which gives for even and odd degrees, respectively:

$$\Sigma(\beta^\mu\beta^\nu-\alpha_1{}^2g^{\mu\nu})(\beta^\sigma\beta^\tau-\alpha_2{}^2g^{\sigma\tau})\cdots=0,$$
$$\Sigma\beta^\mu(\beta^\nu\beta^\sigma-\alpha_1{}^2g^{\nu\sigma})\cdots=0. \quad (26)$$

These are the general relativistic commutation relations previously written symbolically as **G** in (1) and in their irreducible form their degree represents the spin of the elementary particle as given by (8).[8] It may be noted that they are consistent according to definition A and hence satisfy theorem I. From (21) and (25) one gets,

$$(\chi^2-\alpha_1{}^2\mathfrak{B}^2)(\chi^2-\alpha_2{}^2\mathfrak{B}^2)\cdots=0.$$

Hence χ^2/α^2, the eigenvalues of $\mathfrak{B}^2$, are squares of masses of elementary particles.

The matrices $g^{\mu\nu}$ defined by (8) reduce to the nucleus of the representation of Lorentz transformation in a locally Cartesian frame; hence we can define an infinitesimal transformation $\mathcal{T}$ whose representation transforms as $\psi'=\mathrm{T}\psi=[1+\frac{1}{2}\epsilon_{\mu\nu}g^{\mu\nu}]\psi$, where T is a representation matrix of $\mathcal{T}$ which may be called a local Lorentz transformation, and $g_{\mu\nu}$ the nucleus of representation of $\mathcal{T}$. T forms a local group embedded in the general transformation group S and (9) shows us that S can be built up by successive variation of $\Gamma_{\alpha\tau}{}^\mu$ if we know the infinitesimal algebra of $g^{\mu\nu}$ and β^μ defined by (8) and (26). In the particular case of spin $\frac{1}{2}$, $g^{\mu\nu}=\frac{1}{4}[\beta^\mu,\beta^\nu]$, and for spin 0 and 1, $g^{\mu\nu}=[\beta^\mu,\beta^\nu]$.

In the case of spin 2, ψ in (20), (21), (22) represents the wave function of gravitons (gravitational quanta). Now the metrical structure of physical space has been considered to be due to gravitation. One may ask how to reconcile this representation of gravitation with the one in terms of gravitational quanta. The apparent contradiction is resolved by considering matter with its field as causing a (topological) deformation of physical space-time and having the duality of metricity and discreteness, as observed by experiments which also deform space-time. We can observe metrical properties only on the macroscopic level, where approximate rigid bodies and local frames of reference exist, but in the case of the microscopic world where there exist no rigid rods to define distance (since the uncertainty principle applies) the other aspect of duality (discreteness) becomes apparent.

[8] For degrees two and three, these commutation rules reduce to the generalizations of Dirac's and Kemmer's commutation relations:

$$\beta^\mu\beta^\nu+\beta^\nu\beta^\mu=2g^{\mu\nu}$$

for spin $\frac{1}{2}$, and

$$\beta^\mu\beta^\sigma\beta^\nu+\beta^\nu\beta^\sigma\beta^\mu=g^{\mu\sigma}\beta^\nu+g^{\sigma\nu}\beta^\mu$$

for spins 0 and 1.

PHYSICAL REVIEW VOLUME 94, NUMBER 6 JUNE 15, 1954

Electron Energy Distributions in Stationary Discharges

IRA B. BERNSTEIN* AND T. HOLSTEIN
Westinghouse Research Laboratories, East Pittsburgh, Pennsylvania
(Received March 3, 1954)

Calculations of the electron distribution function are presented for some simple examples of a stationary discharge in a dc space charge field. The treatment is valid when the predominant mechanism of energy exchange arises from motion in the dc space charge field. The computations indicate that the effect of dc space charge is, for a given external field strength, to increase the proportion of high energy electrons over that computed neglecting space charge. This results in a larger specific ionization rate, but the effect is not so great as to account for the low maintenance potentials observed in positive columns and in microwave discharges in inert gases.

I. INTRODUCTION

PAST theoretical analyses[1] of the energy distribution of electrons in gases have generally ignored the presence of space charge fields. In the microwave discharge, for example, the electric field is usually assumed to be of external origin; in positive columns of dc discharges, the relevant field is taken to be the longitudinal gradient. Now, in both these examples, the removal of charged particles takes place via the mechanism of ambipolar diffusion. This process requires the presence of a space charge field sufficiently strong to retard the motion of electrons, and to accelerate that of the positive ions to the boundary. Such fields are often comparable to or even larger than the external fields.

In order to obtain some idea of the effect of a space charge field (of the type prevalent in ambipolar diffusion) on electron energy distributions and associated quantities, such an average ionization rates, it has been deemed of interest to investigate the situation in which the space charge field is much larger than the applied field. This case represents the opposite extreme to that already treated, namely, space charge field very much less than external field. By this procedure one may hope to achieve an understanding of the generally encountered intermediate case by interpolation between the two extremes.

* Present address: Forrestal Research Center, Princeton, New Jersey.

[1] Morse, Allis, and Lamar, Phys. Rev. **48**, 412 (1935); J. A. Smit, Physica **3**, 543 (1936); T. Holstein, Phys. Rev. **70**, 367 (1946), to be referred to hereafter as I.

A special feature of interest arises in connection with the question of low maintenance voltages in positive columns and microwave discharges.[2] In these discharges the essential problem is to account for the extraordinary ionization efficiency. A possible contributing mechanism might involve just the phenomenon under discussion. This possibility has in fact been investigated by Davydov[3] who comes to the conclusion that an ambipolar field large compared to the longitudinal gradient "Maxwellianizes" the energy distribution, thereby providing a relatively large number of "tail" (i.e., high-energy) electrons capable of direct ionization. In this paper the analysis of Davydov will be shown to be incorrect; furthermore, it will turn out that the effect of space charge fields upon average ionization rates, while not negligible, is completely unable to account for the observed low maintenance voltages.

