,y)]dxdy, \tag{48}$$

where σ represents the matrix

$$\sigma=\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix},$$

and the surface Σ contains all sources.[6] This formula differs from Eq. (1) but can be applied in the same way: it represents the same physical observable. Modifications similar to Eqs. (43) and (44) apply if a and b are at the same place.

Returning to the case where a and b are at different places, let S represent any surface which separates a and b (e.g., all parts of a are inside of S and all parts of b are outside of S). Then

$$\langle a,b\rangle=\int\int_S[\mathbf{H}(a)\times\mathbf{E}(b)-\mathbf{H}(b)\times\mathbf{E}(a)]\cdot\mathbf{n}dS, \tag{49}$$

[5] T. H. Crowley, J. Appl. Phys. **25**, 119 (1954).
[6] V. H. Rumsey, J. Appl. Phys. **24**, 1358 (1953).

where **n** points away from a towards b. This illustrates another aspect of the reaction concept, for Eq. (49) states that the flux of the vector in brackets is conserved (provided S separates a and b), and $\langle a,b\rangle$ represents the conserved property. This principle of conservation is self-evident from a physical point of view, for Eq. (42) can be interpreted as stating that the reaction between a and b is the same as the reaction between a and the equivalent sources $\mathbf{n}\times\mathbf{E}(b)=\mathbf{K}(b)$ and $\mathbf{H}(b)\times\mathbf{n}=\mathbf{J}(b)$ which generate the same field as b at a. It is apparent that the reaction between a and b is the same as the reaction between a and any source which produces the same field as b at a.

The reaction concept can be extended to anisotropic media by using a more general form of the reciprocity theorem (which was brought to the author's attention by M. H. Cohen). Let $[\mathbf{E}(a),\mathbf{H}(a)]$ represent the field generated by the source distribution $[d\mathbf{J}(a),d\mathbf{K}(a)]$ as before. Let $[\mathcal{E}(a),\mathcal{H}(a)]$ be the field generated by the same source when all media are replaced by the corresponding media whose macroscopic constants μ (permeability), ϵ (dielectric constant), and σ (conductivity), are the transposes of the original constants. [If $\tilde{\epsilon}$ represents the transpose of ϵ, and **A** and **B** represent any two vectors, then the scalar product $\mathbf{A}\cdot(\epsilon\mathbf{B})\equiv\mathbf{B}\cdot(\tilde{\epsilon}\mathbf{A})$]. Then the formula for the reaction is

$$\langle a,b\rangle=\int\int\int[\mathbf{E}(b)\cdot d\mathbf{J}(a)-\mathbf{H}(b)\cdot d\mathbf{K}(a)], \quad (50)$$

subject to the same conditions as Eq. (1) except that the media need not be isotropic.

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Point Source Kernel for Diffusion with Small-Angle Scattering

W. E. Drummond and C. S. Gardner
Livermore Research Laboratory, California Research and Development Company, Livermore, California

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Analytical expressions are derived for the particle flux from a point source which emits particles with an angular distribution $(1/\pi\beta)\exp[-\theta^2/\beta]$. The emission is into an infinite medium characterized by a strongly forward differential scattering cross section which can be approximated by a Gaussian $\Sigma(\theta)=(\Sigma/\pi\alpha)\exp[-\theta^2/\alpha]$, and in which all cross sections are energy independent. In particular, an asymptotic expression is obtained for large Σr, *viz.*

$$\phi(r,\theta)\sim\frac{\exp[-(\Sigma_t-\Sigma)r]}{\pi r^2(\beta+\frac{1}{3}\alpha\Sigma r)}\exp\left\{-\frac{\theta^2}{\beta+\frac{1}{3}\alpha\Sigma r}\right\}.$$

INTRODUCTION

A DIFFERENTIAL scattering cross section which has a strong forward scattering peak can in some cases be approximated by a Gaussian, $\sigma(\theta)=[\sigma/(\pi\alpha)]\times\exp[-\theta^2/\alpha]$. This approximation is physically realized in several different cases, among which are the differential Compton scattering cross section for γ rays and the differential scattering cross section of high Z materials for very high-energy neutrons ($E>100$ Mev).

It is the purpose of this paper to show that under the small-angle approximation, $\theta^2\ll1$, for the energy-independent case an expression can be derived for the particle flux at x, y, z, from a point source emitting particles with an angular distribution $[1/\pi\beta]\exp[-\theta^2/\beta]$ into an infinite medium which is characterized by a total cross section, Σ_t, and a differential scattering cross section, $\Sigma(\theta)=[\Sigma/(\pi\alpha)]\exp[-\theta^2/\alpha]$. It will further be shown that an analytic expression for the distribution function at a distance z from an infinite plane source can be derived.

SMALL-ANGLE APPROXIMATION

The differential elastic scattering cross section of several materials has been measured for 83-Mev neutrons[1] and for 300-Mev neutrons.[2] The experimental results can be easily fitted by a Gaussian $[\sigma/\pi\alpha]\times\exp[-\theta^2/\alpha]$. The Gaussian character of the scattering is also displayed by the "opaque" theory of scattering. The "opaque" model gives an angular dependence $[J_1(KR\theta)/\theta]^2$ which upon expansion in a power series gives, neglecting terms in θ^4 and higher powers,

$$[\tfrac{1}{2}KR]^2[1-\tfrac{1}{2}(KR\theta)^2]\sim[\tfrac{1}{2}KR]^2\exp[-\tfrac{1}{2}(KR\theta)^2].$$

The total cross section for neutrons of >200 Mev seems to be roughly geometric and therefore independent of energy. Thus, considering the total cross section to be constant and the differential scattering

[1] Bratenahl, Fernbach, Hildebrand, Leith, and Moyer, Phys. Rev. **77**, 597 (1950).
[2] W. P. Ball, University of California Radiation Laboratory Report 1938 (unpublished).