II. BASIC ENERGY DISTRIBUTION EQUATION

The rigorous starting point of any energy distribution treatment is, of course, the Boltzmann transport equation,

$$\frac{\partial f}{\partial t}=-\mathbf{v}\cdot\mathrm{grad}_{\mathbf{r}}f-\mathbf{a}\cdot\mathrm{grad}_{\mathbf{v}}f=\frac{\partial f}{\partial t}\Big)_{\mathrm{coll}}. \tag{1}$$

Here, $f=f(\mathbf{r},\mathbf{v},t)$ gives the distribution of electrons in position and velocity space, $\mathbf{a}=-e\mathcal{E}/m$, where the electric field, $\mathcal{E}$, is in general a function of position $\mathbf{r}$ and time t. The first two terms on the right-hand side represent the change in f arising from the velocities of the electrons and from the action of the electric field. The last term represents the change in f due to collisions.

At this point it is to be remarked that, apart from the consideration of the space-charge field, the present analysis will be based on essentially the same assumptions as the previous treatments. This implies, first of all, that collisions other than those between electrons and normal atoms will be ignored. In this event $\partial f/\partial t$ has the form[4]

$$\begin{aligned}\partial f/\partial t)_{\mathrm{coll}}=&Nv\int_\Omega d\omega'[f(\theta',\varphi',v,\mathbf{r},t)-f(\theta,\varphi,v,\mathbf{r},t)]q_{el}(\psi,v)\\&+(Nv/v^3)(m/M)(\partial/\partial v)\\&\times\int_\Omega d\omega' f(\theta',\varphi',v,\mathbf{r},t)(1-\cos\psi)q_{et}(\psi,v)v^4\\&+\sum_h\int_\Omega d\omega'\{f(\theta',\varphi',[v^2+v_h^2]^{\frac{1}{2}},\mathbf{r},t)\\&\times(1+v_h^2/v^2)q_h(\psi,[v^2+v_h^2]^{\frac{1}{2}})\\&-f(\theta,\varphi,v,\mathbf{r},t)q_h(\psi,v)\}+\partial f/\partial t)_{\mathrm{ion}}.\end{aligned} \tag{2}$$

In Eq. (2), N is the density of normal gas atoms; v, θ, and φ are polar coordinates in velocity space (the integrals being carried out over the total solid angle Ω in this space, i.e., over-all values of θ' and φ'); $q_{el}(\psi,v)$ is the differential cross section for elastic scattering through an angle ψ of an electron of velocity v; and $q_h(\psi,v)$ the corresponding inelastic differential cross section involving excitation of the h'th atomic level. The first term gives the change in f due to elastic scattering with neglect of recoil loss. The second term takes this loss into account. The third term describes the effect of excitation collisions. In it the summation is extended over all discretely excited levels. Finally, $\partial f/\partial t)_{\mathrm{ion}}$ denotes the change in f due to ionizing collisions. This term will be written down explicitly later.

A second assumption which characterizes the existing theoretical treatments of electron energy distributions is that the distribution is essentially isotropic in velocity space, with only small anisotropies arising from the actions of external fields and diffusion gradients. One thus has

$$f(\mathbf{r},\mathbf{v},t)\approx f_0(\mathbf{r},v,t)+\mathbf{v}\cdot\mathbf{f}_1(\mathbf{r},v,t), \tag{3}$$

where $f_0(\mathbf{r},v,t)$ and $\mathbf{f}_1(\mathbf{r},v,t)$ are scalar and vector functions of $\mathbf{r}$, t and the magnitude of $\mathbf{v}$, and where it is to be presumed that

$$f_0(\mathbf{r},v,t)\gg|\mathbf{f}_1(\mathbf{r},v,t)|. \tag{4}$$

Upon substituting (3) into (2), multiplying the result by $1/4\pi$ or by $3\mathbf{v}/4\pi$, integrating over the velocity angles (as is done in I, pp. 371–372), and transforming the velocity variable from v itself to $u=v^2$, one obtains the following equations[5] for $\partial f_0/\partial t$ and $\partial\mathbf{f}_1/\partial t$.

$$\begin{aligned}u^{\frac{1}{2}}\partial f_0/\partial t=&-(u/3)\,\mathrm{div}_{\mathbf{r}}\mathbf{f}_1-(2\mathbf{a}/3)\cdot(\partial/\partial u)(u\mathbf{f}_1)\\&+(2m/M)(\partial/\partial u)(u^2f_0/\lambda_e)\\&+\sum_h\{(u+u_h)f_0(u+u_h,\mathbf{r})/\lambda_h(u+u_h)\\&-uf_0(u,\mathbf{r})/\lambda_h(u)\}+u^{\frac{1}{2}}\partial f_0/\partial t)_{\mathrm{ion}},\end{aligned} \tag{5}$$

$$u^{\frac{1}{2}}\partial\mathbf{f}_1/\partial t=-u\,\mathrm{grad}_{\mathbf{r}}f_0-2\mathbf{a}u\partial f_0/\partial u-u\mathbf{f}_1/\lambda_e, \tag{6}$$

where

$$1/\lambda_e(u)=2\pi N\int_0^\pi d\psi\,\sin\psi(1-\cos\psi)q_{et}(\psi,u), \tag{7}$$

and

$$1/\lambda_h(u)=2\pi N\int_0^\pi d\psi\,\sin\psi q_h(\psi,u). \tag{8}$$

$\lambda_e(u)$ and $\lambda_h(u)$ are, respectively, the mean free paths for elastic scattering and for excitation of the h'th atomic level.

Equations (5) and (6) will be applied to the positive column of a dc discharge. For the sake of simplicity the usual cylindrical geometry will be replaced by plane

[2] See, for example, Krasik, Alpert, and McCoubrey, Phys. Rev. 76, 722 (1949), especially p. 730 and Fig. 12.
[3] B. Davydov, Physik Z. Sowjetunion 12, 269 (1937).
[4] See, for example, I, Eqs. (10), (11), and (13).
[5] These equations are vector generalizations of (21) and (22) of I, the only further change being the use of $u=v^2$ rather than v itself as the velocity variable.

parallel geometry, i.e., the boundary will be assumed to consist of two parallel walls separated by a distance $2d$. The vector acceleration $\mathbf{a}$ consists of a longitudinal component a_l parallel to the walls, and a transverse component a_t perpendicular to the walls, arising from the ambipolar space charge field. Whereas a_l is constant, a_t must be considered a function of x (the x axis being directed perpendicular to the walls). In addition, the positive column is assumed homogeneous in directions parallel to the walls, i.e., the dependence of f_0 and $\mathbf{f}_1$ on $\mathbf{r}$ is limited to the x coordinate. Under these conditions (6) reduces to

$$o=u^{\frac{1}{2}}\partial f_{1t}/\partial t=-2a_t u\partial f_0/\partial u-uf_{1t}/\lambda_e-u\partial f_0/\partial x, \quad (9)$$

$$o=u^{\frac{1}{2}}\partial f_{1l}/\partial t=-2a_l u\partial f_0/\partial u-uf_{1l}/\lambda_e, \quad (10)$$

where f_{1t} and f_{1l} are the transverse and longitudinal components of $\mathbf{f}_1$. These equations may be used immediately to eliminate $\mathbf{f}_1$ from (5), whence one obtains

$$o=u^{\frac{1}{2}}\partial f_0/\partial t=\tfrac{1}{3}(\partial/\partial x+2a_t\partial/\partial u)(u\lambda_e)\times(\partial/\partial x+2a_t\partial/\partial u)f_0(x,u,t)+L\{f_0(x,u,t)\}, \quad (11)$$

where

$$L\{f_0(x,u,t)\}=(4/3)a_l^2(\partial/\partial u)(u\lambda_e\partial f_0/\partial u)+(2m/M)(\partial/\partial u)(u^2 f_0/\lambda_e)+\sum_h\{(u+u_h)f_0(x,u+u_h,t)/\lambda_h(u+u_h)-uf_0(x,u,t)/\lambda_h(u)\}+u^{\frac{1}{2}}\partial f_0/\partial t)_{\text{ion}}. \quad (12)$$

For the sake of reference the explicit form of $u^{\frac{1}{2}}\partial f_0/\partial t)_{\text{ion}}$ given in I, Eq. (32), will be recorded here. It is

$$u^{\frac{1}{2}}\partial f_0/\partial t)_{\text{ion}}=\int_{u+u_i}^{\infty}du'u'f_0(u')/\lambda_i(u',u)-uf_0(u)\int_0^{(u-u_i)/2}du'/\lambda_i(u,u'), \quad (13)$$

where u_i is the ionization potential, and where $1/\lambda_i(u,u')$ is the "differential" probability per cm for ionization in which one of the two resultant electrons has an energy between u' and $u'+du'$ (the energy of the other being necessarily contained between $u-u_i-u'$ and $u-u_i-u'-du'$). As is the case in the standard treatments,[1] $u^{\frac{1}{2}}\partial f_0/\partial t)_{\text{ion}}$ plays a negligible role in determining the energy distribution, and will henceforth be neglected. It is to be noted that $u^{\frac{1}{2}}f_0 du dx$ is proportional to the fraction of electrons at x with kinetic energy u.

The development of (11) from (5) and (6), which has been given in the preceding paragraphs for the case of the positive column, can also be applied under certain conditions to the microwave discharge. The geometry of the discharge, as before, will be assumed plane-parallel. However, both the ambipolar space charge field $\mathcal{E}_l$, and the applied microwave field $\mathcal{E}_0\cos\omega t$, are in the x direction, perpendicular to the walls. Just as in I, it will be assumed that the microwave frequency ω is high enough so that f_0 does not change appreciably in a cycle of field oscillation: For the steady state, (6) then yields

$$f_1=-\lambda_e\{\partial f_0/\partial x+2a_t\partial f_0/\partial u\}-2a_0\lambda_e\{(\cos\omega t+\omega\lambda_e u^{-\frac{1}{2}}\sin\omega t)/(1+\omega^2\lambda_e^2u^{-1})\}\partial f_0/\partial u. \quad (14)$$

Substituting (14) into Eq. (5) and time-averaging over a cycle of field oscillation (in which f_0 is still to be considered time-independent), one obtains, with the further assumption that $\omega^2\lambda_e^2u^{-1}\ll 1$ (many elastic collisions per oscillation), an equation of precisely the same form as (11). The sole difference is that a_l^2 is to be replaced by $\frac{1}{2}a_0^2$, i.e., the longitudinal field of the dc positive column is equivalent to the root-mean-square field of the microwave discharge.[6]

Equation (11) constitutes the basic equation for the electron energy distribution in the presence of an external and an ambipolar space charge field; its approximate solution for the case of external field very much less than ambipolar field will be developed in the next section. Henceforth the subscript o will be suppressed.

III. METHOD OF SOLUTION OF BASIC EQUATION

An integral feature of the method for solving (11), to be given here, is the transformation from u to a new variable

$$w=u+\varphi(x), \quad (15)$$

where

$$\varphi(x)=-2\int_0^x dx a_t(x). \quad (16)$$

Physically (apart from the factor $\frac{1}{2}m$), $\varphi(x)$ is the potential energy associated with the space charge field. Thus w represents the sum of kinetic energy and space charge potential energy. In what follows, w will be referred to as "total energy."

From (15) and (16) one readily derives the relationships

$$\partial/\partial u)_x=\partial/\partial w)_x, \quad (17)$$

$$\partial/\partial x)_u=\partial/\partial x)_w-2a_t\partial/\partial w)_x, \quad (18)$$

the combination of which yields

$$\partial/\partial x)_w=\partial/\partial x)_u+2a_t\partial/\partial u)_x. \quad (19)$$

With the use of these relations, Eq. (12) assumes the form

$$o=\tfrac{1}{3}(\partial/\partial x)[(w-\varphi)\lambda_e\partial f/\partial x]+L\{f(w,x)\}, \quad (20)$$

where f is now to be considered a function of x and w, and where

$$L\{f(w,x)\}=(4/3)a_l^2(\partial/\partial w)[(w-\varphi)\lambda_e\partial f/\partial w]+(2m/M)(\partial/\partial w)[(w-\varphi)^2 f/\lambda_e]+\sum_h\{(w-\varphi+u_h)f(w+u_h,x)/\lambda_h(w-\varphi+u_h)-(w-\varphi)f(w,x)/\lambda_h(w-\varphi)\} \quad (21)$$

with λ_e a function of $(w-\varphi)$.

[6] This result is also given in I.

Equations (20) and (21) describe a type of diffusion-drift motion in w,x space. In fact, by defining two diffusion coefficients

$$D_x=\tfrac{1}{3}(w-\varphi)\lambda_e(w-\varphi), \tag{22}$$

and

$$D_w=(4/3)a_l^2(w-\varphi)\lambda_e(w-\varphi), \tag{23}$$

one may write (20) and (21) in the form

$$\begin{aligned}o=&(\partial/\partial x)[D_x(\partial f/\partial x)]+(\partial/\partial w)[D_w(\partial f/\partial w)]\\&+(2m/M)(\partial/\partial w)[(w-\varphi)^2 f/\lambda_e]\\&+\textstyle\sum_h\{(w-\varphi+u_h)f(w+u_h,x)/\lambda_h(w-\varphi+u_h)\\&-(w-\varphi)f(w,x)/\lambda_h(w-\varphi)\},\end{aligned} \tag{24}$$

in which it is seen that the first term describes a diffusion in x, the second a diffusion in w, and the remaining terms a degradation in total energy w, which is continuous for the recoil term and takes place in jumps of u_h in the case of the inelastic collision term.

The diffusion coefficients, D_x and D_w are of course functions of the coordinates. In particular, as illustrated in Fig. 1, they vanish on the curve of zero kinetic energy $w=\varphi(x)$ which, since electrons must have positive kinetic energies, constitutes one of the boundaries of the available region of w,x space.

In an analogous fashion, curves of $\varphi(x)$, uniformly displaced upwards by amounts u_h, define the lower boundaries of regions of inelastic collisions of various types. Finally, the curve

$$w=\varphi(x)+u_{\text{ion}}$$

[where $u_{\text{ion}}=(2/m)$ ionization potential] gives the lower boundary of the ionization region.

To complete the description in w,x space, it is to be considered that $\varphi(x)$, as pictured in Fig. 1, is symmetrically disposed with respect to the enclosure geometry —an assumption which will be employed in all that follows—and attains a finite value φ on the enclosure

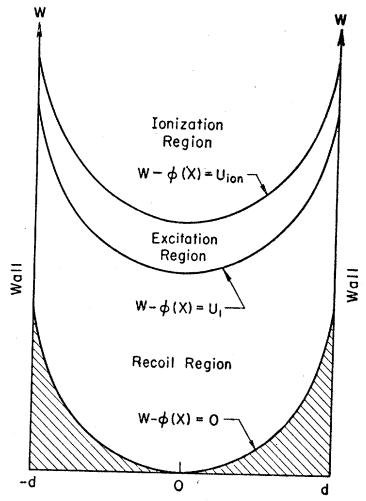

FIG. 1. Diagram of total energy w *versus* position x showing regions of the various classes of collisions.

walls. For $w>\varphi_{\text{wall}}$ the boundary is no longer $w=\varphi(x)$, but the walls themselves. If the latter are assumed absorbing, as is usually the case, the boundary condition

$$f(w,\pm d)=0, \tag{26}$$

($d=\frac{1}{2}$ wall spacing) appropriate to diffusion theory, is to be employed.[7]

It turns out that the inclusion of wall effects leads to special difficulties in the analysis of Eq. (24). The treatment will therefore be limited to situations in which the significant range of w lies below φ_{wall}; this restriction is equivalent to assuming an infinite "space charge well."

With the transformation from the u,x to the w,x system of variables, the stage has been reached where one may profitably make use of the basic assumption, discussed in the introduction of this paper, that the ambipolar field $\mathcal{E}_t$ is much larger than the "applied" longitudinal field $\mathcal{E}_l$. As a preliminary step, it will be helpful to consider the extreme case in which it is imagined that there is a longitudinal field and no collision-induced energy losses, i.e., the electrons move in a space charge field of the type illustrated in Fig. 1, undergoing elastic collisions with infinitely massive atoms. In this case (20) reduces to

$$0=\tfrac{1}{3}(\partial/\partial x)[(w-\varphi)\lambda_e\partial f/\partial x]. \tag{27}$$

A first integration yields

$$\partial f/\partial x=C(w)/(w-\varphi)\lambda_e(w-\varphi).$$

Let it be supposed that $\lambda_e(0)$ is finite. Then the requirement of regularity of f (and $\partial f/\partial x$) at $w=\varphi(x)$ leads to the conclusion that $C(w)$ is zero. Thus $\partial f/\partial x=0$ and

$$f(w,x)=F(w), \tag{29}$$

where $F(w)$ is an arbitrary function of w.

The arbitrariness of $F(w)$ arises from the circumstance that, with neglect of the longitudinal field and collision losses, the sole mechanism for changing kinetic energy is the motion in the ambipolar field. Obviously, a constant of this motion is the total energy, w. Thus, in the approximation represented by Eq. (27), transitions between different w levels do not take place with the consequence that an arbitrary distribution in w of electrons is stationary.

It has been remarked above that the consideration of (27) as an approximation to (20) is in line with the basic point of view of this paper, which regards the effect of the space charge field in determining the electron energy distribution as "dominant." The results just obtained demonstrate that this dominance is necessarily incomplete in the sense that the space charge field cannot determine the distribution in w. It is therefore necessary to proceed to a higher stage of

[7] In contrast to the situation at the physical boundary $x=\pm d$, only regularity of $f(x,w)$ is required at $w=\varphi(x)$.

approximation. This is done by considering the terms designated in (21) by the notation $L\{f(w,x)\}$ as a small but nevertheless nonvanishing perturbation. The introduction of these terms will correspond physically to permitting transitions to take place between the different w levels; these transitions will then determine[8] the form of $F(w)$.

Considering, then, $L\{f(w,x)\}$ in (20) as a perturbation, one attempts a solution of the form

$$f(w,x)=F(w)+G(w,x), \tag{30}$$

where $G(w,x)$ is to be presumed small. Inserting (30) into (20), one has, approximately,

$$0=\tfrac{1}{3}(\partial/\partial x)[(w-\varphi)\lambda_e\partial G/\partial x]+L\{F(w)\}, \tag{31}$$

where the higher-order term $L\{G(w,x)\}$ has been dropped. Upon integration of (31) from $-x_0$ to $+x_0$, x_0 being defined by the relation[9]

$$\varphi(x_0)=\varphi(-x_0)=w, \tag{32}$$

the first term of (31) drops out, and one is left with

$$0=\int_{-x_0}^{x_0} dx L\{F(w)\} \tag{33}$$

as the basic equation[10] for the determination of $F(w)$. Equation (33) may be written more explicitly as

$$o=(4/3)a_i^2(d/dw)[p(w)dF/dw] + (2m/M)(d/dw)[q(w)F] + \sum_h\{r_h(w+u_h)F(w+u_h)-r_h(w)F(w)\}, \tag{34}$$

where

$$p(w)\equiv\int_{-x_0}^{x_0} dx(w-\varphi)\lambda_e(w-\varphi), \tag{35}$$

$$q(w)\equiv\int_{-x_0}^{x_0} dx(w-\varphi)^2\lambda_e(w-\varphi), \tag{36}$$

$$r_h(w)\equiv\int_{-x_0}^{x_0} dx(w-\varphi)/\lambda_h(w-\varphi). \tag{37}$$

It is to be noticed that the form of (34) is quite similar to that of the corresponding equation for the case of no space charge field [see I, (54′)]. Hence, the general technique developed in I for solving that equation will also be applicable here. In the following section (34) will be solved for a number of specific examples and the results compared with those obtained in I for the case of no space charge field.

IV. ILLUSTRATIVE EXAMPLES

1. Elastic Collision Case

The first example to be considered is that in which the electron energies are low enough so that inelastic collisions may be neglected. Equation (34) then reduces to

$$(4/3)a_i^2(d/dw)[p(w)dF/dw] + (2m/M)(d/dw)[q(w)F]=0. \tag{38}$$

A first integration yields

$$d\ln F/dw=-(3m/2Ma_i^2)[q(w)/p(w)], \tag{39}$$

the constant of integration being eliminated by the requirement of regularity of F and dF/dw at the origin. A second integration gives

$$F=C\exp\left[-(3m/2a_i^2)\int_0^w dwq(w)/p(w)\right]. \tag{40}$$

In order to proceed further, it is necessary to make some assumptions concerning both the energy dependence of the mean free path, λ_e, and the spatial dependence of the space charge potential φ. In what follows, it shall be assumed that (a) λ_e=constant, and (b) φ varies parabolically with distance,[11] i.e.,

$$\varphi=\alpha x^2. \tag{41}$$

[8] In the language of perturbation theory the situation may be described as follows. The solutions of the equation $(1/3)(\partial/\partial x)\times[(\omega-\varphi)\lambda_e\partial f/\partial x]=-\beta f$ which belong to the "eigenvalue" $\beta=0$ are degenerate. A complete set of the eigenstates, for example, is given in terms of the Dirac delta function, namely $f_{w_1}(w,x)=\delta(w-w_1)$. The general solution, equivalent to (29) is then, $f(w,x)=\int dw_1F(w_1)f_{w_1}(w,x)$. According to the general methods of perturbation theory, the degeneracy is to be removed by a secular perturbation procedure which considers $L\{f(w,x)\}$ as the perturbation. In such a procedure, one will expect to have to evaluate integrals of the type $\int dxdwf_{w_1}(w,x)L\{f(w,x)\}$. This in fact is just what is done in the text below. (See also reference 10.)

[9] It has already been assumed that $\varphi(x)$ is an even function of x, with the plane $x=0$ chosen to lie midway between the two walls.

[10] With reference to the remarks of footnote 8, the above procedure is fully equivalent to that followed in secular perturbation theory. Namely, one constructs the solution as a linear superposition of degenerate unperturbed functions, i.e., $f(w,x)\approx\int dw_1F(w_1)f_{w_1}(w,x)=F(w)$ [since $f_{w_1}(w,x)=\delta(w-w_1)$] in accordance with footnote 8, and substitutes into the basic Eq. (20), thereby obtaining $o=L\{F(w)\}$. Then, upon multiplying with one of the "eigenfunctions" and integrating with respect to x and w, one obtains

$$o=\int_0^\infty dw\int_{-x_0}^{x_0} dx\delta(w'-w)L\{F(w)\}=\int_{-x_0(w')}^{x_0(w')} dxL\{F(w')\},$$

as is given by (33) of the text.

[11] It is felt that the actual functional form of the space charge potential is contained between two extremes, one of which is represented by (41); the other extreme is a step-like potential given by $\varphi=0$ for $|x|<d$, followed by a discontinuous jump at $x=\pm d$ to the value $\varphi=\varphi_{\text{wall}}$. In particular, it is readily shown that a field of the type occurring in the conventional theory of ambipolar diffusion is contained between these two extremes (apart from a small region in the neighborhood of the wall in which ambipolar theory is not applicable anyway). Now, the "step" field, when applied to Eq. (34), is essentially equivalent to no space charge field. Hence (41) should give an extreme indication of the effect of space charge fields on the electron energy distribution and related statistical quantities.

With these assumptions, (35) and (36) are readily evaluated and yield

$$p(w)=(4/3)\lambda_e w^{\frac{3}{2}}/\alpha^{\frac{1}{2}}, \tag{42}$$

$$q(w)=(16/15)w^{5/2}/\lambda_e\alpha^{1/2}, \tag{43}$$

which, when substituted into (40), give

$$F=C\exp[-(3m/5M)(w^2/a_l^2\lambda_e^2)]. \tag{44}$$

(44) is to be compared to the so-called "Druyvesteyn" distribution obtained for the elastic collision case in the absence of the space charge field. This distribution is[12]

$$F_D=C_D\exp[-(3m/4M)(u^2/a_l^2\lambda_e^2)]. \tag{45}$$

The comparison of (44) with (45) demonstrates a characteristic effect of the space charge field on electron energy distributions, namely, a relative enhancement of the tail. This feature will, in particular, be again encountered in the next example to be treated, in which inelastic collisions are taken into account.

2. Inelastic Collisions

The extension of the electron energy distribution into the inelastic collision region will now be considered. The specific example to be treated is that for which: (a) λ_e=constant (as in the previous example); (b) there exists only one excitation level, $u_1\equiv u_{ex}$ with the corresponding free path, $\lambda_1\equiv\lambda_{ex}$ independent of electron energy;[13] (c) the energy loss due to recoil collisions is negligible; (d) the space charge well is of the form[14]

$$\varphi(x)=\alpha x^{2r};\quad r=1, 2, \cdots. \tag{46}$$

It should be pointed out that (a), (b), and (c) characterize the first example treated in I, Sec. 7. The results of the present treatment will therefore be directly comparable to those of I.

Utilizing these assumptions in the evaluation of the integrals in (35) and (37), one obtains

$$p(w)=2\alpha^{-1/2r}[2r/(2r+1)]w^{1+1/2r}, \tag{47}$$

$$r_{ex}(w)=r_1(w)=2\alpha^{-1/2r}[2r/(2r+1)](w+u_{ex}/2r)\times(w-u_{ex})^{\frac{1}{2}}S(w-u_{ex}), \tag{48}$$

where $S(x)$ is the conventional step function. Substituting (47) and (48) into (34) and dropping the recoil term, one has

$$O=(4/3)\lambda_e\lambda_{ex}a_l^2(d/dw)(w^{1+1/2r}dF/dw)+\{(w+u_{ex}/2r+u_{ex})w^{1/2r}F(w+u_{ex})-(w+u_{ex}/2r)(w-u_{ex})^{1/2r}F(w)S(w-u_{ex})\}. \tag{49}$$

As in the treatment of I, (49) is first solved for the excitation region $w>u_{ex}$. Here, for fields which are not too strong, the distribution function falls off sufficiently rapidly with increasing energy so that the "production" term, proportional to $F(w+u_{ex})$, may be neglected. In this case, and with the introduction of the dimensionless quantities

$$\zeta=w/u_{ex}, \tag{50}$$

$$\gamma^2=\tfrac{3}{4}(u_{ex}/a_l\lambda_e)^2(\lambda_e/\lambda_{ex}), \tag{51}$$

Eq. (49) may be written as

$$o=(d/d\zeta)[\zeta^{1+1/2r}dF(\zeta)/d\zeta]+\gamma^2\zeta^{1/2r}(\zeta+1+1/2r)F(\zeta+1),\quad \zeta\leqslant 1, \tag{52}$$

$$o=(d/d\zeta)[\zeta^{1+1/2r}dF(\zeta)/d\zeta]-\gamma^2(\zeta-1)^{1/2r}(\zeta+1/2r)F(\zeta),\quad \zeta\geqslant 1. \tag{53}$$

The region $\zeta\geqslant 1$ will be considered first. Introducing the substitution

$$G(\zeta)=\zeta^{\frac{1}{2}+1/2r}F(\zeta), \tag{54}$$

one may write (53) as

$$G''(\zeta)=[\gamma^2(1-1/\zeta)^{1/2r}(1+1/2r\zeta)-(2r+1)/16r\zeta^2]G(\zeta). \tag{55}$$

Here primes indicate differentiation with respect to ζ.

A considerable simplification of (55) may be achieved if the last term in the square bracket is neglected, a step which is permissible for sufficiently large γ (low field). Actually in the present paper, the range of field for which specific calculations is such that[15]

$$\gamma\geqslant 10. \tag{56}$$

For the range defined by (56) the second term in the bracket of (55) is quite small and will be dropped forthwith. Equation (55) then becomes

$$G''(\zeta)-k^2(\zeta)G(\zeta)=0, \tag{57}$$

where

$$k(\zeta)=\gamma(1-1/\zeta)^{1/4r}(1+1/2r\zeta)^{\frac{1}{2}}. \tag{58}$$

Equation (57) will be solved by an approximate method based on the circumstance that, because of (56), $G(\zeta)$ attains its asymptotic form, as given by the W.K.B. method[16] for $\zeta-1\ll 1$, which solution is then readily joined to one which is specially valid for the

[12] Equation (45) is readily obtained by taking $\varphi=0$ in (35) and (36) [the procedure being equivalent to assuming the "step field" of reference 10].

[13] This assumption corresponds to a "step function" cross section, the step occurring at $u=u_{ex}$. An energy dependence of this type is actually not unreasonable. In particular, the excitation cross sections of metastable levels (which is a number of gases are the lowest excited levels) often exhibit a rather sharp initial rise, followed by a plateau, or even by a diminishing characteristic.

[14] It will be noticed that the previously employed parabolic space charge potential $\varphi=\alpha x^2$ is a special case of (46) with $r=1$. Higher values of r represent cases intermediate between the parabolic potential and the square well potential of footnote 10.

[15] In a typical case such as argon, $\lambda_e/\lambda_{ex}\sim 50$, $\lambda_e p\sim(1/60)$ cm mm (where p is the gas pressure in mm Hg), and the excitation potential, $V_{ex}\sim 15$ volts. With these numbers, and with $\mathcal{E}$=longitudinal field in volts/cm, the limitation expressed by (56) is equivalent to $\mathcal{E}/p\leqslant 23$. This limit is considerably higher than the values of $\mathcal{E}/p$ commonly encountered in maintenance discharges (see Fig. 12 of reference (2)).

[16] H. F. Jeffries and B. S. Jeffries, *Methods of Mathematical Physics* (Cambridge University Press, Cambridge, 1950), second edition, p. 522.

immediate neighborhood of $\zeta=1$. Thus, the asymptotic solution of (57), as given by the W.K.B. method,[16] is

$$G(\zeta)=Ak(\zeta)^{-1/2}\exp\left[-\int_1^{\zeta}d\zeta' k(\zeta')\right]. \quad (59)$$

On the other hand, in the immediate neighborhood of $\zeta=1$, (57) may be approximated by

$$G''(\zeta)-(1+1/2r)\gamma^2(\zeta-1)^{1/2r}G(\zeta)=0, \quad (60)$$

the appropriate solution of which is

$$G(\zeta)=(\zeta-1)^{\frac{1}{2}}K_{2r/(4r+1)}[(1+1/2r)^{\frac{1}{2}} \times(1+1/4r)^{-1}\gamma(\zeta-1)^{1+1/4r}], \quad (61)$$

where $K_\nu(z)$ denotes the modified Bessel function of the second kind.[17]

Comparison of (60) with the asymptotic form of (61) in the neighborhood of $\zeta=1$ then gives

$$A=(\pi/2)^{\frac{1}{2}}(1+1/4r)^{\frac{1}{2}}. \quad (62)$$

Equations (54), (59), (61), and (62) give $F(\zeta)$ in the region $\zeta\geqslant 1$. For $\zeta\leqslant 1$, the combination of (52) and (53) yields

$$\zeta^{1+1/2r}F'(\zeta)=F'(1)-(\zeta+1)^{1+1/2r}F'(\zeta+1), \quad (63)$$

from which

$$F(\zeta)=F(1)-\int_{\zeta}^{1}d\zeta\zeta^{-(1+1/2r)}F'(1) -(\zeta+1)^{1+1/2r}F'(\zeta+1), \quad (64)$$

where $F(1)$ and $F'(1)$ are obtained[18] from the solution in (61). Thus, $F(\zeta)$ is given for the whole energy region. Specific calculations are carried out by numerical evaluation of (64).

The fraction of electrons with energy between w and $w+dw$ is given by

$$\begin{aligned}N(w)&=\int_{-x_0}^{x_0}dx(w-\varphi)^{\frac{1}{2}}f(w,x)\\&=\int_{-x_0}^{x_0}dx(w-\alpha x^{2r})^{\frac{1}{2}}F(w)\\&=2[\Gamma(1+1/2r)/\Gamma(\tfrac{3}{2}+1/2r)]w^{\frac{1}{2}+1/2r}\alpha^{-1/2r} \times\alpha w^{\frac{1}{2}+1/2r}F(w). \quad (65)\end{aligned}$$

Curves of $N(w)$, normalized to unity, are given in Fig. 2 for the case of $r=1$ (parabolic space charge field), and for various values for γ (corresponding to various values of the applied longitudinal field).

[17] G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, Cambridge, 1948), second edition, p. 78. The solution involving the modified Bessel function of the first kind, $I_\nu(z)$, is excluded because it would represent an energy distribution which increases exponentially with increasing ζ.

[18] This is done conveniently with the aid of the series representation of $K_\nu(z)$.

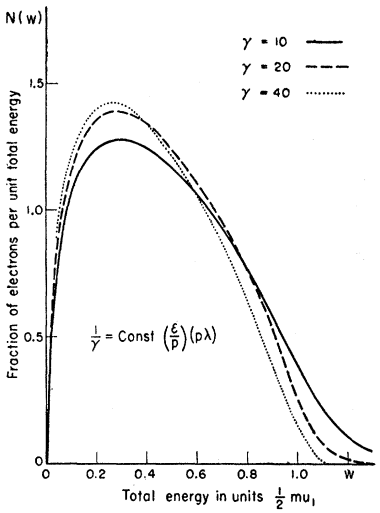

FIG. 2. Normalized distribution in total energy $N(w)$ *versus* w for various values of ε/p. The space charge well has been assumed to be parabolic.

Now that the distribution function has been determined, it is of interest to calculate the specific ionization rate β. For this computation, it is assumed that the ionization cross section is a linear function of the electron kinetic energy, an assumption that is quite good near threshold. Thus

$$1/\lambda_{\text{ion}}=(K_{\text{ion}}/\lambda_e)[(u-u_{\text{ion}})/u_{\text{ion}}]S(u-u_{\text{ion}}). \quad (66)$$

β is then given by

$$\beta=\frac{\int_{u_{\text{ion}}}^{\infty}dw\int_{-x_{\text{ion}}}^{x_{\text{ion}}}dx(w-\varphi)f(w,x)/\lambda_{\text{ion}}}{\int_0^{\infty}dw\int_{-x_0}^{x_0}dx(w-\varphi)^{\frac{1}{2}}f(w,x)}, \quad (67)$$

where x_{ion} is defined by the relation

$$w-\varphi(x_{\text{ion}})=u_{\text{ion}}. \quad (68)$$

Taking $f(x,w)=F(w)$ and using (46), one obtains, after some manipulation, the expression

$$\frac{\beta}{K_{\text{ion}}u_{\text{ion}}^{\frac{1}{2}}}=\frac{I}{J} \quad (69)$$

where

$$I=\int_{u_{\text{ion}}/u_{ex}}^{\infty}d\zeta F(\zeta)\{8r^2(\zeta-u_{\text{ion}}/u_{ex})^{2+1/2r} +2r(4r+1)(u_{\text{ion}}/u_{ex})(u-u_{\text{ion}}/u_{ex})^{1+1/2r}\}, \quad (69a)$$

and

$$J=[2(2r+1)(4r+1)\Gamma(1+1/2r)/\Gamma(\tfrac{3}{2}+1/2r)] \times\int_0^{\infty}d\zeta\zeta^{1+1/2r}F(\zeta), \quad (69b)$$

which may be evaluated numerically.

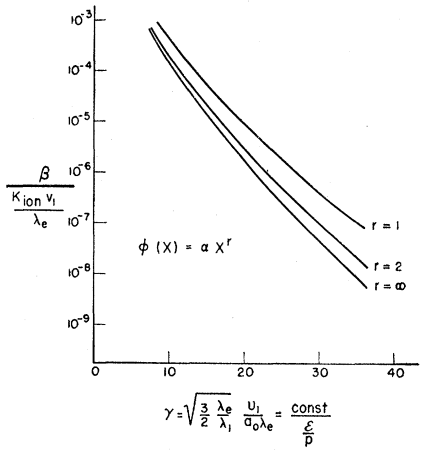

FIG. 3. Specific ionization rate β *versus* external field parameter $\gamma \propto p/\mathcal{E}$. The space charge well has been assumed to be of the form $\varphi(x)=\alpha x^{2r}$. The curve for $r=\infty$ corresponds to the case of vanishing space charge field, that for $r=0$ a square potential well.

The results of the computation are presented in Fig. 3. The principal feature to be noted is that, for a given external field, i.e., a given value of γ, the specific ionization increases as the parameter r is varied from ∞—the case of a square-well potential, equivalent to no space charge field—to unity, the parabolic case, representing the extreme manifestation of a space charge field. However, and this qualification is of particular significance with regard to the low-voltage problem mentioned in the introduction of this paper, the "space charge" reduction of the longitudinal field required to achieve a *given* rate of ionization, is rather limited. Specifically, inspection of Fig. 3 indicates that this reduction is at most of the order of thirty percent. With reference to Fig. 12 of the paper of Krasik, Alpert, and McCoubrey,[2] it is immediately obvious that the effect is much too small to explain the discrepancy between theory and experiment. It is therefore necessary to invoke some other mechanism, such as cumulative or secondary ionization.

V. VALIDITY OF GENERAL METHOD

At this point some remarks on the validity of the general method are in order. It will be recalled that the basis of this method is contained in the assumption that, for a given total energy w, the distribution function is, to a sufficiently good approximation, independent of the spatial coordinate, x, i.e.,

$$f(w,x) \approx F(w). \tag{70}$$

It was implied in the text that (70) is a suitable approximation provided that the space charge field is large compared to the applied field. When this situation prevails, one may infer that the rate of energy change due to motion in the space charge field is large compared to that arising from the applied field. Now, as is exemplified in Sec. IV, it is necessary (for equilibrium) that the energy gain associated with motion in the applied field be compensated by loss due to collisions. Hence, one would expect (70) to be valid provided that the rate of energy change due to motion in the space charge field is large compared to the rate of energy loss due to collisions.

In the case of inelastic collisions, in which most of the energy is lost in one encounter, this condition is equivalent to the requirement that the time for an electron to diffuse throughout the accessible region of the space charge well ($w \gtrsim \varphi$) be small compared to the mean time between inelastic collisions, λ_{ex}/v.

In the case of recoil loss, an electron loses, on the average, a fraction $2m/M$ of its energy per collision. Hence, the requirement would be that the electron diffuses throughout the well in a time small compared to $(M/2m)(\lambda_e/v)$.

These intuitive expectations have been confirmed by a detailed analysis presented in a Westinghouse Research Report.[19]

VI. SUMMARY

In this paper the theoretical treatment of electron energy distributions has been extended to include the case in which—in addition to a spatially homogeneous applied field—there exists a dc space charge field of the type associated with ambipolar diffusion. The method is valid in the limiting case that the dominant mechanism for changing the energy of an electron is its motion in the space charge field. The equation determining the energy distribution [Eq. (34)] is then found to be of a form similar to that prevalent in the absence of the space charge field [Eq. (54′), I]. Illustrative calculations (Sec. V) demonstrate that the principal effect of the space charge field is an enhancement of the tail of the distribution function, with a consequent increase in specific ionization. The magnitude of the latter effect, however, is found to be much too small to account for the large ionization rates required in steady state discharges, such as the positive column of a noble gas dc discharge.

The treatment presented above applies both to the positive column and to low-frequency microwave discharges. The sole difference is that in the latter case one uses the rms microwave field in place of the dc longitudinal field of the positive column.

[19] T. Holstein, Westinghouse Research Report, R-94411-9-0, Higher-Order Approximations to Electron Energy Distributions in D. C. Space Charge Fields; reprints available upon written request.